Rudiments of Conformal Field Theory

Conformal invariance on the worldsheet is a key consistency requirement in string theory. CFT also arise in many problems in statistical mechanics, in particular near second order phase transitions.

Here we shall introduce some basic techniques for dealing with CFT, with emphasis on $D=2$.

Rk CFT's are QFT without mass scale so S matrix is ill-defined. Rather we focus on correlation function of composite operators

1. Conformal group in D dimensions

* Conformal transformations are coordinate transformations $x \mapsto x'$ which preserve the metric up to a scale factor, (hence preserve angles)

$$g_{\mu\nu}(x) \longmapsto g'_{\mu\nu}(x) = \Omega(x') \, g_{\mu\nu}(x')$$

$$\parallel$$

$$\frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} g_{\alpha\beta}(x')$$

In flat $\mathbb{R}^D$ with metric $\eta_{\mu\nu}$ of signature (p,q), these transformations generate the group $SO(p+1, q+1)$.

To see this, consider infinitesimal transf. $\delta g_{\mu\nu} = \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu$

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \underbrace{\frac{2}{D}\left(\partial_\rho \varepsilon^\rho\right)}_{\text{fixed by trace}} \eta_{\mu\nu}$$

Acting with ∂_μ: $\quad \Box \varepsilon_\nu = \left(\frac{2}{D} - 1\right) \partial_\nu (\partial\varepsilon)$

$\Box$: $\quad \partial_\mu \Box \varepsilon_\nu + \partial_\nu \Box \varepsilon_\mu = \frac{2}{D} \Box(\partial\varepsilon) \eta_{\mu\nu}$

hence $\quad \left(\frac{2}{d} - 1\right) \partial_\mu \partial_\nu \left[\partial\varepsilon\right] = \frac{2}{D} \Box(\partial\varepsilon) \eta_{\mu\nu}$

hence $\quad \left[\eta_{\mu\nu} \Box + (D-2) \partial_\mu \partial_\nu\right](\partial\varepsilon) = 0$

For $D \neq 2$, ε must be at most quadratic in x

General solutions are linear combinations of

a) $\varepsilon_\mu = a_\mu$ constant : translations

b) $\varepsilon_\mu = \omega_{\mu\nu} x^\nu$, ω antisym : rotations

c) $\varepsilon_\mu = \lambda x^\mu$: scale transformations

d) $\varepsilon_\mu = b_\mu x^2 - 2 x_\mu (b \cdot x)$: special conformal

linearizations of $\underset{c)}{\mathbb{R}^+} \times \underset{b)}{SO(p,q)} \ltimes \underset{a)}{\mathbb{R}^{p+q}}$

and
$$x \mapsto x' = \frac{x + b x^2}{1 + 2\, b \cdot x + b^2 x^2}$$

or equivalently $\quad x'^\mu / x'^2 = \dfrac{x^\mu}{x^2} + b^\mu$

In fact, $x \mapsto x^\mu / x^2$ is also a conformal transformation

* Conformally invariant theories in $d (>2)$ dimensions are characterized by a basis of fields $\Phi_i(x)$, called 'quasi primary', which transform homogeneously under conformal transf,

$$\Phi_i(x) \longrightarrow \Phi_i'(x) = \Omega(x')^{+\Delta/2}\, \Phi_i(x')$$

ex: $x' = \lambda x$
$\Phi'(x) = \lambda^\Delta \Phi(\lambda x)$

where $\Omega(x') = \left| \partial x' / \partial x \right|^{+2/d}$

such that correlation functions are invariant:

$$G_N \equiv \langle \Phi_1(x_1) \cdots \Phi_N(x_N) \rangle$$
$$= \Omega(x_1')^{+\Delta/2} \cdots \Omega(x_N')^{+\Delta_N/2}\; G_N(x_1', \ldots, x_N')$$

All other fields are linear combinations of Φ_i and their derivatives.

This implies that G_N depends only on the $N(N-3)/4$ cross ratio

$$\frac{r_{ij}\, r_{kl}}{r_{ik}\, r_{jl}}$$, up to some powers of r_{ij} fixed by the conformal weight

(invariant since $r_{ij} \to \dfrac{r_{ij}}{r_i r_j}$ under special conformal transformations)

Conventions for conformal transformations:

We want $\langle \phi_1(x_1)\ \phi_2(x_2) \rangle = \dfrac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}$

$$\langle \phi_1(\lambda x_1)\ \phi_2(\lambda x_2)\rangle = \frac{C_{12}}{\lambda^{\Delta_1+\Delta_2}} \frac{1}{|x_1-x_2|^{\Delta_1+\Delta_2}}$$
$$= \langle \lambda^{-\Delta_1}\phi_1(x_1)\quad \lambda^{-\Delta_2}\phi_2(x_2)\rangle$$

$x_1' = \lambda x_1 \qquad \partial x_1'/\partial x_1 = \lambda^D$

$x_2' = \lambda x_2 \qquad \partial x_2'/\partial x_2 = \lambda^D$

$$\langle \phi_1(x_1')\,\phi_2(x_2')\rangle = \left(\partial x_1'/\partial x_1\right)^{-\Delta_1/D}\left(\partial x_2'/\partial x_2\right)^{-\Delta_2/D} \times \langle \phi_1(x_1)\,\phi_2(x_2)\rangle$$

