Large N methods in QFT

Perturbation theory is efficient for describing high energy phenomena in asymptotically free QFT such as QCD, but fails to capture the low energy / infrared dynamics such as confinement, chiral sym breaking, spectrum of hadrons, glueballs etc.

The main difficulty is that QCD has no dimensionless parameter, and all such observables are (pure number) $\times$ Λ_{QCD}

Eventually lattice computations may allow to compute these observables with a reasonable accuracy.

In the meantime, a nice source of insight has been to think of the number of colors N as a free parameter, and try and derive the physically relevant $N=3$ case as an expansion in $1/N$ near $N=\infty$, where the theory greatly simplifies.

Before discussing QCD, and more generally models with fields transforming in adjoint rep of $U(N)$ [ie with N^2 perturbative d.o.f] we shall consider the large N limit of vector models [with N perturbative d.o.f] in low dimension.

1. <u>O(N) vector model</u> [Moshe Zinn-Justin hep-th/0306133] ②

$$S(\phi) = \int \frac{1}{2}\left\{ \left(\partial_\mu \phi^i\right)^2 + m_0^2 \left(\phi^i\right)^2 + \frac{1}{8}\,\frac{\lambda}{N}\left[\left(\phi^i\right)^2\right]^2 \right\} d^d x \qquad i = 1 \ldots N$$

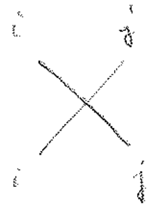
$\hookrightarrow$ convenient normalization

(useful in description of second order phase transitions in vapor/liquid systems, or superfluid He, or ferromagnets, or even polymers) $\quad [\phi] = \frac{D-2}{2} \quad [\lambda] = D - 2(D-2) = 4-D$

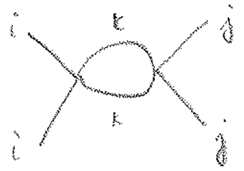
$\hookrightarrow N \to 0$ limit

This model is perturbatively renormalizable in $d \leq 4$ at fixed N, $\lambda \ll 1$

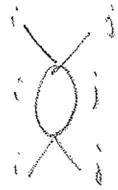
but it turns out to admit a $\frac{1}{N}$ expansion at fixed m_0, λ



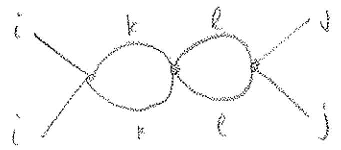
$\sim \frac{\lambda}{N}$



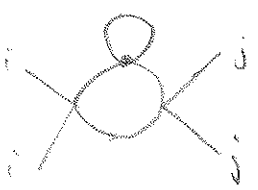
$\sim \frac{\lambda^2}{N}$

(

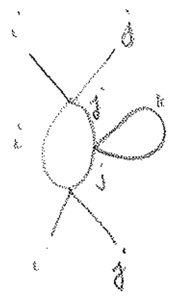
$\sim \frac{\lambda^2}{N^2}$ subdominant)



$\sim \frac{\lambda^3}{N}$



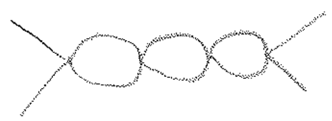
$\sim \frac{\lambda^3}{N}$

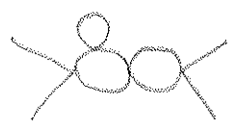
(

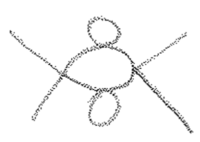
$\sim \frac{\lambda^3}{N^2}$ subdominant)

$\rightarrow$ all diagrams carry a power $1/N$ or larger

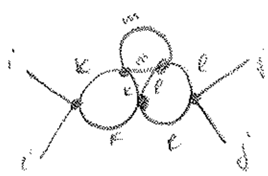
. only 'bubble diagrams' remain at leading order in $1/N$







$\frac{\lambda^4}{N}$



$\frac{\lambda^5}{N^2}$ etc

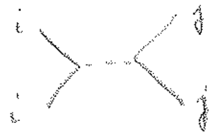
To discuss the limit $N \to \infty$, it is advantageous to introduce an auxiliary field σ mediating the quartic interaction:

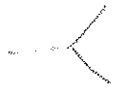
$$Z = \int D\phi^i \, e^{-S(\phi)} = \int D\phi^i \, D\sigma \, e^{-S(\phi^i,\sigma)}$$

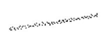
where $S(\phi^i,\sigma) = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{m_0^2}{2}(\phi^i)^2 - \frac{1}{2}\frac{N}{\lambda}\sigma^2 + \frac{1}{2}\sigma(\phi^i)^2$

(e.o.m of σ: $\frac{N}{\lambda}\sigma = \frac{1}{2}(\phi^i)^2$

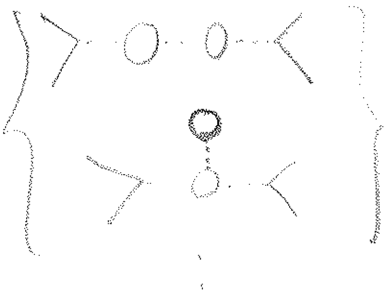
$\frac{N}{2\lambda}\sigma^2 - \frac{1}{2}\sigma(\phi^i)^2 = \frac{\lambda}{N}\left(\frac{1}{8}-\frac{1}{4}\right)(\phi^i)^2$)

 is now resolved to  ; = $\frac{2}{N}$

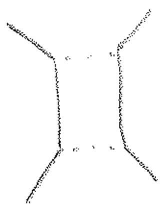
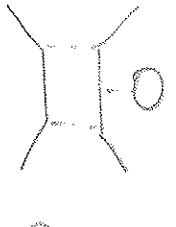
 : 1

