Compactification

We have seen that closed bosonic strings are consistent only when the embedding coordinates $X^\mu(\tau,\sigma)$ are described by a conformal field theory of central charge $c=26$, so as to cancel the conformal anomaly of the b,c ghosts.

Choosing X^μ to be flat coordinates in $\mathbb{R}^{1,25}$ is one possibility but not the only one. That's fortunate, since we don't live in 26 dimensions!

<u>More generally</u>, one could choose $X^{\mu=0\ldots D-1}$ to be flat coordinates in $\mathbb{R}^{1,D-1}$, and the remaining coordinates $X^{i=D\ldots 26}$ live in $(26-D)$ dimensional curved space $\mathcal{M}$ with metric $G_{ij}(x^m)$, two-form $B_{ij}(x^m)$, dilaton $\phi(x^m)$ satisfying the equations of motion.
For energies less than $1/R$ where R is the characteristic scale of $\mathcal{M}$, the physics would be effectively D-dimensional.

<u>Even more generally</u>, one could replace $X^{i=D\ldots 26}$ by an abstract CFT of central charge $c=26-D$, with no geometric description, for example $(\text{Ising})^{\frac{26-D}{2}}$, where Ising is the 2D Ising model with $c=\frac{1}{2}$!!

Here we shall restrict ourselves to the case where M is locally flat, but not globally so.

<u>The simplest example is the circle</u> $M=S_1=\mathbb{R}/\mathbb{Z}$ with radius R.
A striking result is that closed strings cannot distinguish between $S_1(R)$ and $S_1(1/R)$!

1 - Circle compactification

1.1. Field theory on $\mathbb{R}^{1,D-2} \times S_1(R)$

$S_1(R)$ can be constructed as the quotient $\mathbb{R}/\mathbb{Z}$ where $\mathbb{Z}$ acts by $x^{D-1} \to x^{D-1} + 2\pi R$

Field theory on $\mathbb{R}^{1,D-2} \times S_1(R)$ is the same as field theory on $\mathbb{R}^{1,D-1}$, restricted to the sector where fields are periodic under $x^{D-1} \to x^{D-1} + 2\pi R$.

In particular, plane waves $\psi = e^{i p_\mu x^\mu + i p_{D-1} x^{D-1}}$ must satisfy $p_{D-1} = \frac{m}{R}$ with $m \in \mathbb{Z}$

The mass-shell condition $p_\mu^2 + p_{D-1}^2 = -M^2$ for a field ϕ of mass M in dimension D becomes

$$p_\mu^2 = -M^2 - p_{D-1}^2 = -M_{D-1}^2 \,, \qquad M_{D-1} = \sqrt{M^2 + \left(\frac{m}{R}\right)^2}$$

Thus, ϕ gives rise to an infinite tower of fields ϕ_m, in dimension $D-1$, with mass $M_{D-1} = \sqrt{M^2 + \left(\frac{m}{R}\right)^2}$;

The lightest state is ϕ_0, and the other states ϕ_m are its '<u>Kaluza-Klein excitations</u>'.

If $M=0$, the masses of the KK states are evenly separated by $\frac{1}{R}$.

If ϕ carries spin, the spin of ϕ_m can be obtained by decomposing the $SO(1, D-1)$ rep under $SO(1, D-2)$

E.g.
- scalar $\longrightarrow$ scalar
- vector $A_M \longrightarrow$ vector A_μ, scalar A_{D-1}
- spin-2 $g_{MN} \longrightarrow$ spin-2 $g_{\mu\nu}$, vector $g_{\mu,D-1}$, scalar $g_{D-1,D-1}$, etc

- The effective theory for the lightest modes is obtained by dimensional reduction, ie by assuming that ϕ is independent of x_{D-1}

e.g: gravity in dimension D

indep' on $y = x_{D-1}$

$$S_D = \int d^Dx \frac{\sqrt{-G}\ R[G]}{\kappa^2}$$

$$G_{mn}\, dx^m dx^n = g_{\mu\nu}\, dx^\mu dx^\nu + R^2 (dy + \omega_\mu dx^\mu)^2$$

'Kaluza-Klein Ansatz':

$$R[G] = R[g_{-1}] - 2\frac{\Box R}{R} - \frac{R^2}{4} F_{\mu\nu}^2$$

where $F_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu A_\mu$

'Kaluza-Klein vector field'

$$S_D = \int dx^{D-1} \frac{R}{\kappa^2}\sqrt{-g}\left(R[g_{-1}] - 2\frac{\Box R}{R} - \frac{R^2}{4} F_{\mu\nu}^2\right)$$