Define $\boxed{\phi_i'(x) = \left(\frac{\partial x}{\partial x'}\right)^{-\Delta_i/D} \phi_i(x')} = \left[\Omega(x')\right]^{\Delta_i/2}\phi_i(x')$

then $\langle \phi_1'(x_1)\,\phi_2'(x_2)\rangle = \langle \phi_1(x_1)\,\phi_2(x_2)\rangle$

Take $x' = x/|x|^2$: $\quad (dx')^2 = \dfrac{(dx)^2}{(x^2)^2} \qquad \left|\dfrac{\partial x'}{\partial x}\right| = \left(\dfrac{1}{x^2}\right)^D$

$(x'-y')^2 = \dfrac{(x-y)^2}{|x|^2|y|^2}$

$$\phi_i'(x) = (x^2)^{-\Delta_i}\,\phi_i(x/|x|^2)$$

$$\langle \phi_1'(x_1)\,\phi_2'(x_2)\rangle = (x_1^2)^{-\Delta_1}(x_2^2)^{-\Delta_2}\frac{C_{12}}{\left(\frac{x_1}{|x_1|^2}-\frac{x_2}{|x_2|^2}\right)^{\Delta_1+\Delta_2}}$$
$$= (x_1^2)^{-\Delta_1}(x_2^2)^{-\Delta_2}\frac{C_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}}\cdot (x_1^2 x_2^2)^{\frac{\Delta_1+\Delta_2}{2}}$$
$$= (x_1^2)^{\frac{\Delta_2-\Delta_1}{2}}(x_2^2)^{\frac{\Delta_1-\Delta_2}{2}}\langle \phi_1(x_1)\,\phi_2(x_2)\rangle$$

so $C_{12} = 0$ unless $\Delta_1 = \Delta_2$.

Transf of the metric. $\quad g'_{\mu\nu}(x)\,dx^\mu dx^\nu = g_{\mu\nu}(x')\,dx'^\mu dx'^\nu$

$$g'_{\mu\nu}(x) = g_{\alpha\beta}(x')\frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x'^\nu} = \Omega(x)\,g_{\mu\nu}(x)$$

$$\Omega(x') = |\partial x'/\partial x|^{2/D}$$

Eg $N=2$, (no cross ratio)

$$\langle \phi_1(x_1)\,\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}}$$ using translation + scale

$$\langle \phi_1(\lambda x_1)\,\phi_2(\lambda x_2)\rangle = \lambda^{-\Delta_1-\Delta_2}\langle \phi_1(x_1)\,\phi_2(x_2)\rangle$$

Under special conformal trans:

$$\langle \phi_1(x_1/x_1^2\,,\ x_2/x_2^2)\rangle = \frac{C_{12}(r_1 r_2)^{\Delta_1+\Delta_2}}{|r_{12}|^{\Delta_1+\Delta_2}} = \frac{C_{12}\,r_1^{2\Delta_1}\,r_2^{2\Delta_2}}{|r_{12}|^{\Delta_1+\Delta_2}}$$

only if $\Delta_1=\Delta_2$ or $C_{12}=0$

Eg $N=3$: (no cross ratio)

$$\langle \phi_1(x_1)\,\phi_2(x_2)\,\phi_3(x_3)\rangle = \frac{C_{123}}{r_{12}^{\Delta_1+\Delta_2-\Delta_3}\ r_{23}^{\Delta_2+\Delta_3-\Delta_1}\ r_{31}^{\Delta_3+\Delta_1-\Delta_2}}$$

Eg $N=4$; 2 cross ratio

$$G_4 = \prod_{i<j} r_{ij}^{-(\Delta_i+\Delta_j)+\Delta/3}\ \underbrace{f}_{\text{arbitrary}}\left(\frac{r_{12}r_{34}}{r_{13}r_{24}}\,,\ \frac{r_{13}r_{34}}{r_{23}r_{41}}\right)$$

* Invariance under scale transformations imply that the stress-energy tensor $T_{\mu\nu} = \frac{1}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}}$ is <u>traceless</u>, at least in flat space.

In general, there can be quantum anomalies, eg in $D=4$

$$T_\mu{}^\mu = a\left[R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R\right] \qquad \text{Euler density}$$

$$\quad - c\,W^2_{\mu\nu\rho\sigma} \qquad \text{Weyl tensor}^2$$

($\partial_\mu T^{\mu\nu}=0$ by definition)