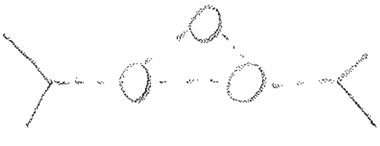
 : $\frac{1}{p^2+m_0^2}$

Dominant diagrams:

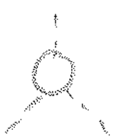
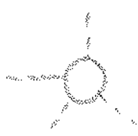
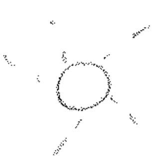
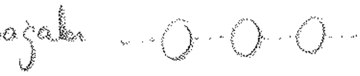


Subdominant:

 ,  ,



Dominant diagrams are trees of σ lines, with an infinite number of vertices

 ,  ,  , etc. ; propagator 

tadpole 

To construct the effective action generating these graphs, one just needs to perform the Gaussian integral wrt to ϕ^i. It is convenient to integrate out only $N-1$ of the ϕ^i's, so that we remain with one φ:

$$Z = \int D\sigma\, D\phi \;\exp\left[-S(\sigma,\phi)\right]$$

$$S(\phi,\sigma) = \int\left(\frac{1}{2}(\partial_\mu\phi)^2 + \frac{(\sigma+m_0^2)}{2}\phi^2 - \frac{1}{2}\frac{N}{\lambda}\sigma^2\right)d^dx + \frac{N-1}{2}\,\mathrm{Tr}\,\log\left[-\nabla^2+m_0^2+\sigma\right]$$

cf. 4'

Thus, the large N limit corresponds to the semi-classical limit for the field σ — ie the field $\frac{\sum(\phi^i)^2}{N}$ self averages (central limit thm)

The classical value of σ, ϕ is obtained by minimizing

$\sigma \sim O(1)$
$\phi \sim O(\sqrt{N})$

$$\mathcal{E} = \frac{1}{2}(\sigma+m_0^2)\phi^2 - \frac{1}{2}\frac{N}{\lambda}\sigma^2 + \frac{N}{2}\int^\Lambda \frac{d^dk}{(2\pi)^d}\log(k^2+m_0^2+\sigma)$$

letting $\boxed{m^2 = \sigma + m_0^2}$ be the mass of ϕ (and of all ϕ^i's):

ϕ: $\quad m^2\phi = 0$

σ: $\quad \frac{1}{2}\phi^2 - \frac{N\sigma}{\lambda} + \frac{N}{2}\int_0^\Lambda \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2} = 0 \quad$ 'gap equation'

Define $\quad \Omega_d(m^2) = \int^\Lambda \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2}$

$$= \Lambda^{d-2}\left[a_d + \frac{1}{(4\pi)^{d/2}}\Gamma\left(1-\frac{d}{2}\right)\left(\frac{m}{\Lambda}\right)^{d-2} + b_d\left(\frac{m}{\Lambda}\right)^2 + O\left(\left(\frac{m}{\Lambda}\right)^4, \left(\frac{m}{\Lambda}\right)^d\right)\right]$$

cut-off independent

$a_d = \infty$ for $d \le 2$
due to infrared divergence

Rk $\mathrm{Tr}\left[\log\left(-D^2+m^2+\sigma\right)\right]$:

$$= \mathrm{Tr}\log(-D^2+m^2) + \mathrm{Tr}\log\left(1+(-D^2+m^2)^{-1}\sigma\right)$$

$$= \int\frac{d^dk}{(2\pi)^d}\log(k^2+m^2)$$

$$+\ \mathrm{Tr}\log\ (-D^2+m^2)^{-1}\sigma$$

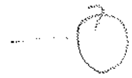
$$+\frac{1}{2}\mathrm{Tr}\log\ (-D^2+m^2)^{-1}\sigma\,(-D^2+m^2)^{-1}\sigma$$

$$+\ \dots$$

In momentum rep: $\sigma\cdot|k\rangle = \int\sigma(p)\frac{d^dp}{(2\pi)^d}\,|k+p\rangle$

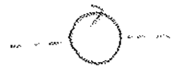
hence $\mathrm{Tr}\log\ (-D^2+m^2)^{-1}\sigma = \int\frac{d^dk}{(2\pi)^d}\cdot\int\frac{d^dp}{(2\pi)^d}\cdot\frac{1}{k^2+m^2}\sigma(p)\ \langle k|k+p\rangle$

$$= \sigma(0)\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2}$$



$$\mathrm{Tr}\left[\log(-D^2+m^2)^{-1}\sigma(-D^2+m^2)^{-1}\sigma\right] = \int\frac{d^dk}{(2\pi)^d}\int\frac{d^dp}{(2\pi)^d}\frac{d^dp'}{(2\pi)^d}\frac{\sigma(p)\sigma(p')\ \langle k|k+p+p'\rangle}{\left[(k+p)^2+m^2\right]\left[(k+p+p')^2+m^2\right]}$$

$$= \int\frac{d^dp}{(2\pi)^d}\underbrace{\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2+m^2)\,((k+p)^2+m^2)}}\ \sigma(p)\,\sigma(-p)$$



etc.

