Rudiments of Conformal Field Theory

We have seen that conformal invariance on the worldsheet is a key consistency requirement in string theory. Conformal field theories also arise in many problems in statistical mechanics, e.g. in the study of phase transitions.

In this lecture we shall introduce some basic techniques for dealing with 2D CFT

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1. Conformal invariance in two dimensions

* Conformal transformations are diffeomorphisms which preserve the metric up to a (possibly position dependent) scale factor.

Eg for the flat metric $ds^2 = \sum_i dx_i^2$ on $\mathbb{R}^d$, infinitesimal transformations include $\delta x^\mu = \varepsilon^\mu$ with $\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \frac{2}{d}(\partial\cdot\varepsilon)\,\delta_{\mu\nu}$

- translations: $\varepsilon^\mu = a^\mu$ (cte)
- rotations: $\varepsilon^\mu = \omega^{\mu\nu} x^\nu$ $\quad (\omega^{\mu\nu} = -\omega^{\nu\mu})$
- scale transformations: $\varepsilon^\mu = \lambda x^\mu$
- special conformal transformations $\varepsilon^\mu = b^\mu (x\cdot x) - 2(b\cdot x)\, x^\mu$

These $d + \frac{d(d-1)}{2} + 1 + d = \frac{(d+1)(d+2)}{2}$ generators satisfy the Lie algebra of $SO(1, d+1)$

$$\left(\begin{array}{cc|c} 0 & \lambda & a-b \\ -\lambda & 0 & a+b \\ \hline b-a & a+b & \omega_{\mu\nu} \end{array}\right)$$

In $d=2$, the space of (local) conformal transformations is much larger, including all $z \to f(z)$, $\bar{z} \to \bar{f}(\bar{z})$ for any analytic function f

Taking $f(z) = 1 - \varepsilon z^{n+1}$ gives generators

$$l_n = -z^{n+1}\partial_z \qquad \tilde{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}$$

$$[l_n, l_m] = (m-n)\, l_{m+n}$$

The generators $l_0, l_{\pm 1}, \tilde{l}_0, \tilde{l}_{\pm}$ form a closed subalgebra, corresponding to global conformal transformations of the Riemann sphere.

$$ds^2 = \frac{dz\,d\bar{z}}{(1+z\bar{z})^2} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$$

$$z \longmapsto \frac{az+b}{cz+d} \qquad ad-bc=1, \quad a,b,c,d \in \mathbb{C}$$

Given 4 pts z_1, z_2, z_3, z_4, the 'cross ratio'

$$x = [z_1, z_2, z_3, z_4] = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}$$ is invariant under $SL(2,\mathbb{C})$

Under permutations of z_i, $[z_i, z_j, z_k, z_l] \in \left\{ x, 1-x, \frac{x}{1-x}, \frac{1}{x}, \frac{1}{1-x}, \frac{1-x}{x} \right\}$

Any set of 4 pts can always be mapped to e.g. $[\infty, 1, x, 0]$

(hence 3 pts can always be mapped to $[\infty, 1, 0]$
2 pts $[\infty, 0]$)

* A conformally invariant field theory is characterized by

- a set of 'primary fields' Φ transforming as

$$\Phi(z,\bar{z}) \longrightarrow \left(\frac{\partial f}{\partial z}\right)^{\Delta} \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{\Delta}} \Phi(f(z), \bar{f}(\bar{z}))$$

under any local conformal transformation

- a larger set of 'quasi primary fields' transforming in the same way under global conformal transformations. This in particular contains all derivatives of primary fields

– Correlation functions of quasi-primary fields are invariant under global conformal transformations (on the sphere)

$G_N \equiv \left\langle \prod_{i=1}^{N} \Phi_i(z_i, \bar{z}_i) \right\rangle$ satisfies,

$$G_N(z_i, \bar{z}_i) = \prod_{i=1}^{N} \left(\frac{\partial f}{\partial z}\right)_{z \to z_i}^{\Delta_i} \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)_{\bar{z} \to \bar{z}_i}^{\bar{\Delta}_i} G_N(f(z_i), \bar{f}(\bar{z}_i))$$

or infinitesimally:

$$\sum_{i=1}^{N} \partial_i G_N = 0$$

$$\sum_{i=1}^{N} (z_i \partial_i + \Delta_i)\, G_N = 0$$

$$\sum_{i=1}^{N} (z_i^2 \partial_i + 2 z_i \Delta_i)\, G_N = 0$$

In particular, 2pt and 3pt functions are fixed up to a constant.