2. Conformal invariance in 2 dimensions

For $d=2$, in Euclidean signature say, local conformal transformations are generated by any holomorphic function :

$$z' = f(z) \quad , \quad \bar{z}' = \bar{f}(\bar{z})$$

$$g'_{z\bar{z}}\, dz d\bar{z} = g_{z\bar{z}}\, dz' d\bar{z}' = g_{z\bar{z}} |f'(z)|^2 dz d\bar{z}$$

$$\Rightarrow \Omega(z') = |f'|^2$$

Taking $f(z) = 1 - \varepsilon z^{n+1}$ gives generators

$$\ell_n = -z^{n+1} \partial_z \quad , \quad \tilde{\ell}_n = -\bar{z}^{n+1} \partial_{\bar{z}}$$

satisfying $\quad [\ell_m, \ell_n] = (m-n)\, \ell_{m+n}$, etc

The generators $\ell_0, \ell_{\pm 1}, \bar{\ell}_0, \bar{\ell}_{\pm 1}$ form a closed subalgebra, corresponding to global conformal transformations of the Riemann sphere

$$ds^2 = \frac{dz\, d\bar{z}}{(1+z\bar{z})^2} \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{C}) = SO(1,3)$$

$$z \longmapsto \frac{\alpha z + \beta}{\gamma z + \delta}$$

A conformally invariant field theory in $d=2$ is characterised by

- a set of quasi primary fields Φ, with conformal dimension $\Delta, \bar{\Delta}$, transforming as

$$\Phi'(z,\bar{z}) = \left(\frac{\partial f}{\partial z}\right)^{\Delta} \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{\Delta}} \Phi(f(z), \bar{f}(\bar{z})) \qquad \begin{cases} \Delta + \bar{\Delta} : \text{scaling weight} \\ \Delta - \bar{\Delta} = \underline{\text{spin}} \end{cases}$$

  under <u>global</u> conformal transformations

- a smaller set of primary fields Φ_i, transforming the same way under all <u>local</u> conformal transformations.

- correlation function of quasi-primary fields on the sphere are invariant,

$$G_N(z_i, \bar{z}_i) = \left\langle \prod_{i=1}^{N} \Phi_i(z_i, \bar{z}_i) \right\rangle = \prod_{i=1}^{N} \left(\frac{\partial f}{\partial z}\right)^{\Delta_i} \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{\Delta}_i} G_N\left(f(z_i), \bar{f}(\bar{z}_i)\right)$$

This fixes the form of 2pt, 3pt functions:

$$G_2 = \frac{C_{12}}{(z_{12})^{2\Delta}(\bar{z}_{12})^{2\bar{\Delta}}} \quad \text{if } \begin{matrix}\Delta_1=\Delta_2\equiv\Delta \\ \bar{\Delta}_1=\bar{\Delta}_2\equiv\bar{\Delta}\end{matrix}, \quad 0 \text{ otherwise}$$

$$G_3 = \frac{C_{123}}{z_{12}^{\Delta_1+\Delta_2-\Delta_3}\, z_{23}^{\Delta_2+\Delta_3-\Delta_1}\, z_{13}^{\Delta_1+\Delta_3-2\Delta_2}} \times cc$$

Higher pt functions are functions of the cross ratio

$$[z_1,z_2,z_3,z_4] \equiv \frac{(z_i-z_j)(z_k-z_l)}{(z_i-z_k)(z_j-z_k)}, \quad cc$$

Note however that there are relations among them, eg for $N=4$ there is only one cross ratio x, mapped to $\left\{x, 1-x, \frac{x}{1-x}, \frac{1}{x}, \frac{1}{1-x}, \frac{1-x}{x}\right\}$ under permutations of 4 pts.

In general, 3 pts can be mapped to $0, 1, \infty$, leaving $N-3$ indep^t. ratios

- The stress tensor $T_{\mu\nu}$ is traceless on flat worldsheet,

$$\partial^\mu T_{\mu\nu} = 0, \quad T_{z\bar{z}} = 0 \quad \Rightarrow \quad \begin{matrix}\bar{\partial}_z T_{zz} = 0 \\ \partial_z T_{\bar{z}\bar{z}} = 0\end{matrix}$$

so $T \equiv T_{zz}$ is holomorphic; quasi primary with $(\Delta,\bar{\Delta}) = (2,0)$

$\bar{T} = T_{\bar{z}\bar{z}}$ is antiholomorphic; " $(0,2)$

keep this for later

On a curved worldsheet however,

$$T^\mu{}_\mu = -\frac{c}{12} R \qquad \text{where } c \text{ is the 'central charge'}$$

As we shall see, this leads to anomalies in the Virasoro algebra.

It also implies that T does not transform as a primary field, rather

$$T(z) = (f'(z))^2\, T(f(z)) + \frac{c}{12}\left(\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right)$$

$S(w,z) = (\partial_z f)^2 S(w,f) + S(f,z)$ Schwarz derivative $S(f,z)$

The stress tensor generates conformal transformations:

$$T_{\alpha\beta} = \frac{-4\pi}{\sqrt{g}} \frac{\delta S}{\delta g^{\alpha\beta}}$$

$\delta z = \varepsilon(z)$

$\delta \bar{z} = \bar{\varepsilon}(\bar{z})$

$$j = \int d\sigma \left(T_{0z}\, \varepsilon + T_{0\bar{z}}\, \bar{\varepsilon} \right)$$

$$= \int dz \left(T_{zz}\, \varepsilon + T_{\bar{z}\bar{z}}\, \bar{\varepsilon} \right)$$

since $T_{z\bar{z}} = 0$

$z = \tau + i\sigma$

$d\tau = \frac{dz + d\bar{z}}{2}$ $\quad \partial_\tau = \partial_z + \partial_{\bar{z}}$

