Classical strings

1. Relativistic Point Particle

* Particles in a fixed gravitational background propagate along geodesics. These trajectories can be derived from an action

$g_{\mu\nu}$: sig$(d-1, 1)$ $\oplus$ $\ominus$
'target space metric'

$$S(x^\mu, e) = -\frac{m}{2}\int d\tau \left[\frac{\dot{x}^\mu \dot{x}^\nu}{e} g_{\mu\nu}(x) - e \right]$$

$\delta S/\delta e = 0 \longrightarrow \quad e^2 = -\dot{x}^\mu g_{\mu\nu} \dot{x}^\nu$

$\delta S/\delta x^\mu = 0 \longrightarrow \quad \frac{d}{d\tau}\left(\frac{1}{e} g_{\mu\nu} \frac{dx^\nu}{d\tau} \right) = 0$

Plugging back in $S(x^\mu, e)$, an equivalent action is

$$S(x^\mu) = m \int d\tau \sqrt{-\dot{x}^\mu g_{\mu\nu} \dot{x}^\nu} = m \times \int ds$$

length

indeed, $ds^2 = -g_{\mu\nu} dx^\mu dx^\nu$

* Both actions are manifestly invariant under reparametrization of proper time τ. Infinitesimally,

$$\delta x^\mu(\tau) = \xi(\tau)\, \dot{x}^\mu(\tau) + O(\xi^2)$$
$$\delta e(\tau) = \partial_\tau [\xi(\tau) e(\tau)] + O(\xi^2)$$

In both cases, S varies by a total derivative.

In fact, one can view $e^2 = \gamma_{\tau\tau}$ as a worldline metric,

$$S(x^\mu, \gamma) = -\frac{m}{2}\int d\tau \sqrt{\gamma_{\tau\tau}} \left(\gamma^{\tau\tau} \partial_\tau x^\mu \partial_\tau x^\nu g_{\mu\nu}(x) - 1 \right)$$

Going back to $S(x^\mu, e)$, the momentum conjugate to x^μ is

$$p_\mu = -\frac{m}{e} g_{\mu\nu} \mathring{x}^\nu$$

Hamiltonian: $H = \mathcal{L} - p_\mu \mathring{x}^\mu = \frac{e}{2m}\left(p_\mu g^{\mu\nu} p_\nu + m^2\right)$

p_μ is constrained by the eom of e to be on the mass-shell,

$$H = 0 \Leftrightarrow p_\mu g^{\mu\nu} p_\nu + m^2 = 0 \qquad (g^{\mu\nu} = \text{inverse of } g_{\mu\nu})$$

This is a general feature of reparametrization-invariant actions.

One can fix reparametrization eg by setting $e = m$, however one must still impose $H = 0$. Another possibility is static gauge, $\tau = X^0$, particularly useful for studying non-relativistic limit.

* If the background metric is invariant under some isometry,

$$\delta x^\mu = K^\mu(x) \quad \text{where } K^\mu \text{ satisfies the Killing equation}$$

$$\mathcal{L}_K g = 0 \Leftrightarrow \nabla_\mu K_\nu + \nabla_\nu K_\mu = 0$$

then there is an additional conserved quantity, the momentum along K,

$$P = -K^\mu p_\mu = K_\mu \mathring{x}^\mu$$

For example, for Minkowski space, $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, any constant K^μ is a Killing vector, and therefore

$$P_\mu = -p_\mu = \mathring{x}^\mu \quad \text{is conserved}$$

Also $k_\mu = \omega_{\mu\nu} x^\nu$ is a Killing vector for any antisym $\omega_{\mu\nu}$, with conserved quantity

$$J_{\mu\nu} = x^\mu \mathring{x}^\nu - x^\nu \mathring{x}^\mu$$

2. Relativistic strings

The worldline $X^\mu(\tau)$ is replaced by a worldsheet $X^\mu(\tau,\sigma)$

* A natural generalization of the point particle action is the 'Polyakov action' [Reparam invariant both on WS and TS]

$$S[X^\mu, \gamma_{\alpha\beta}] = -\frac{T}{2} \int d^2\xi \, \sqrt{-\det \gamma_{\alpha\beta}} \left(\gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X) - \Lambda \right)$$

where $\xi^0 = \tau$, $\xi^1 = \sigma$, $\alpha = 0,1$ $\qquad \gamma^{\alpha\beta} = (\gamma_{\alpha\beta})^{-1}$ worldsheet metric, sig $(-,+)$