$$G_2 = \frac{C_{12}}{(z_{12})^{2\Delta} (\bar{z}_{12})^{2\bar{\Delta}}} \quad \text{if} \quad \begin{array}{l} \Delta_1 = \Delta_2 \equiv \Delta \\ \bar{\Delta}_1 = \bar{\Delta}_2 \equiv \bar{\Delta} \end{array}, \ 0 \text{ otherwise}$$

$$G_3 = \frac{C_{123}}{z_{12}^{\Delta_1 + \Delta_2 - \Delta_3}\, z_{23}^{\Delta_2 + \Delta_3 - \Delta_1}\, z_{13}^{\Delta_1 + \Delta_3 - \Delta_2} \times cc}$$

– a conserved traceless energy momentum tensor $T_{\mu\nu}$

$$T_{z\bar{z}} = 0, \quad \partial_\mu T^{\mu\nu} = 0 \quad \Rightarrow \quad \begin{array}{l} \bar{\partial}_z T_{zz} = 0 \\ \partial_{\bar{z}} T_{\bar{z}\bar{z}} = 0 \end{array}$$

$T \equiv T_{zz}$ is holomorphic, T is quasiprimary with $(\Delta, \bar{\Delta}) = (2, 0)$
$\tilde{T} = T_{\bar{z}\bar{z}}$ is antiholomorphic

Satisfying conformal Ward identities: for any primary fields $\{\phi_i\}$,

$$\left\langle T(z) \prod_{i=1}^{n} \phi_i(z_i, \bar{z}_i) \right\rangle = \sum_{i=1}^{n} \left(\frac{\Delta_i}{(z - z_i)^2} + \frac{1}{(z - z_i)} \frac{\partial}{\partial z_i} \right) \left\langle \prod_{i=i}^{n} \phi_i(z_i, \bar{z}_i) \right\rangle$$

Ex free boson on the sphere

$$S = \frac{1}{2\pi \ell_s^2} \int d^2z \; \partial X \bar{\partial} X$$

The propagator is $\langle X(z,\bar{z})\, X(w,\bar{w}) \rangle = -\frac{\ell_s^2}{2} \log\left(|z-w|^2 \mu^2 \right)$

where μ = IR cut-off

X is not a conformal field, but ∂X is a primary field:

$$\langle \partial X(z)\, \partial X(w) \rangle = \frac{-\ell_s^2}{2(z-w)^2} \qquad (\Delta, \bar{\Delta}) = (1, 0)$$

The energy momentum tensor is

$$T(z) = -\frac{1}{\ell_s^2} :\partial X \partial X:(z)$$

$: \, :$ is the conformal normal ordering, defined by

$$:\partial X \partial X:(z) = \lim_{z \to w} \left(\partial X(z) \partial X(w) - \langle \partial X(z)\, \partial X(w) \rangle \right)$$

This ensures in particular that $\langle T(z) \rangle = 0$

More generally, one must subtract all Wick contractions

Eg, $\langle \partial X(z) \rangle = 0$ (due to $X \to -X$ symmetry)

$\langle T(z)\, \partial X(w) \rangle = 0$

$\vdots$

$$\begin{aligned}
&\langle T(z)\, \partial X(z_1)\, \partial X(z_2) \rangle \\
&= -\frac{1}{\ell_s^2} \langle :\partial X \partial X:(z)\; \partial X(z_1)\, \partial X(z_2) \rangle \\
&= -\frac{2}{\ell_s^2} \langle \partial X(z)\, \partial X(z_1) \rangle \langle \partial X(z)\, \partial X(z_2) \rangle \\
&= -\frac{\ell_s^2}{2} \frac{1}{(z-z_1)^2 \; (z-z_2)^2}
\end{aligned}$$

in agreement with Ward identity

Moreover, at the operator level:

$$\begin{aligned}
T(z)\, \partial X(w) &= -\frac{2}{\ell_s^2} \partial X(z) \langle \partial X(z) \partial X(w) \rangle + :\partial X \partial X \partial X:(z) \\
&= \frac{\partial X(z)}{(z-w)^2} + :\partial X \partial X \partial X: \\
&= \frac{\partial X(w)}{(z-w)^2} + \frac{1}{z-w} \partial^2 X(w) + \text{regular}
\end{aligned}$$

This is in fact (as we shall see) the defining property of primary fields:

$$T(z)\,\Phi(w) = \frac{\Delta\,\Phi(w)}{(z-w)^2} + \frac{1}{z-w}\partial_w\Phi(w) + \text{reg}$$