$d\sigma = \frac{dz - d\bar{z}}{2i}$ $\quad \partial_\sigma = \partial_z - \partial_{\bar{z}}$

- The existence of a convergent operator product expansion (OPE):

$$O_i(z,\bar z)\, O_j(w,\bar w) = \sum_k \underbrace{C_{ij}^k}_{\text{same as in 3pt function}} (z-w)^{\Delta_k-\Delta_i-\Delta_j} (\bar z-\bar w)^{\bar\Delta_k-\bar\Delta_i-\bar\Delta_j} \underbrace{O_k(w,\bar w)}_{\text{quasi-primaries}}$$

as operator insertions in correlation functions, as long as other operators in correlator stay at finite distance

- Singular terms determine the algebra of operators, in particular an operator Φ_i is primary iff its OPE with $T(z)$ is of the form

$$T(z)\,\Phi_i(w) = \frac{\Delta_i\,\Phi_i(w)}{(z-w)^2} + \frac{1}{(z-w)}\,\partial_w \Phi_i(w) + \text{reg}$$

Quasi primary fields can have singularities of higher order as $z \to w$.

Recall $\bar{\partial}_z \frac{1}{z-w} = 2\pi\delta$

$\partial_z\bar{\partial}_{\bar z} \langle X(z) X(w)\rangle = \bar{\partial}_z\left(-\frac{\ell_s^2}{2}\frac{1}{z-w}\right) = -\frac{\ell_s^2}{2}\cdot 2\pi\delta$

Ex free boson on the sphere

$$S = \frac{1}{2\pi \ell_s^2}\int d^2z \; \partial X \bar{\partial} X$$

The propagator is $\langle X(z,\bar z)\, X(w,\bar w)\rangle = -\frac{\ell_s^2}{2}\log\left(|z-w|^2 \mu^2\right)$

where μ = IR cut-off

X is not a conformal field, but ∂X is a primary field:

$$\langle \partial X(z)\, \partial X(w)\rangle = \frac{-\ell_s^2}{2(z-w)^2} \qquad (\Delta,\bar\Delta) = (1,0)$$

The energy momentum tensor is

$$T(z) = -\frac{1}{\ell_s^2} :\partial X \partial X:(z)$$

$: :$ is the conformal normal ordering, defined by

$$:\partial X\partial X:(z) = \lim_{z\to w}\left(\partial X(z)\partial X(w) - \langle \partial X(z)\partial X(w)\rangle\right)$$

This ensures in particular that $\langle T(z)\rangle = 0$

More generally, one must subtract all Wick contractions

Eg, $\langle \partial X(z)\rangle = 0$ (due to $X \to -X$ symmetry)

$\langle T(z)\, \partial X(w)\rangle = 0$

$\vdots$

$$\begin{aligned}
&\langle T(z)\, \partial X(z_1)\, \partial X(z_2)\rangle \\
&= -\frac{1}{\ell_s^2}\langle :\partial X \partial X:(z)\; \partial X(z_1)\, \partial X(z_2)\rangle \\
&= -\frac{2}{\ell_s^2}\langle \partial X(z)\,\partial X(z_1)\rangle \langle \partial X(z)\,\partial X(z_2)\rangle \\
&= -\frac{\ell_s^2}{2}\frac{1}{(z-z_1)^2\,(z-z_2)^2}
\end{aligned}$$

in agreement with Ward identity

Moreover, at the operator level:

$$\begin{aligned}
T(z)\,\partial X(w) &= -\frac{2}{\ell_s^2}\,\partial X(z)\,\langle \partial X(z)\partial X(w)\rangle + :\partial X\partial X\partial X:(z) \\
&= \frac{\partial X(z)}{(z-w)^2} + :\partial X\partial X\partial X:(z) \\
&= \frac{\partial X(w)}{(z-w)^2} + \frac{1}{z-w}\,\partial^2 X(w) + \text{regular}
\end{aligned}$$

This is in fact (as we shall see) the defining property of primary fields:

$$T(z)\,\Phi(w) = \frac{\Delta\,\Phi(w)}{(z-w)^2} + \frac{1}{z-w}\partial_w\Phi(w) + \text{reg}$$

Another set of primary fields in the free boson CFT are the vertex operators,

$$V_p = :e^{ipX}: \qquad \partial_z V_p = :ip\partial X\, e^{ipX}:$$

$$T(z)\,V_p(w) = -\frac{1}{\ell_s^2}\sum_{n=0}^{\infty}\frac{(ip)^n}{n!} :\partial X\partial X(z): \; :X^n(w,\bar w):$$

$$= -\frac{1}{\ell_s^2}\sum_{n=0}^{\infty}\frac{(ip)^n}{n!}\Big\{ n(n-1)\langle \partial X(z)\,X(w)\rangle^2 :X^{n-2}(w,\bar w): + 2n\,\langle \partial X(z)\,X(w)\rangle :\partial X(z) X^{n-1}(w,\bar w): \Big\} + \text{reg}$$