Two phases are possible in principle:

- Broken phase (low T): $m^2 = 0$, $\phi \neq 0$, $\sigma = -m_0^2$

$$\frac{1}{2}\frac{\phi^2}{N} - \frac{\sigma}{\lambda} + \frac{1}{2} a_d \Lambda^{d-2} = 0$$

$$\Rightarrow \frac{\sigma}{\lambda} \geqslant \frac{1}{2} a_d \Lambda^{d-2} \qquad (m_0^2 \text{ negative and large})$$

impossible for $d=2$ since $a_d = \infty$

[Coleman Mermin Wagner theorem: no spontaneous breaking of continuous sym in $d=2$]

— critical value at $-\frac{m_0^2}{\lambda} = \frac{1}{2} a_d \Lambda^{d-2}$

- Unbroken phase (high T): $\phi = 0$, $m^2 \neq 0$

$$\frac{m^2 - m_0^2}{\lambda} = \frac{1}{2}\Omega_d(m^2) \quad , \quad m \ll \Lambda$$

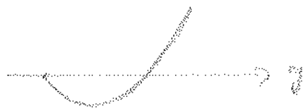
Free energy: $\mathcal{E} = \frac{1}{2} m^2 \phi^2 - \frac{N}{2\lambda}(m^2 - m_0^2)^2 + \frac{N}{2}\int_0^m s\,ds\, \Omega_d(s)$

One can go on and compute two-point functions for σ, φ, specific heat, etc. Critical exponents agree with $(4-\varepsilon)$ expansion at large N, $d \to 4$.

- Note that in the large N limit, σ has dimension 2. Thus the term $\frac{N\sigma^2}{\lambda}$ is irrelevant in $d \leq 4$. Upon dropping it, and replacing m_0^2 by $m^2 - \sigma$, σ becomes a Lagrange multiplier imposing the constraint $\sum \phi_i^2 = 1$; hence the $O(N)$ vector model is equivalent (at criticality) with the $O(N)$ σ-model. For finite N, the latter is renormalizable in $D=2$

Recall β function in dimension $d = 4 - \varepsilon$:

$$\beta = -\varepsilon g + \frac{N+8}{48\pi^2} g^2 + O(g^3)$$



This has a stable IR fixed point at $\quad g^* = \dfrac{48\pi^2\varepsilon}{N+8}$

which is weakly coupled as $\varepsilon \to 0$ <u>or</u> $N \to \infty$

Stability: $\quad \beta' = -\varepsilon + \dfrac{N+8}{24\pi^2} \cdot \dfrac{48\pi^2\varepsilon}{N+8} = +\varepsilon > 0$

2. Gross-Neveu model in $D=1+1$

Consider a model with $\tilde N$ Dirac-fermions ; $N = \tilde N \,\mathrm{Tr}\, 1 = 2^{D/2}\tilde N$

$$S(\psi) = \int d^2x \left[\sum_{i=1}^{\tilde N} \bar\psi\, i\partial\!\!\!/\, \psi + \frac{G}{2N} (\bar\psi\psi)^2 \right] ; \quad S_E = -\int \bar\psi\partial\!\!\!/\psi + \frac{G}{2N}(\bar\psi\psi)^2$$

$G>0$

In $D=2$, Dirac matrices are 2×2

$\gamma^0 = \sigma_3$

$\gamma^1 = i\sigma_2$

$\gamma^5 = \gamma^0\gamma^1 = \sigma_1$

$[\psi] = \frac{D-1}{2}$, $[G] = D - 2(D-1) = 2-D$

so renormalizable in $D\le 2$

The model is manifestly invariant under $U(\tilde N)$ (in fact $O(N)$ in $D=2$)

moreover it has a discrete chiral symmetry

$\psi^i \to \gamma_5 \psi^i$

$\bar\psi^i \to -\bar\psi^i \gamma_5$

$\bar\psi\psi \to -\bar\psi\psi$

which forbids a mass term.

Perturbatively, to all orders in G (fixed N), ψ^i are massless fermions, and the chiral sym is unbroken.

At large N, like in the $O(N)$ vector model, it is convenient to introduce an auxiliary field σ:

$$S(\psi^i,\sigma) = -\int d^Dx \quad \bar\psi^i \partial\!\!\!/ \psi^i - \frac{N}{2G}\sigma^2 + \sigma\sum\bar\psi\psi$$

eom: $\frac{N\sigma}{G} = \sum\bar\psi\psi$

$-\frac{N\sigma^2}{2G} + \sigma\bar\psi\psi = \frac{G}{N}\left(1-\frac{1}{2}\right)$

Integrating out all ψ^i's except one:

$$S(\psi,\sigma) = -\int d^2x \left(\bar\psi\, i\partial\!\!\!/\,\psi - \frac{N}{2G}\sigma^2 + \sigma\bar\psi\psi\right) - \frac{\tilde N - 1}{2}\,\mathrm{Tr}\log(\partial\!\!\!/ + \sigma)$$

$$\approx -\frac{N}{2}\,\mathrm{tr}\log(\partial^2 - \sigma^2)$$

$\mathrm{Tr}\, 1 = \frac{N}{\tilde{N}}$

For constant σ

$$\mathrm{Tr} \log(\not{\partial} + \sigma) = \mathrm{Tr} \log(\not{\partial} - \sigma)$$
$$= \frac{1}{2} \mathrm{Tr} \log\left[(\not{\partial}+\sigma)(\not{\partial}-\sigma)\right]$$
$$= \frac{N}{\tilde{N}} \mathrm{tr} \log(\partial^2 - \sigma^2)$$

⚠ this is true only at zero momentum

we recover the same structure as before, except that $m_0^2 = 0$ and the sign of Tr has flipped.

In particular, the field $\sigma \approx \frac{G}{N} \sum_i \bar{\psi}^i \psi^i$ becomes semi-classical,

$\psi \bar{\psi} = O(1)$

Let $\langle \sigma \rangle = M$, the mass of $\bar{\psi}\psi$:

$$\mathcal{E} = N \left[+ \frac{M^2}{2G} - \frac{1}{2} \int^\Lambda \frac{d^d k}{(2\pi)^d} \log(k^2 + M^2) \right]$$

Extremal value lies at

$$\frac{M}{G} = M \int^\Lambda \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + M^2} = M\, \Omega_d(M) \qquad \text{(gap equation)}$$

One solution lies at $M = 0$, provided $\Omega_d(0)$ is finite. [unbroken phase]

Another solution is at $G\, \Omega_d(M) = 1$.