Another set of primary fields in the free boson CFT are the 'vertex operators'

$$V_p = :e^{ipX}:$$

$$T(z)\,V_p(w) = \sum_{n=0}^{\infty}\frac{(ip)^n}{n!} :\partial X\partial X(z): \; :X^n(w,\bar{w}):$$

$$= \sum_{n=0}^{\infty}\frac{(ip)^n}{n!}\Big\{ n(n-1)\langle\partial X(z)\,X(w)\rangle^2 :X^{n-2}(w,\bar{w}): + 2n\,\langle\partial X(z)\,X(w)\rangle\,\partial X(z) :X^{n-1}(w,\bar{w}):\Big\} + \text{reg}$$

$$= \left(\frac{-l_s^2}{2}\,\frac{1}{z-w}\right)^2 (ip)^2 \cdot :e^{ipX}:(w) + 2ip\cdot\frac{-l_s^2}{2}\frac{\partial X(z)}{(z-w)} :e^{ipX}(w): + \text{reg}$$

$$= \frac{l_s^2 p^2\,V_p}{4(z-w)^2} + \frac{1}{z-w}\partial_p V_p + \text{reg}$$

so $\Delta = \dfrac{l_s^2 p^2}{4}$

necessary for cancelling IR divergences

Moreover, $\left\langle \prod_{i=1}^{n} V_{p_i}(z_i,\bar{z}_i)\right\rangle = 2\pi \prod_{i<j} |z_{ij}|^{l_s^2 p_i p_j}\,\delta\left(\Sigma p_i\right)$ from Wick's theorem

On the other hand, $T(z)$ is not a primary field, rather it satisfies

$$T(z)\,T(w) = \frac{1}{l_s^4} :\partial X\partial X(z): \; :\partial X\partial X(w):$$

$$= \frac{2}{l_s^4}\langle\partial X(z)\partial X(w)\rangle^2 + \frac{4}{l_s^2}\langle\partial X(z)\partial X(w)\rangle :\partial X(z)\partial X(w): + \text{reg}$$

$$= \frac{1/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial_w T + \dots$$

we shall read off the conformal anomaly from this term!

2. Radial quantization and operator / state correspondence

Correlation functions can be reinterpreted as matrix elements in the Hilbert space of the CFT on the cylinder as follows.

The conformal transformation $w = \tau + i\sigma \longrightarrow z = e^{w}$ maps the cylinder $\tau \in \mathbb{R}$, $\sigma \in [0, 2\pi[$ to the Riemann sphere punctured at $z=0$ and $z=\infty$, or z plane $\setminus \{0\}$

Fixed time slices $\tau = \text{cte}$ on the cylinder correspond to circles of fixed radius in the z plane.

An insertion of a field A at $z=0$ creates an 'in' state at $\tau = -\infty$:

$$|A_{in}\rangle = \lim_{z\to 0} A(z,\bar{z})\,|0\rangle$$

Similarly, an 'out' state is created by an insertion at $z=\infty$:

$$\langle A_{out}| = \lim_{z\to\infty} \langle 0|\, A(z,\bar{z})\, z^{2\Delta}\, \bar{z}^{2\bar{\Delta}}$$

in such a way that C_{ij} is a transition amplitude,

$$\langle A^2_{out} \,|\, A^1_{in}\rangle = \lim_{z_1\to 0,\, z_2\to\infty} \langle A_2(z_2,\bar{z}_2)\, A_1(z_1,\bar{z}_1)\, z_2^{2\Delta}\, \bar{z}_2^{2\bar{\Delta}}\rangle$$

$$= C_{12}$$

The state $|0\rangle$ is created by the identity operator and must be $SL(2,\mathbb{C})$ invariant.

More generally, n-point functions can be related as matrix elements of time ordered (in w plane) or radially ordered (in z plane) operators.

Conserved charges $Q = \int_{\sigma=0}^{2\pi} j_\tau \, d\sigma$ on the cylinder can be rewritten as contour integrals $Q = \frac{1}{2\pi i}\oint j(z)\, dz$ in the z plane around $z=0$.

In particular, conformal transformations $z \to z + \varepsilon(z)$

are generated by $\quad Q_\varepsilon = \frac{1}{2\pi i} \oint dz\, \varepsilon(z)\, T(z)$

(or rather, $Q_\varepsilon + \tilde{Q}_{\bar{\varepsilon}}$)

The condition $\quad \delta_\varepsilon \Phi = \left(\Delta \partial \varepsilon + \varepsilon \partial\right) \Phi$

for primary fields will hold provided

$$[Q_\varepsilon, \Phi(w)] = \frac{1}{2\pi i} \int dz\, \varepsilon(z)\, T(z)\, \Phi(w) = \frac{1}{2\pi i} \int_{w} dz\, \varepsilon(z)\, T(z)\, \phi(w)$$

$$\overset{?}{=} \left(\Delta \partial \varepsilon + \varepsilon \partial\right) \Phi(w)$$

hence

$$\boxed{T(z)\, \phi(w, \bar{w}) = \frac{\Delta}{(z-w)^2}\, \phi(w, \bar{w}) + \frac{1}{z-w}\, \partial_w \phi(w, \bar{w}) + \text{reg}}$$

As announced,

The Ward identity follows from this eq by contour manipulations.