$$= -\frac{1}{\ell_s^2}\Big\{\Big(-\frac{\ell_s^2}{2}\,\frac{1}{z-w}\Big)^2(ip)^2 \cdot :e^{ipX}:(w) + 2ip\cdot -\frac{\ell_s^2}{2}\frac{1}{(z-w)} :\partial X(w)\, e^{ipX}(w): \Big\} + \text{reg}$$

$$= \frac{\ell_s^2 p^2}{4(z-w)^2}V_p + \frac{1}{z-w}\partial_p V_p + \text{reg}$$

so $\Delta = \dfrac{\ell_s^2 p^2}{4}$

Moreover, $\langle \prod_{i=1}^{n} V_{p_i}(z_i,\bar z_i)\rangle = 2\pi \prod_{i<j} |z_{ij}|^{\ell_s^2 p_i p_j}\,\delta(\Sigma p_i)$ from Wick's theorem

necessary for cancelling IR divergences

On the other hand, $T(z)$ is not a primary field, rather it satisfies

$$T(z)\,T(w) = \frac{1}{\ell_s^4} :\partial X\partial X(z): \; :\partial X\partial X(w):$$

$$= \frac{2}{\ell_s^4}\langle \partial X(z)\partial X(w)\rangle^2 + \frac{4}{\ell_s^2}\langle \partial X(z)\partial X(w)\rangle :\partial X(z)\partial X(w): + \text{reg}$$

$$= \frac{1/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial_w T + \dots$$

we shall read off the conformal anomaly from this term!

2. Radial quantization and operator / state correspondence

Correlation functions can be reinterpreted as matrix elements in the Hilbert space of the CFT on the cylinder as follows.

The conformal transformation $w = \tau + i\sigma \longrightarrow z = e^{w}$ maps the cylinder $\tau \in \mathbb{R}$, $\sigma \in [0, 2\pi[$ to the Riemann sphere punctured at $z=0$ and $z=\infty$, or z plane $\setminus \{0\}$

Fixed time slices $\tau = cte$ on the cylinder correspond to circles of fixed radius in the z plane.

An insertion of a field A at $z=0$ creates an 'in' state at $\tau = -\infty$:

$$|A_{in}\rangle = \lim_{z \to 0} A(z, \bar{z})\, |0\rangle$$

Similarly, an 'out' state is created by an insertion at $z = \infty$:

$$\langle A_{out}| = \lim_{z \to \infty} \langle 0|\, A(z, \bar{z})\, z^{2\Delta}\, \bar{z}^{2\bar{\Delta}}$$

in such a way that C_{ij} is a transition amplitude,

$$\langle A^2_{out} \,|\, A^1_{in} \rangle = \lim_{z_1 \to 0,\, z_2 \to \infty} \langle A_2(z_2, \bar{z}_2)\, A_1(z_1, \bar{z}_1)\, z_2^{2\Delta}\, \bar{z}_2^{2\bar{\Delta}} \rangle$$

$$= C_{12}$$

The state $|0\rangle$ is created by the identity operator and must be $SL(2,\mathbb{C})$ invariant.

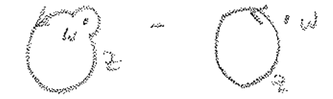
More generally, n-point functions can be related as matrix elements of time ordered (in w plane) or radially ordered (in z plane) operators.

Conserved charges $Q = \frac{1}{2\pi}\int_{\sigma=0}^{2\pi} j_\tau \, d\sigma$ on the cylinder can be rewritten as contour integrals $Q = \frac{1}{2\pi i} \oint j(z)\, dz$ in the z plane around $z=0$.

(Rk: only if j is a chiral current, i.e. $\bar{\partial} j = \partial_{\bar{z}} \bar{j} = 0$)

* Commutators are thus nested contour integrals:

$$[Q, \Phi(w)] = \frac{1}{2\pi i}\oint \frac{dz}{2\pi i}\, J(z)\,\phi(w) = \frac{1}{2\pi i}\oint_{\circlearrowleft w} dz\, J(z)\,\phi(w)$$



This can be computed in terms of OPE of J and ϕ.

In particular, conformal transf are generated by

$$Q_\varepsilon = \frac{1}{2\pi i}\oint dz\, \varepsilon(z)\, T(z) \qquad \text{for } \delta z = \varepsilon(z)$$

The transformation property $\delta_\varepsilon \phi = (\Delta\,\partial\varepsilon + \varepsilon\,\partial)\phi$
for primary fields requires the OPE

$$T(z)\,\phi(w) = \frac{\Delta}{(z-w)^2}\phi(w) + \frac{1}{z-w}\partial_w \phi(w) + \text{reg},$$

For quasiprimary fields, one requires above property for

$\varepsilon = 1$: $\quad \delta_\varepsilon \phi = \partial\phi$

$\varepsilon = z$: $\quad\quad\quad z\,\partial\phi + \Delta\phi$

$\varepsilon = z^2$: $\quad\quad\quad z^2\partial\phi + 2\Delta z\,\phi$

So one can allow more singular terms $\frac{O_n}{(z-w)^{n}}{}_{n\geq 4}$ in OPE
where O_n must have dimension $\Delta_n = \Delta + 2 - n$.