This solution only exists if $G > G_c$, $\quad G_c\, \Omega_d(0) = 1$.

The energy of the broken phase is $<$ than that of unbroken, so the symmetry is broken for $G > G_c$, and fermions are <u>massive</u>.

d=2

$$\int_0^\Lambda \frac{d^2k}{(2\pi)^2} \log(k^2+m^2) = \frac{1}{2}\int_0^{\Lambda^2} \frac{dx}{2\pi} \log(x+m^2)$$

$$= \frac{1}{4\pi}\left[(\Lambda^2+m^2)\log(\Lambda^2+m^2) - (\Lambda^2+m^2) - m^2\log m^2 + m^2\right]$$

$$= \frac{1}{4\pi}\left[(\Lambda^2+m^2)\log\Lambda^2 + (\Lambda^2+m^2)\cdot\frac{m^2}{\Lambda^2} - \Lambda^2 - m^2\log m^2\right]$$

$$\underset{\Lambda\to\infty}{\sim} \frac{1}{4\pi}\left[\Lambda^2\log\Lambda^2 - \Lambda^2 + m^2\left(1+\log\frac{\Lambda^2}{m^2}\right)\right]$$

so $$\mathcal{E} = N\left[\frac{M^2}{2G} + \frac{1}{8\pi} M^2\left(\log\frac{M^2}{\Lambda^2} - 1\right) + \text{infinite constant}\right]$$

To take the continuum limit $\Lambda \to \infty$, we must renormalize the coupling G.

Let us define $$\frac{1}{G(\mu)} = \frac{1}{G} + \frac{1}{4\pi}\log\frac{\mu^2}{\Lambda^2}$$

so that $$\mathcal{E} = N\left[\frac{M^2}{2G(\mu)} + \frac{1}{8\pi} M^2\left(\log\frac{M^2}{\mu^2} - 1\right)\right]$$

This shows that the Gross Neveu model in $d=2$ is renormalizable and asymptotically free: keeping g fixed and sending $\mu \to \infty$, $g(\mu) \to 0$.

The vev of $\sigma = M$ is obtained by minimizing $\mathcal{E}$:

$$\frac{1}{G(\mu)} + \frac{1}{4\pi}\log\frac{M^2}{\mu^2} = 0 \quad\Rightarrow\quad M^2 = \mu^2 e^{-2\pi/g}$$

$$\mathcal{E} = N\left(M^2\left(\frac{1}{2G(\mu)} - \frac{1}{8\pi}\right) - \frac{M^2}{2g(\mu)}\right) = -\frac{N M^2}{8\pi} < \mathcal{E}(M=0)$$

$\longrightarrow$ the chiral symmetry is spontaneously broken at the non perturbative level

σ - propagator:


$$= \frac{G^2}{N} \int \frac{d^dk}{(2\pi)^k} \frac{\mathrm{Tr}\left[(\not{k}+M) \cdot ((\not{p}-\not{k})+M)\right]}{(k^2+M^2)\left[(p-k)^2+M^2\right]}$$

$$\mathrm{Tr}\, \not{k}(\not{p}-\not{k}) = \frac{N}{N}\left(k.p - k^2\right)$$

$$k^2 - k.p = \alpha\left(k^2+M^2\right) + \beta\left((p-k)^2+M^2\right)$$

$$\alpha+\beta = 1\ ,\ \beta = -\frac{1}{2} \quad \alpha = \frac{3}{2}$$

$$k^2 - k.p = \frac{3}{2}\left(k^2+M^2\right) - \frac{1}{2}\left[(p-k)^2+M^2\right] + \frac{1}{2}p^2 - M^2$$

eventually

$$\Delta_\sigma^{-1}(p) = \frac{1}{2} N\left(p^2+4M^2\right) B_n(p,m) \qquad \sim p^{d-2} \quad , \quad \Delta \sim \frac{1}{p^{2-d}}$$

$$B_n = \frac{1}{(2\pi)^d} \int^\Lambda \frac{d^dq}{\left[q^2+m^2\right]\left[(p-q)^2+m^2\right]}$$

At large p: $\displaystyle \int \frac{d^dq}{q^2\left[(p-q)^2\right]} \sim p^{d-4}$

$d=3$: $\displaystyle \sim \frac{1}{p}$

So Δ grows like $\dfrac{1}{p}$ rather than $\dfrac{1}{}$

$\longrightarrow$ no worse divergences than in $D=2$

pole at $p^2 = 4M^2$

$\longrightarrow$ threshold bound state of two fermions

3. Large N adjoint models

Consider a QFT with fields transforming in adjoint rep of a symmetry group $G = U(N)$ (eg $U(N)$ Yang Mills)

$$\mathcal{L} = \mathrm{Tr}\left[-\frac{1}{2} \partial^\mu B \, \partial_\mu B + \frac{1}{3} g B^3 - \frac{\lambda}{4} B^4 \right]$$

Srednicki Chap 80.