This can be recast in the usual Einstein-Hilbert form by 'going to Einstein frame', i.e rescale the metric

$$g = e^{2\alpha\phi}\,\hat{g} \qquad \alpha^2 = \frac{1}{2(D-2)(D-3)}$$

$$R = e^{\beta\phi} \qquad \beta = -(D-3)\alpha$$

So that

$$S_D = \int dx^{D-1}\sqrt{-g}\left(R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4}e^{-2(D-2)\alpha\phi} F^2\right)$$

Thus we get gravity in dim $D-1$, plus a massless scalar field ϕ (sometimes called 'dilaton', but its not the same as the string theory dilaton!)

and a massless gauge field ω_μ.

The gauge symmetry $\omega_\mu \to \omega_\mu + \partial_\mu\lambda$ is the remnant of the diffeo symmetry in dim D $g_{mn} \to g_{mn} + \partial_m \lambda_n + \partial_n \lambda_m$ where $\lambda_{D-1} = 0$, λ_μ independent of y.

- Importantly, dimensional reduction is a <u>consistant</u> reduction, ie any solution of the (D-1)-dimensional action gives a solution of the D-dim action, invariant under translations along S_1.

- In the presence of fermionic fields, one must choose periodicity conditions for the fermions:

  - periodic boundary conditions lead to $p_{D-1} = \frac{m}{R}$, $m \in \mathbb{Z}$
    'Ramond bc'

  - anti-periodic boundary conditions lead to $p_{D-1} = \frac{m+\frac{1}{2}}{R}$, $m \in \mathbb{Z}$
    'Neveu-Schwarz bc'

In addition,
- spinors in odd dim D decompose into left and right chiral spinors in dim D-1
- Chiral spinors in even dim D reduce to (non chiral) spinors in dim D-1

$\rightsquigarrow$ the spectrum in D-1 dimensions is always non-chiral.

1.2. Circle compactification in bosonic closed string

The mode expansion of the coordinate $X \equiv X^{D-1}$ along S^1 must take into the fact that $X(\tau, \sigma + 2\pi)$ can differ from $X(\tau, \sigma)$ by a multiple of $2\pi R$.

$$X(\tau, \sigma) = X(\tau) + 2\pi n R$$

$n \in \mathbb{Z}$
'winding number'

The most general solution is

$$X = x + \ell_s^2 p\tau + nR\sigma + \frac{i\ell_s}{\sqrt{2}} \sum_{k \neq 0} \left(\frac{\alpha_k}{k} e^{-ik(\tau-\sigma)} + \frac{\tilde{\alpha}_k}{k} e^{-ik(\tau+\sigma)} \right)$$

$$= X_L + X_R$$

with

$$X_L = \frac{x}{2} + \frac{\ell_s^2}{2} p_L(\tau+\sigma) + \dots$$

$$X_R = \frac{x}{2} + \frac{\ell_s^2}{2} p_R(\tau-\sigma) + \dots$$

with

$$\left.\begin{aligned} p_L &= \frac{m}{R} + \frac{nR}{\ell_s^2} \\ p_R &= \frac{m}{R} - \frac{nR}{\ell_s^2} \end{aligned}\right\} \text{ the only difference is in the zero-modes}$$

$$L_0 = \frac{1}{4}\left(\frac{m\ell_s}{R} + \frac{nR}{\ell_s} \right)^2 + \sum_{k=1}^{\infty} \alpha_{-k}\alpha_k$$

$$\hat{L}_0 = \frac{1}{4}\left(\frac{m\ell_s}{R} - \frac{nR}{\ell_s} \right)^2 + \sum_{k=1}^{\infty} \alpha_{-k}\alpha_k$$