In unitary CFT's, we shall see that $\Delta \geq 0$, so n is bounded.

* Eg for $\phi = T$, one can have at most

$$T(z)\,T(w) = \frac{c/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg}$$

implying
$$\delta_\varepsilon T = 2\,\partial_\varepsilon T + \varepsilon\,\partial T + \frac{c}{12}\partial^3\varepsilon$$

which exponentiates to $T'(z) = \left(f'(z)\right)^2 T(f(z)) + \frac{c}{12}\left(\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right)$

* The Hilbert space must form a representation of the Virasoro algebra:

The state $|\phi\rangle$ associated to a primary operator transforms under L_m as

$$\frac{1}{2\pi i}\oint_0 dz\, z^{m+1} \underbrace{T(z)\,\phi(0)}_{\frac{\Delta\phi}{(z-w)^2}+\frac{\partial\phi}{z-w}} |0\rangle$$

so $\quad L_{m>0}|\phi\rangle = 0 \quad ; \quad$ "highest weight"

$\quad\quad L_0|\phi\rangle = \Delta$

Other states are obtained by acting with $L_{m<0}$: "descendants"

$$L_{-1}|\phi\rangle = |\partial\phi\rangle$$ quasi primaries only

In particular, the state created by operator $\mathbb{1}$ is invariant under $SL(2,\mathbb{C})$; $\quad L_{-2}|\mathbb{1}\rangle = |T\rangle$

The whole tower of descendants forms a 'Verma module'

* Hermiticity on the cylinder requires $L_n = L_{-n}^{+}$

It follows that $\Delta \geq 0$ for any state / operator : (quasi primary)

$$\| L_{-1}|\phi\rangle \|^2 = \langle\phi|L_1 L_{-1}|\phi\rangle$$
$$= \langle\phi|[L_1, L_{-1}]|\phi\rangle$$
$$= 2\Delta \langle\phi,\phi\rangle$$

Moreover $c>0$:

$$\| L_{-n}|0\rangle \|^2 = \langle 0|[L_n, L_{-n}]|0\rangle$$
$$= \frac{c}{12} n(n^2-1) \geq 0$$

* In particular, due to the conformal anomaly,

$$T_{plane}(z) = \frac{1}{z^2}\left(T_{cyl}(w) + \frac{c}{24} \right) \qquad z = e^{\tau + i\sigma} = e^{w}$$

The Taylor expansion on the plane is then related to Fourier expansion on cylinder: ← in general, choose Δ here

$$T_{plane} = \sum_{n\in\mathbb{Z}} L_n z^{n-2}$$

$$T_{cyl} = \sum_{n\in\mathbb{Z}} L_n^{cyl} e^{-n(\tau+i\sigma)}$$

$$L_n^{cyl} = L_n - \frac{c}{24}\delta_{n,0}$$

* The Virasoro algebra for the modes $L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z)\, dz$ is then

$$[L_m, L_n] = \oint \frac{dw}{2\pi i} \oint_{C_w} \frac{dz}{2\pi i}\; z^{n+1}\, T(z)\; w^{m+1}\, T(w)$$

$$= \quad " \quad " \quad z^{n+1} w^{m+1} \left(\frac{c}{2}\frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T}{z-w} \right)$$

$$= \oint \frac{dw}{2\pi i} \left(\frac{c}{12}\, n(n+1)(n-1)\, w^{n-2} w^{m+1} + 2(n+1)\, w^{n} w^{m+1}\, T(w) + w^{n+1} w^{m+1}\, \partial T(w) \right)$$

← integ. by parts

$$= (n-m) L_{n+m} + \frac{c}{12} n(n^2-1)\, \delta_{m+n,0}$$

To relate this to the trace anomaly.

use energy conservation $\quad \partial_z T_{z\bar{z}} = -\bar{\partial}_{\bar{z}} T_{zz}$

hence

$$\partial_z T_{z\bar{z}} \cdot \partial_w T_{w\bar{w}} = \bar{\partial}_{\bar{z}} \bar{\partial}_{\bar{w}} \left[\frac{c/2}{(z-w)^4} + \cdots \right]$$

In the sense of distributions, $\quad \bar{\partial}_{\bar{z}} \dfrac{1}{z-w} = 2\pi\, \delta(z-w, \bar{z}-\bar{w})$

$$\frac{1}{(z-w)^4} = \frac{1}{6} \partial_z^2 \partial_w \frac{1}{z-w}$$

so

$$\partial_z T_{z\bar{z}}\ \partial_w T_{w\bar{w}} = \frac{c}{12} \partial_z^2 \partial_w \partial_{\bar{w}} \partial_{\bar{z}} \frac{1}{z-w}$$

$$T_{z\bar{z}}\ T_{w\bar{w}} = \frac{c}{12} \partial_z \partial_{\bar{w}} \cdot 2\pi\, \delta(z-w, \bar{z}-\bar{w}) + \cdots$$

$$T^\alpha{}_\alpha(\sigma)\ T^\beta{}_\beta(\sigma') = -\frac{\pi c}{3} \partial^2 \delta(\sigma-\sigma') + \cdots$$