Decomposing $B(x) = \sum_{a=1..N^2} B^a(x) T^a$

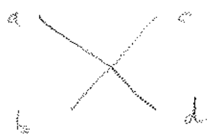
where T^a are generators of G normalized such that

$\mathrm{Tr}\, T^a T^b = \delta^{ab}$

then we can expand $\mathcal{L}$ in terms of B^a and find the Feynman rules

$$\frac{\delta^{ab}}{p^2 - i\varepsilon}$$

$2ig \; \mathrm{Tr}(T^a T^b T^c)$



$-6i\lambda \; \mathrm{Tr}(T^a T^b T^c T^d)$

For $G = U(N)$, it is more efficient to use the components $B_i{}^j$

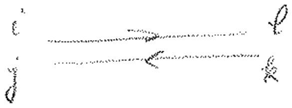
(the adjoint rep is the tensor product of fundamental and antif. rep)

$B_i{}^j = B^a (T^a)_i{}^j$ has propagator $\dfrac{(T^a)_i{}^j (T^a)_k{}^\ell}{p^2 - i\varepsilon}$

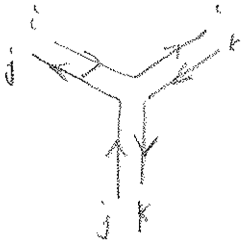
but $(T^a)_i{}^j (T^a)_k{}^\ell$ must be proportional to the unique G-invariant in $\mathrm{adj} \otimes \mathrm{adj}$, ie $\alpha \, \delta_i{}^\ell \delta_k{}^j$

The prop constant is obtained by contracting : $\frac{1}{2} \mathrm{Tr}\, \delta^{ab} = \frac{N^2}{2} = \alpha N^2 \Rightarrow \alpha = 1$

The propagator can be represented in double line notation:



vertices are



(ig) from $B_i^{\ j} B_j^{\ k} B_k^{\ i} \cdot \frac{g}{3}$ in Lagrangian

$(-i\lambda)$ → planar diagrams

→ non planar diagrams

Now, rescale the fields and coupling constants such that a factor of N appears in front:

$$B = \sqrt{N}\,\tilde{B}$$

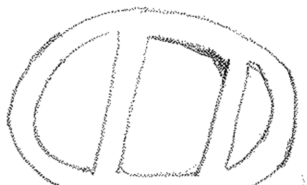
$$g = \frac{\tilde{g}}{\sqrt{N}}, \quad \lambda = \frac{\tilde{\lambda}}{N}$$

$$\mathcal{L} = N \,\mathrm{Tr}\left(-\frac{1}{2}(\partial\tilde{B})^2 + \frac{1}{3}\tilde{g}\,\tilde{B}^3 - \frac{\tilde{\lambda}}{4}\tilde{B}^4\right)$$

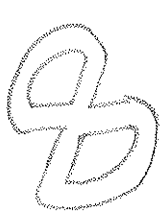
A diagram with E external legs, I internal legs, V_3 cubic vertices, V_4 quartic vertices $\quad 3V_3 + 4V_4 = E + 2I$

and therefore $L = I - V + 1$ loops [$E = 0$ for simplicity]

$$(\tilde{g}N)^{V_3}\,(\tilde{\lambda}N)^{V_4}\;N^{-I}\cdot N^{F} \propto N^{V-I+F}$$

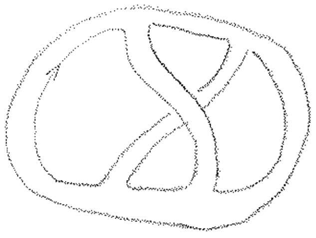


$F=4$, $V=4$, $I=6$, $L=3$: N^2

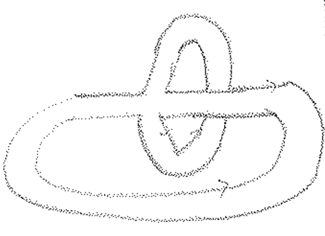


$F=3$, $V=1$, $I=2$, $L=2$ $\quad N^2$

$F=4$, $V=4$, $I=6$, $L=3$ $\quad N^2$



$F=2$, $V=4$, $I=6$, $L=3$: N^0



$F=1$, $V=1$, $I=2$, $L=2$ $\quad N^0$

$\chi = F - I + V$ is a topological invariant known as the Euler number of the surface (of which the flattened Feynman diagram gives a simplicial decomposition

$$\chi = 2 - 2h \qquad h = \text{'genus'} = \# \text{ of handles.}$$

Thus, in the limit $N \to \infty$ keeping $\hat{g} = g\sqrt{N}$ fixed, $\hat{\lambda} = \lambda N$

the dominant diagrams are planar;

$$Z = \sum_{h=0}^{\infty} N^{2-2h} \underbrace{Z_h(\hat{g}, \hat{\lambda})}_{\text{still involves diagrams with arbitrary \# of loops}}$$

Notice that the limit $N \to \infty$, $g \to 0$ with $\hat{g}$ fixed preserves the QCD scale

$$\mu \frac{dg}{d\mu} = -\left(\frac{11}{3}N - \frac{2}{3}N_f\right)\frac{g^3}{16\pi^2} + O(g^5) = -\frac{b_0}{16\pi^2} g^3$$

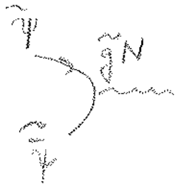
$$\to \quad \mu \frac{d\tilde{g}}{d\mu} \sim -\left(\frac{11}{3} - \frac{2}{3}\frac{N_f}{N}\right)\frac{\tilde{g}^3}{16\pi^2} + \cdots$$

$$\mu \frac{d}{d\mu}\left[\frac{1}{g^2}\right] = \frac{-2}{g^3}\,\mu\frac{d}{d\mu} g = \frac{b_0}{8\pi^2} \quad \Rightarrow \quad \frac{1}{g^2(\mu)} = \frac{1}{g_0^2} + \frac{b_0}{8\pi^2}\ln\mu = \frac{b_0}{8\pi^2}\ln\frac{\mu}{\Lambda}$$

$$\Lambda = \mu \exp\left[-\frac{8\pi^2}{b_0 g^2(\mu)}\right]$$

In the presence of fundamental matter, eg N_f quarks species (N_f fixed);