The spectrum can be constructed, e.g. in light-cone gauge:

$$p^+ p^- = \sum_{i=2}^{D-2} p_i^2 + p_L^2 + \frac{4}{\ell_s^2}\left(\sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i - \frac{D-2}{24} \right)$$

$$= \sum_{i=2}^{D-2} p_i^2 + p_R^2 + \frac{4}{\ell_s^2}\left(\sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i - \frac{D-2}{24} \right)$$

here $i = 2 \dots D-1$

Using $\frac{1}{2}\left(p_L^2 + p_R^2\right) = \left(\frac{m}{R}\right)^2 + \left(\frac{nR}{\ell_s^2}\right)^2$

$$\frac{1}{2}\left(p_L^2 - p_R^2\right) = 2\,\frac{mn}{\ell_s^2}$$

we find the mass-shell condition

$$M^2_{D-1} \equiv p^+p^- - \sum_{i=2}^{D-2} p_i^2$$

$$= \left(\frac{m}{R}\right)^2 + \left(\frac{nR}{\ell_s^2}\right)^2 + \frac{2}{\ell_s^2}\left(\sum_{i=2}^{D-1}\sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n + \tilde\alpha^i_{-n}\tilde\alpha^i_n - \frac{D-2}{12}\right)$$

$$= \left(\frac{m}{R}\right)^2 + \left(\frac{nR}{\ell_s^2}\right)^2 + M^2$$

while the matching condition is now

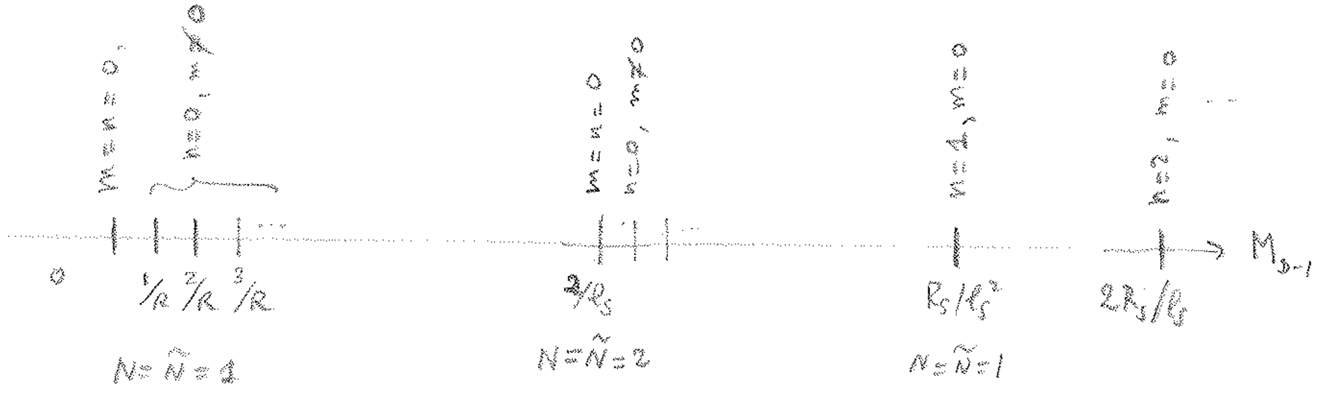
$$mn = \sum \tilde\alpha^i_{-n}\tilde\alpha^i_n - \sum \alpha^i_{-n}\alpha^i_n = \tilde N - N$$

For $n=0$ (no winding), the result is the same as in field theory:

$$M^2_{D-1} = \left(\frac{m}{R}\right)^2 + M^2 \quad : \text{ tower of Kaluza-Klein states}$$

For $n \neq 0$ however, there is an additive contribution to the mass, corresponding to the tension of the string wound around the circle. Moreover, if $mn \neq 0$, there are physical states in $(D-1)$ dimension which originate from 'unphysical states' in dimension D.

* If $R \gg \ell_s$, the energy spectrum looks like (ignoring the tachyon)



If $R \ll l_s$, the winding states are instead lighter than the Kaluza-Klein states

In fact, the spectrum is symmetric under

$$R \to \frac{l_s^2}{R} \quad , \quad m \leftrightarrow n \quad ; \quad \text{`T-duality'} \qquad T = \text{target space}$$

This is an exact symmetry of the conformal field theory of a compact boson, valid at each order in g_s (i.e on surfaces of any genus)

It acts by
$$p_L \to p_L \quad , \quad p_R \to -p_R$$
$$\alpha_k \to \alpha_k \quad , \quad \tilde{\alpha}_k \to -\tilde{\alpha}_k$$
$$X = X_L + X_R \quad \longrightarrow \quad \hat{X} = X_L - X_R$$

More generally, $\partial_\alpha X = \varepsilon_{\alpha\beta}\, \partial^\beta \tilde{X}$ on any worldsheet

Thus, the bosonic closed string cannot distinguish between $S_1(R)$ and $S_1(1/R)$!