Now the variation of $\langle T^\alpha{}_\alpha \rangle$ under Weyl transformation $\quad \delta\gamma_{\alpha\beta} = 2\omega\, \gamma_{\alpha\beta}$

$\delta\gamma^{\alpha\beta} = -2\omega\, \gamma^{\alpha\beta}$

is $\quad \dfrac{1}{4\pi} \displaystyle\int D\phi\, e^{-S}\, T^\alpha{}_\alpha(\sigma) \int d^2\sigma' \sqrt{\gamma}\, \delta\gamma^{\beta\gamma}\, T_{\beta\gamma}(\sigma')$

$$= -\frac{1}{2\pi} \int D\phi\, e^{-S}\, \omega(\sigma')\, T^\alpha{}_\alpha(\sigma)\, T^\beta{}_\beta(\sigma')$$

$$= \frac{c}{6} \partial^2 \omega$$

But for a metric $\gamma_{\alpha\beta} = e^{2\omega} \delta_{\alpha\beta}$, the Ricci scalar is $R = -2 e^{-2\omega} \partial^2 \omega$

so

$$\langle T^\alpha{}_\alpha \rangle = -\frac{c}{12} R$$

Ex free scalar:

On cylinder. $\partial X = \frac{i\ell_s}{\sqrt{2}} \sum_{k\in\mathbb{Z}} e^{-k(\tau+i\sigma)} \alpha_k^\mu$

On plane. $\partial X = \frac{i\ell_s}{\sqrt{2}} \sum_{k\in\mathbb{Z}} z^{-m-1} \alpha_m^\mu$

$$\alpha_m^\mu = \frac{\sqrt{2}}{\ell_s} \oint \frac{dz}{2\pi} z^m \, \partial X^\mu(z)$$

$$\Rightarrow \quad [\alpha_m^\mu, \alpha_n^\nu] = m\,\delta_{m+n}$$

$$\alpha_{-m}^\mu |0\rangle = \frac{\sqrt{2}}{\ell_s} \oint \frac{dz}{2\pi} z^{-m} \partial X^\mu(z)|0\rangle$$

$$= \frac{i\sqrt{2}}{\ell_s} \frac{\partial^m X^\mu}{(m-1)!}(0)|0\rangle \qquad \text{if } m \geq 1 \text{, } 0 \text{ if } m \leq 0$$

The operator associated to state $|p\rangle$ is e^{ipX}.

$$\alpha_{-m}^\mu |p\rangle = \frac{\sqrt{2}}{\ell_s} \oint \frac{dz}{2\pi} z^{-m} \underbrace{\partial X^\mu(z)\, e^{ipX}(0)}_{\frac{ip\, e^{ipX}}{z} + \text{reg}} |0\rangle$$

$$= 0 \text{ if } m < 1, \quad \frac{\sqrt{2}\, p^\mu}{\ell_s} \text{ if } m = 0$$

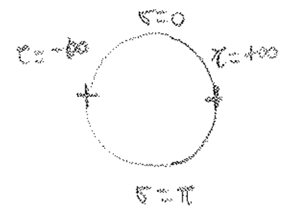
3. Conformal field theory on the disk

The free propagation of an open string is described by the strip

$$0 \leq \sigma \leq \pi \quad , \quad -\infty < \tau < \infty$$

This is mapped by $z = e^{w} = e^{\tau + i\sigma}$ to the upper half plane

and further to the disk by $u = re^{i\theta} = \dfrac{z+i}{z-i}$



The condition that no momentum should flow through the boundary requires (in z plane) $\quad T_{zz} = T_{\bar{z}\bar{z}} = 0 \quad$ at $z = \bar{z}$

Similarly, for a free boson with Neumann bc, $\partial X = \bar{\partial} X$ at $z = \bar{z}$

Dirichlet bc, $\partial X = -\bar{\partial} X$ at $z = \bar{z}$

It is convenient to obtain the UHP by identifying with the antiholomorphic involution $z \to \bar{z}$ (or $u \to +1/\bar{u}$)

Propagators in the presence of boundaries can be obtained by method of images, eg for a free boson

$$G = \langle X(z,\bar{z})\, X(w,\bar{w}) \rangle$$

$$= -\frac{l_s^2}{2}\left[\log|z-w|^2 \pm \log|z-\bar{w}|^2 \right]$$

$\ominus$ ensures $G = 0$ if $w = \bar{w}$ or $z = \bar{z}$; D

$\oplus$ ensures $\partial_z \partial_w G = 0$ if $\quad$ " $\quad$; N

The radial quantization & state/operator correspondance proceed as before, except that local operators are inserted on the boundary.

Ex show that conformal dim of $e^{ipX}\big|_{z=\bar{z}}$ is $l_s^2 p^2$, unlike in the bulk $\dfrac{l_s^2 p^2}{4}$

Exercices 6. Conformal field theory

1. Using the mode expansion and canonical commutation relations, compute the two-point function of fermions in the Ramond sector, on the sphere. Answer: $\langle S | \psi^i(z) \psi^j(w) | S \rangle$
$$= \delta^{ij} \frac{z+w}{2\sqrt{zw}} \cdot \frac{1}{z-w} .$$
Extract the vev of the stress energy tensor $\langle S | T(z) | S \rangle$ and conclude that the conformal dimension of S is $N/16$.