One should also rescale $\psi \to \sqrt{N}\,\psi$ such that



$$\mathcal{L} = N\left[-\frac{1}{2}\mathrm{Tr}\,\tilde F_{\mu\nu}^2 + \bar{\tilde\psi}\, i\not{D}\,\tilde\psi\right]$$

The propagator of $\tilde\psi$ is now a single line, weighted by $\frac{1}{N}$

$\sim (\tilde g N)^4\, N^{-6}\cdot N^3 \sim N$

more generally $N^{V-I+F-b} = N^{\chi}$

$b = \#$ boundaries $\qquad \chi = 2-2h-b$

$\longrightarrow$ simplicial decomposition of a surface with h handles and b boundaries

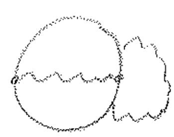
Quark loops are suppressed;

$O(N)$

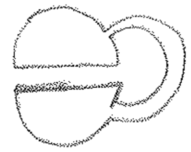
compared to

$O(1)$

Gluon loops must be inside quark loop:



$\frac{(gN)^4}{N^6}\cdot N \sim \frac{1}{N}$



is $O\left(\frac{1}{N^2}\right)$ compared to

=

$\frac{(gN)^2 N^2\, N}{N^3}$

More generally let

$\hat{G}_i$ be any purely gluonic operator eg $\mathrm{Tr}\, \hat{F}^{\alpha n}_{\mu\nu}$ (gauge invariant)

$\hat{H}_i$ be a gauge invariant operator bilinear in quarks ('mesonic')

A connected n-point functions involving r gluonic operators and s mesonic ops scales like N^{2-r} if $s=0$, or N^{1-r} if $s>0$

(to see this, add source terms $N(J_i \hat{O}_i)$ to the rescaled Lagrangian

$$\langle \hat{O}_1 \cdots \hat{O}_r \rangle = \frac{1}{iN}\frac{\partial}{\partial J_1} \cdots \frac{1}{iN}\frac{\partial}{\partial J_r} W(J) \quad , \quad W(J) = O(N^2)\cdot J_G + O(N)\cdot J_H$$

The operators $\hat{G}_i$ create glueballs from vacuum
$\hat{H}_i$ mesons

They can be normalized by looking at two-pt functions:

$\langle \hat{G}_1 \hat{G}_2 \rangle_c \sim 1$ so $|\text{glueball}_i\rangle = \hat{G}_i |0\rangle$

$\langle \hat{H}_1 \hat{H}_2 \rangle_c \sim \frac{1}{N}$ $\quad |\text{meson}_i\rangle = \sqrt{N}\, \hat{H}_i |0\rangle$

Thus s-meson interactions are of order $\langle (\sqrt{N}\hat{H})^s \rangle = N^{1-s/2}$: $g_{open} \sim \frac{1}{\sqrt{N}}$

r-glueball $\langle (\hat{G})^r \rangle \sim N^{2-r}$: $g_{closed} \sim \frac{1}{N}$

r-glueballs + s mesons $\langle (\sqrt{N}\hat{H})^s (\hat{G})^r \rangle \sim N^{1-\frac{s}{2}-r}$

eg $\langle \sqrt{N}\hat{H}\, \hat{G} \rangle \sim \frac{1}{\sqrt{N}}$: meson-glueball mixing is suppressed as $N \to \infty$

(Pion decay constant $\sim \sqrt{N}$

$$\langle 0 | \underbrace{\bar{q}\gamma^\mu \gamma_5 T^a q}_{N\hat{H}} | \pi^b \rangle = i f_\pi\, p^\mu \delta^{ab} \sim N \langle \hat{H}_1 \cdot \sqrt{N}\, \hat{H}_2 \rangle \sim \sqrt{N}$$

$i\mathcal{M} = G_F f_\pi \bar{u}(q) \not{p} (1-\gamma_5) v$)

$\longrightarrow$ mesons are stable and non-interacting in the $N \to \infty$ limit, and have finite mass; 2body decay amplitude $\sim \frac{1}{\sqrt{N}}$, meson-meson elastic scattering $O(\frac{1}{N})$

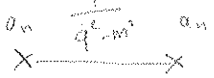
Still the coupling constant runs logarithmically:

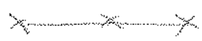
$$\int d^4x \, e^{iqx} \langle J(x) J(0) \rangle_c \sim \log q^2 \quad \text{at high energy}$$

$$= \sum_i \frac{Z_i}{q^2 - m_i^2} \qquad \text{(single meson exchange dominates)}$$

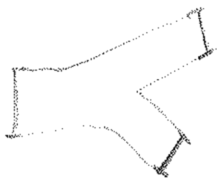
This can only happen if there is an ∞ number of mesons in large N limit.

This is because, upon cutting the diagram, the only intermediate states contains $\underline{1}$ $q\bar{q}$ pair plus gluons in a singlet – hence, assuming that the theory confines, one (and only one) meson.