This can also be used to compute the variation of T under conformal transf.

Suppose, as in the free boson case, that

$$T(z)\, T(w) = \frac{c/2}{(z-w)^4} + \frac{2\, T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg}$$

then

$$\delta_\varepsilon T = \frac{1}{2\pi i} \oint dw\, \varepsilon(w)\, T(w)\, T(z) = \underbrace{\boxed{\frac{c}{12}\, \partial^3 \varepsilon}}_{\text{conformal anomaly}} + 2\, \partial\varepsilon\, T + \varepsilon\, \partial T$$

For finite local conformal transf, this can be integrated to

$$T(z) = \left(f'(z)\right)^2 T(z') + \frac{c}{12} \underbrace{\left(\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right)}_{\text{known as Schwarzian derivative}}$$

It is sometimes useful to translate this in terms of mode expansions on the cylinder:

If Φ is a primary / chiral (= holomorphic) of dimension Δ, let

$$\phi_{cyl}(w) = \sum_{n\in\mathbb{Z}} \phi_n \, e^{\underbrace{-n(\tau+i\sigma)}_{w}}$$

$$\phi_{plane}(z) = \sum_{n\in\mathbb{Z}} \phi_n \, z^{-n-\Delta}$$

For the stress tensor,

$$T_{plane}(z) = \sum_{n\in\mathbb{Z}} L_n \, z^{-n-2}$$

but then, due to the conformal anomaly,

$$T_{plane} = \frac{1}{z^2}\left(T_{cyl} + \frac{c}{24}\right)$$

$$T_{cyl} = \sum_{n\in\mathbb{Z}} \left(L_n - \frac{c}{24}\delta_{n,0}\right) e^{-n(\tau+i\sigma)}$$

The Virasoro algebra for the modes $L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z) dz$ is then

$$[L_m, L_n] = \oint \frac{dw}{2\pi i} \oint_{C_w} \frac{dz}{2\pi i} z^{n+1} T(z) \, w^{m+1} T(w)$$

$$= (n-m) L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}$$

where c is the same constant which appears in

$$T(z) T(w) = \frac{c/2}{(z-w)^4} + \cdots$$

CFT of ghosts (rappel)

The b, c ghosts of diffeomorphism invariance have $\Delta = 2, -1$, resp.
Soon we will encounter the β, γ superghosts of local supersymmetry, with $\Delta = 3/2, -1/2$, resp.

We can treat both uniformly by taking
$$b : \Delta = \lambda$$
$$c : \Delta = 1 - \lambda$$

and $\varepsilon = 1$ (anticommuting)
$\quad\;\; -1$ (commuting)

Then $S = \frac{1}{\pi} \int d^2z \; b \bar{\partial} c$ is independant of λ,

but $T = -\lambda \, b \, \partial c + (1-\lambda) \, \partial b \,.\, c$

$$c(z)\, b(w) = \frac{1}{z-w} + \text{reg} \qquad\qquad c(z) = \sum_n z^{-n-(1-\lambda)} c_n$$

$$b(z)\, c(w) = \frac{\varepsilon}{z-w} + \text{reg} \qquad\qquad b(z) = \sum_n z^{-n-\lambda} \; b_n$$

$$c_n^+ = c_{-n} \,, \quad b_n^+ = \varepsilon \, b_{-n}$$

$$c_m b_n + \varepsilon \, b_n c_m = \delta_{m+n}$$

NS sector ( ) : $\quad b_n : n \in \mathbb{Z} - \lambda$
$\qquad\qquad\qquad\qquad\; c_n : n \in \mathbb{Z} + \lambda$

R sector : $\quad b_n : n \in \mathbb{Z} + \frac{1}{2} - \lambda$
$\qquad\qquad\quad\;\; c_n : n \in \mathbb{Z} + \frac{1}{2} + \lambda$

$$c = -2\varepsilon \left(6\lambda^2 - 6\lambda + 1 \right) \qquad\qquad Q \equiv \varepsilon(1 - 2\lambda)$$
$$\;\; = \varepsilon \left(1 - 3Q^2 \right)$$

$$\begin{pmatrix} \lambda = 2, \; \varepsilon = 1 : & \boxed{c = -26} \\ \lambda = 3/2, \; \varepsilon = -1 : & c = 11 \end{pmatrix} \qquad \begin{matrix} Q = -3 \\ Q = 1 \end{matrix}$$

$J(z) = - : b(z)\, c(z) :$ is anomalous :

$$T(z)\, J(w) = \frac{Q}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial_w J}{z-w}$$