2. Consider a free massless scalar field X, with deformed stress energy tensor
$$T = -\frac{1}{\ell_s^2} :\partial X \partial X: + \frac{Q}{\ell_s \sqrt{2}} \partial^2 X$$
and usual propagator
$$\langle X(z,\bar{z}) \, X(w,\bar{w}) \rangle = -\frac{\ell_s^2}{2} \log |z-w|^2$$
Compute the central charge and dimension of the vertex operator $:e^{ipX}:$

Answers: $c = 1 + 3Q^2$, $\Delta(p) = \frac{\ell_s^2}{4} p^2 + iQ \frac{\ell_s p}{2\sqrt{2}}$

[Optional: show that the $u(1)$ symmetry associated to $J = \frac{i\sqrt{2}}{\ell_s} \partial X$ is anomalous, in the sense that
$$T(z) J(w) = \frac{iQ}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{z-w} + \text{reg.}$$
Conclude that $\langle \prod_{i=1}^n V_{p_i} \rangle = 0$ unless $i\sqrt{2}\,\ell_s \sum_{i=1}^n p_i = Q\chi$ on a Riemann surface of genus g.]

String Theory - Exercises 3

* Verify that the correlator of 4 quasiprimary fields Φ_i on the sphere is constrained by global conformal invariance to take the form

$$G_4 = \langle \prod_{i=1}^{4} \Phi_i(z_i,\bar z_i) \rangle = f(x,\bar x) \prod_{i<j} (z_i - z_j)^{-\Delta_i - \Delta_j + \frac{\Delta}{3}} (\bar z_i - \bar z_j)^{-\bar\Delta_i - \bar\Delta_j + \frac{\bar\Delta}{3}}$$

where $(\Delta_i, \bar\Delta_i)$ are the conformal dimensions of Φ_i,

$$\Delta = \sum_{i=1}^{4} \Delta_i \quad , \quad \bar\Delta = \sum_{i=1}^{4} \bar\Delta_i$$

and $f(x,\bar x)$ is an arbitrary function of the conformal cross ratio

$$x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}$$

* Consider a free massless scalar field X, with deformed stress energy tensor

$$T = -\frac{1}{\ell_s^2} :\partial X \partial X: + \frac{Q}{\ell_s \sqrt{2}} \partial^2 X$$

and standard propagator $\langle X(z,\bar z), X(w,\bar w) \rangle = -\frac{\ell_s^2}{2} \log |z-w|^2$

Show that the central charge and dimension of vertex operator $:e^{ipX}:$ are

$$c = 1 + 3Q^2 \quad , \quad \Delta(p) = \frac{\ell_s^2}{4} p^2 + iQ \frac{\ell_s p}{2\sqrt{2}}$$

Show that the OPE of T with the $U(1)$ current $J = \frac{i\sqrt{2}}{\ell_s} \partial X$ is

$$T(z) J(w) = \frac{iQ}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{z-w} + \text{reg}$$

Conclude that the $U(1)$ symmetry is anomalous on curved worldsheets,

$$\partial_\alpha J^\alpha = \frac{iQ}{8} R^{(2)}$$

hence that correlators $\langle \prod_{i=1}^{N} V_{p_i}(z_i) \rangle$ on a Riemann surface of genus g vanish unless

$$\sum_{i=1}^{n} p_i = \frac{Q (2-2g)}{i\sqrt{2}}$$

[Use Gauss Bonnet formula $\frac{1}{4\pi} \int d^2\sigma \sqrt{g} R^{(2)} = 2 - 2g$]

String theory - Exercises 4

* Show that the consistency of the transformation of the stress tensor

$$T(z) \rightarrow T'(z) = \left(\frac{\partial f}{\partial z}\right)^2 T(f(z)) + \frac{c}{12} S(f,z),$$

under successive conformal transformations requires that

$$S(f \circ g, z) = S(g,z) + \left(\frac{\partial g}{\partial z}\right)^2 S(f,g)$$

Verify that the Schwarz derivative

$$S(f,z) = f'''/f' - \frac{3}{2}\left(f''/f'\right)^2$$

satisfies this property; moreover that $S(f,z)$ iff f is a fractional linear transformation $f(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$

* Consider a unitary CFT with $c = \frac{1}{2}$ and a primary field ψ of dimension $1/2$ (as in the 2D Ising Model).
Show that $|\chi\rangle \equiv \left(L_{-2} - \frac{3}{4} L_{-1}^2\right) |\psi\rangle$ is annihilated by all $L_{n>0}$ and has 0 norm.

* Compute the central charge for a free chiral (anticommuting) fermion with propagator $\psi(z)\,\psi(w) = \frac{1}{z-w} = -\psi(w)\,\psi(z)$
and stress tensor $T(z) = -\frac{1}{2} :\psi \partial \psi: (z)$