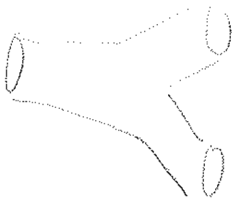
$\langle J J \rangle = \sum$ 

$\langle J J J \rangle = \sum$  $+ \sum$ 

In the large N limit, this suggests a description of mesons as excitations of an open string, which interacts by splitting / joining its ends:

 $\sim g_o \sim \frac{1}{\sqrt{N}}$ $\longrightarrow$ linear potential between q and $\bar{q}$

Similarly, glueballs are closed string excitations,

 $\sim g_{cl} \sim \frac{1}{N}$

Such a string model is also suggested by the Regge trajectories with mass $m^2 \sim J/\alpha'$ $\qquad \alpha' = 1\,(\text{GeV})^{-2}$

Rk In the limit $N \to \infty$, the gluonic operators $\hat{G}_i$ become classical, in the sense that their quantum fluctuations vanish:

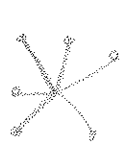
$$\frac{\langle \Delta \hat{G} \rangle}{\langle \hat{G} \rangle} = \frac{\sqrt{\langle \hat{G}^2 \rangle - \langle \hat{G} \rangle^2}}{\langle \hat{G} \rangle} = \frac{\sqrt{\langle \hat{G}^2 \rangle_c}}{\langle \hat{G} \rangle} \sim \frac{1}{N}$$

This suggests that there should exist a classical <u>'master'</u> field A_{cl} such that $\langle G \rangle = G(A_{cl})$

Rk In addition to mesons, there are also baryons involving N quarks. They appear at mass $= O(N)$, but then have $O(1)$ spacing, and $O(1)$ scattering amplitudes against mesons.

$\longrightarrow$ baryons should be some sort of solitons / lumps in the effective theory of mesons / pions [skyrmions]

In string picture:



Rk The string picture becomes precise in the context of the AdS/CFT correspondence $\mathcal{N}=4$ SYM $\Longleftrightarrow$ Type IIB on $AdS_5 \times S_5$

$$ds^2 = \frac{-dx_0^2 + dx_1^2 + \quad + dx_3^2 + dz^2}{z^2} = \frac{SO(2,4)}{SO(1,4)} \qquad z \to 0 \text{ boundary}$$

global sym on boundary $\longleftrightarrow$ local sym in bulk

glueballs $\longleftrightarrow$ gravitons

mesons $\longrightarrow$ m.

baryons $\longleftrightarrow$ D-branes, M

4. Matrix models at large N

Consider $Z = \int dM \exp\left[-N \operatorname{Tr} V(M)\right]$

where V is some polynomial in M, hermitian $N\times N$ matrix.

The measure $dM = \prod_i dM_{ii} \prod_{i<j} d\mathrm{Re}M_{ij}\, d\mathrm{Im}M_{ij}$ and action $S = N \operatorname{Tr} V(M)$

are invariant under adjoint action $M \to U^\dagger M U$, $U \in U(N)$

Let us change coordinates from M, to (U, Λ)

$$\Lambda \doteq \mathrm{diag}(\lambda_i)$$
$$U \in U(N)$$

The Jacobian is $J = \prod_{i\neq j} (\lambda_i - \lambda_j)^2$

Dropping the volume of $U(N)$,

$$Z = \int d\lambda_i \prod_{i\neq j} (\lambda_i - \lambda_j)^2 \exp\left[-N \sum_k V(\lambda_k)\right]$$

$$= \int d\lambda_i \exp\left[-N E(\{\lambda_k\})\right]$$

$$E(\{\lambda_k\}) = \sum_k V(\lambda_k) - \frac{1}{N} \sum \log(\lambda_i - \lambda_j)^2$$

↑ repulsive potential

In large N limit, both terms scale the same way. Stationary pts lie at

$$V'(\lambda_k) = \frac{2}{N} \sum_{n\neq k} \frac{1}{\lambda_k - \lambda_n}$$

Assuming that the particles form a continuous density distribution $\rho(\lambda)$, with $\int \rho(\lambda) = 1$, this becomes

$$V'(\lambda) = 2\, \mathcal{P} \int d\mu \frac{\rho(\mu)}{\lambda - \mu} \qquad \mathcal{P} = \text{principal value.}$$

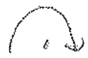
$$= \int d\mu \frac{\rho(\mu)}{\lambda + i\varepsilon - \mu} + \int \frac{d\mu\, \rho(\mu)}{\lambda - i\varepsilon - \mu}$$

Principal value:

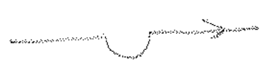
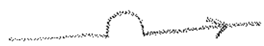
$f(x)$ has simple pole at $x=0$:

$$\mathcal{P}\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{-\varepsilon} f(x)\,dx + \int_{\varepsilon}^{+\infty} f(x)\,dx \qquad x=\varepsilon e^{i\theta}$$

Using the fact that the integral of f along 

and  are opposite

$i\int_{\pi}^{0} d\theta$ $\qquad$ $i\int_{-\pi}^{0} d\theta$

one can rewrite it as  + 

which is the same as shifting the pole up and down,

and taking the average.

Introducing the resolvant

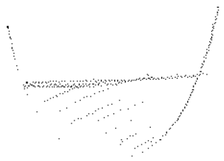
$$G(z) = \left\langle \frac{1}{N} \operatorname{Tr}\left[\frac{1}{z-M}\right] \right\rangle$$

$$= \int d\mu \, \frac{\rho(\mu)}{z-\mu}$$

This becomes $\quad \operatorname{Re} G(\lambda) = \frac{1}{2} V'(\lambda) \quad$ on real axis