* The inclusion of these winding states is actually required for modular invariance; at one-loop, in Schwinger-time rep, the sum over momenta $P_{D-1} = \frac{m}{R}$ can be rewritten as a sum over windings in proper time, by Poisson resummation:

$$\sum_{m \in \mathbb{Z}} \exp\left[-\pi t \left(\frac{m}{R}\right)^2 \right] = \sum_{\tilde{m} \in \mathbb{Z}} \frac{R}{\sqrt{t}} \exp\left[-\pi R^2 \frac{\tilde{m}^2}{t} \right]$$

$\frac{R^2 \tilde{m}^2}{t}$ is the action for the configuration $X(\tau) = 2\pi \tilde{m} R \tau + x$

$\int_0^1 \frac{1}{e} (\partial_\tau X)^2 \, d\tau$, $e(\tau) = t$ $\qquad X(1) = X(0) + 2\pi \tilde{m} R$

but worldsheet diffeo can exchange τ and σ, so one must also include winding in σ.

Indeed, the partition function of the compact boson

$$Z(R) = \mathrm{Tr}\left(q^{L_0 - \frac{c}{24}} \, \bar{q}^{\tilde{L}_0 - \frac{c}{24}} \right)$$

$$= \frac{\sum_{m,n \in \mathbb{Z}^2} q^{\frac{1}{2} p_L^2} \, \bar{q}^{\frac{1}{2} p_R^2}}{\eta \bar{\eta}} = \sum_{m,n} e^{-\pi \tau_2 \left(\frac{m^2 \ell_s^2}{R^2} + \frac{n^2 R^2}{\ell_s^2} \right) + 2\pi i m n \tau_1}$$

is modular invariant. This can be made manifest by Poisson resumming $m \to \tilde{m}$:

$$\left(\text{recall} \ \sum_{m \in \mathbb{Z}} e^{-\pi a m^2 + 2\pi i m b} = \frac{1}{\sqrt{a}} \sum_{\tilde{m} \in \mathbb{Z}} e^{-\frac{\pi}{a} (\tilde{m} - b)^2} \right).$$

$$Z(R) = \frac{1}{\eta \bar{\eta}} \cdot \frac{R}{\ell_s \sqrt{\tau_2}} \cdot \sum_{\tilde{m}, n} \exp\left[-\frac{\pi R^2}{\ell_s^2} \, \frac{(\tilde{m} - n\tau_1)^2 + n^2 \tau_2^2}{\tau_2} \right]$$

$$= \frac{R}{\ell_s} \underbrace{\frac{1}{\sqrt{\tau_2} \, \eta \bar{\eta}}}_{\substack{\text{modular} \\ \text{invariant}}} \cdot \underbrace{\sum_{\tilde{m}, n} \exp\left[-\frac{\pi R^2}{\ell_s^2} \, \frac{|\tilde{m} - n\tau|^2}{\tau_2} \right]}_{\text{invariant under } \tau \to \frac{a\tau + b}{c\tau + d}}$$

$$\begin{pmatrix} \tilde{m} \\ n \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tilde{m} \\ n \end{pmatrix}$$

This can be viewed as a sum over classical configurations

$$X = 2\pi R \left(\underset{\substack{\uparrow \\ \text{worldsheet} \\ \text{time}}}{\tilde{m} \, \sigma_1} + \underset{\substack{\uparrow \\ \text{worldsheet} \\ \text{space.}}}{n \, \sigma_2} \right) + X_{\text{fluct}}.$$

* For $R > \ell_s$ the only massless states are those with $m = n = 0$
  → just the dimensional reduction of the D-dimensional spectrum

$\alpha_{-1}^{\mu} \tilde{\alpha}_{-1}^{\nu} |0\rangle$: graviton, $B_{\mu\nu}$, dilaton in D-1 dimension

$\alpha_{-1}^{D-1} \tilde{\alpha}_{-1}^{\mu} |0\rangle$, $\alpha_{-1}^{\mu}, \tilde{\alpha}_{-1}^{D-1} |0\rangle$: two gauge fields $G_{\mu y}$, $B_{\mu y}$

$\alpha_{-1}^{D-1} \tilde{\alpha}_{-1}^{D-1} |0\rangle$: a scalar field R → radius of S_1