Assuming that $\rho(\lambda)$ has finite support on $[-a, a]$ on the real axis, we find that $G(z)$ must be analytic in z plane away from $[-a, a]$, behaving as $1/|z|$ as $|z| \to \infty$, and with a cut along $[-a, a]$, with discontinuity

$$\operatorname{Im} G(\lambda) = -\pi \rho(\lambda)$$



E.g. for $\quad V = \frac{1}{2} m^2 M^2$

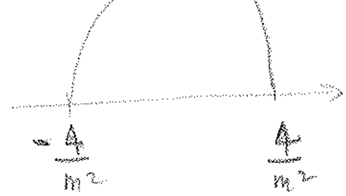
$$G = \frac{m^2}{2}\left(z - \sqrt{z^2 - a^2}\right) \quad = G(1/z) \text{ as } |z| \to \infty$$

for $-a \le z \le a$, $\quad G = \underbrace{\frac{m^2}{2} z}_{\frac{1}{2}V'} \pm i \frac{m^2}{2}\sqrt{z^2 - a^2}$

hence $\quad \rho(z) = \frac{m^2}{2}\sqrt{z^2 - a^2} \quad$: 'Master field'

normalising $\int \rho = 1$ gives $\quad a = \frac{4}{m^2}$

so $\rho(z)$ satisfies the semi-circle law



More generally, for V polynomial of degree d with a single global min, one can write $\quad G(z) = \frac{1}{2}\left[V' - P(z)\sqrt{z^2 - a^2}\right]$ where $P(z)$ is a poly of degree $d-2$.

Gross Witten Wadia phase transition

The partition function Z and Wilson line W in 2D lattice QCD is related to the 1-plaquette integrals ($G = U(N)$)

$$z = \int [dU] \exp\left[\frac{1}{g^2} \mathrm{Tr}(U + U^+) \right]$$

$$w = \int [dU] \; \frac{1}{N} \mathrm{Tr}\, U \cdot \exp\left[\frac{1}{g^2} \mathrm{Tr}(U + U^+) \right] \Big/ z$$

by

$$Z = (z)^{V/a^2}$$

$$W = (w)^{RT/a^2}$$

The integral runs over unitary matrices U

Since the integrand is invariant under $U \to T U T^+$, $T \in U(N)$

one may decompose $U = T \,\mathrm{diag}\, e^{i\alpha_i}\, T^+$

The jacobian is $\displaystyle\prod_{i<j} \sin^2 \left|\frac{\alpha_i - \alpha_j}{2}\right|$

hence

$$Z = \mathrm{cte.} \int_0^{2\pi} d\alpha_i \; \prod \sin^2 \left|\frac{\alpha_i - \alpha_j}{2}\right| \; \exp\left[\frac{2}{g^2} \sum_{i=1}^{N} \cos\alpha_i \right]$$

In large N limit, we need to extremize

$$E = \frac{2}{g^2 N} \sum_{i=1}^{N} \cos\alpha_i + \frac{1}{N} \sum_{i \neq j} \ln \left| \sin \frac{\alpha_i - \alpha_j}{2} \right|$$

$$\Rightarrow \quad \frac{2}{\lambda} \sin\alpha_i = \sum_{j \neq i} \cot \left|\frac{\alpha_i - \alpha_j}{2}\right| \qquad \lambda = g^2 N$$

Introducing a density of eigenvalues $\rho(\alpha)$,

$$\frac{2}{\lambda} \sin\alpha = P \int d\beta \; \rho(\beta) \cot\frac{\alpha - \beta}{2}$$

For λ large, the eigenvalues spread over the whole circle

$$\rho(\alpha) = \frac{1}{2\pi}\left[1 + \left(\frac{2}{\lambda}\right)\cos\alpha\right]$$ is then a solution, positive for $\lambda \geq 2$

[indeed, use $\cot\frac{\alpha-\beta}{2} = 2\sum_{n=1}^{\infty} \sin n\alpha \cos n\beta - \cos n\alpha \sin n\beta$

$$\left(\begin{array}{l} \int \cos\beta \,.\, \cot\frac{\alpha-\beta}{2} = 2\sin\alpha \,.\, \int \cos^2\beta = \sin\alpha \\ \int 1 \,.\, \cot\frac{\alpha-\beta}{2} = 0 \end{array}\right)$$]

For $\lambda < 2$, introduce

$$G(z) = \int_{-\alpha_c}^{\alpha_c} d\beta\, \rho(\beta) \cot\frac{z-\beta}{2}$$

- $G(z + 2\pi) = G(z)$

$G(z)$ analytic outside $[-\alpha_c + 2\pi N, \alpha_c + 2\pi N]$

$G(z)$ real on $\mathbb{R} \setminus$ intervals

$G(\alpha \pm i\varepsilon) = \frac{2}{\lambda}\sin\alpha \mp 2\pi i \rho(\beta)$

$G(z) \rightarrow 1$ in any dir except real axis

$$\hookrightarrow \quad G(z) = \frac{2}{\lambda}\sin\alpha - \frac{4}{\lambda}\cos\frac{\alpha}{2}\sqrt{\sin^2\frac{\alpha}{2} - \underbrace{\sin^2\frac{\alpha_c}{2}}_{=\frac{\lambda}{2}}} \quad \sim \frac{2}{\lambda}\sin\alpha - \frac{4}{\lambda}\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} \rightarrow 0$$

$$\boxed{\rho(\alpha) = \frac{2}{\pi\lambda}\cos\frac{\alpha}{2}\left(\frac{\lambda}{2} - \sin^2\frac{\alpha}{2}\right)^{1/2}}$$

As $\lambda \rightarrow 0$, one recovers the semi-circle law $\quad |\alpha| \leq \sqrt{2\lambda}$

$$-E_0(\lambda) = \begin{cases} 1/\lambda^2 & \lambda \geq 2 \\ \frac{2}{\lambda} + \frac{1}{2}\log\frac{\lambda}{2} - \frac{3}{4} & \lambda \leq 2 \end{cases}$$

: free energy, first + second derivatives are continuous but third derivative is disc.

$$W(\lambda) = -\frac{\lambda^2}{2N^2}\frac{\partial \ln z}{\partial \lambda} = \begin{cases} 1/\lambda & \lambda \geq 2 \\ 1 - \lambda/4 & \leq 2 \end{cases}$$