The conserved charge associated to $G_{\mu y}$ = momentum m

$B_{\mu y}$ = winding n

* For $R = \ell_s$, new massless states appear:

$$\alpha_{-1}^{\mu} \,|\, m = \pm 1, \; n = \mp 1 \rangle = W_{\mu}^{\pm}$$

$$\tilde{\alpha}_{-1}^{\mu} \,|\, m = \pm 1, \; n = \pm 1 \rangle = \tilde{W}_{\mu}^{\pm}$$

have mass $M_{D-1}^2 = \left(\frac{1}{R} - \frac{R}{\ell_s^2} \right)^2 \longrightarrow 0$ at $R = \ell_s$

'self dual radius'

At this point, the Abelian symmetry $U(1) \times U(1)$ is enhanced to $SU(2) \times SU(2)$

↳ $W_{\mu}^{\pm}$ and $\omega_{\mu} - \tilde{\omega}_{\mu}$

↳ $\tilde{\omega}^{\pm}$ and $\omega_{\mu} + \tilde{\omega}_{\mu}$

T-duality $R \to \frac{1}{R}$ can be viewed as a gauge transformation $\mathbb{Z}_2 \subset SU(2)$!

1.3 Circle compactification for open strings

Open strings with Neumann boundary conditions cannot wind,
so the spectrum of open bosonic strings on $S_1(R)$ is just

$$M^2_{D-1} = M^2_D + \left(\frac{m}{R}\right)^2 \qquad m \in \mathbb{Z}$$

momentum

like in QFT.

Thus, the D-dim massless gauge boson A_M reduces to

a (D-1)-dim $\left\{\begin{array}{l} \text{massless gauge boson } A_\mu \\ + \text{ scalar } \chi = \int A_y \, dy \ : \ \text{holonomy of } A_M \text{ around the circle} \end{array}\right.$

+ their Kaluza-Klein towers.

If $X \neq 0$, states which ~~are~~ carry charge q under A_M
must have momentum $\quad p_{D-1} = \frac{m}{R} - \frac{\chi q}{2\pi R}$

such that the wavefunction $e^{i p_{D-1} x^{D-1}}$
transforms with a phase $e^{-iq\chi}$ under $x^{D-1} \to x^{D-1} + 2\pi R$

Yet we can ask what is the effect of replacing
$X_L + X_R$ by $X_L - X_R$:

The mode expansion for Neumann b.c.

$$X = x + 2\ell_s^2 p \tau + i\sqrt{2}\, \ell_s \sum_{n \neq 0} \frac{\alpha_k}{k} \cos k\sigma \, e^{-ik\tau}$$

becomes the one for Dirichlet conditions:

$$\tilde{X} = \tilde{x} + 2\ell_s^2 p \sigma + i\sqrt{2}\, \ell_s \sum_{n \neq 0} \frac{\alpha_k}{k} \sin k\sigma \, e^{-ik\tau}$$

vanishes at $\sigma = 0, \pi$

hence $\tilde{X}(\sigma=0,\tau) = \tilde{x}$

$$\tilde{X}(\sigma=\pi,\tau) = 2\pi l_s^2 p = 2\pi l_s^2 \frac{m}{R} = 2\pi m \tilde{R}$$

Thus the open string winds m times around the dual circle, and its ends are attached on a D-brane located at $X = \hat{x} \mod 2\pi\tilde{R}$!

$\Rightarrow$ D branes are required by T-duality

Dp brane wrapped around $S_1 \xleftrightarrow{T}$ D(p-1) brane at fixed pos on $\tilde{S}_1$

Dp-branes can be viewed as solitons of string theory. They carry a worldvolume gauge field $A_{\mu=0\ldots p}$, and $25-p$ scalar fields corresponding to their transverse fluctuations.

Their dynamics is induced by fluctuations of open strings, and described at low energies by the Born-Infeld action

$$S = -T_p \int d^{p+1}x \; e^{-\phi} \sqrt{-\det\left(G_{\mu\nu} + 2\pi l_s^2 F_{\mu\nu}\right)}$$

target space metric.

follow from disk diagram

 — closed string mode

They can also be described by semi-classical solutions of the low energy effective theory of closed strings

Many applications to - model building (Universe as a brane)
- Strongly coupled gauge theories (AdS/CFT)

ETC