The equation of motion of $\gamma^{\alpha\beta}$ implies the 'Virasoro condition'

$$T_{\alpha\beta} \equiv g_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu - \frac{1}{2} \gamma_{\alpha\beta} \left\{ \gamma^{\rho\sigma} \partial_\rho X^\mu \partial_\sigma X_\mu - \Lambda \right\} = 0$$

$$\delta \sqrt{-\det \gamma_{\alpha\beta}} = \sqrt{-\det \gamma_{\alpha\beta}} \; \delta \ln \sqrt{-\det \gamma_{\alpha\beta}}$$
$$= \frac{1}{2} \; '' \; , \; \gamma_{\alpha\beta} \, \delta \gamma^{\alpha\beta}$$

$T_{\alpha\beta} = -\dfrac{1}{\sqrt{-\det \gamma_{\alpha\beta}}} \dfrac{\delta S}{\delta \gamma^{\alpha\beta}}$ is the response to a variation of the WS metric, hence the W.S energy momentum tensor.

Contracting with $\gamma^{\alpha\beta}$, one finds $\underline{\Lambda = 0}$

Further, $\quad \gamma_{\alpha\beta} = \lambda \, \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}$ where λ is arbitrary,

Plugging back into S, one finds the Nambu-Goto action

$$S(X^\mu) = -T \int d^2\xi \sqrt{-\det\left(\partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}\right)} = -T \times \text{area}$$
$$= -T \int d\tau d\sigma \sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2}$$

Thus the parameter $T \sim \frac{1}{L^2}$ is the string tension.

It is customary to define $T = \dfrac{1}{2\pi\alpha'} = \dfrac{1}{2\pi l_s^2}$, $1/l_s$ = 'string scale'

* The fact that | the trace of $T_{\alpha\beta}$ vanished / the parameter λ was arbitrary | reflects a key property of the string action. <u>Weyl invariance</u>

$$\gamma_{\alpha\beta} \to \gamma_{\alpha\beta}\, e^{2\phi} \quad , \quad \gamma^{\alpha\beta} \to \gamma^{\alpha\beta} e^{-2\phi}$$

$$\sqrt{-\det\gamma} \longrightarrow e^{2\phi}\sqrt{-\det\gamma} \qquad \phi(\sigma,\tau) \text{ arbitrary.}$$

⚠ This would not hold for higher dim objects such as membranes

* Using this property, + reparam invariance, one can always pick $\gamma_{\alpha\beta} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ constant. 'conformal gauge'

The eom of the string are then

$$\partial_\alpha\left(g_{\mu\nu}(X)\, \partial^\alpha X^\nu\right) = 0$$

supplemented by the eom of $\gamma_{\alpha\beta}$, known as <u>Virasoro condition</u>:

$$T_{\alpha\beta} = 0 \qquad \left[\begin{array}{l}\text{analogue of } H=0 \text{ in particle case ;} \\ \text{reflects reparam invariance on WS}\end{array}\right]$$

* For a background with isometry $k_\mu(X)$ there exists a conserved charge

$$Q = T\int d\sigma \;\; k_\mu\, \partial_\tau X^\mu$$

E.g in Minkowski space,

$$P_\mu = T\int d\sigma \;\; \partial_\tau X^\mu \qquad \text{momentum}$$

$$J_{\mu\nu} = T\int d\sigma \;\; X_\mu \partial_\tau X^\nu - X_\nu \partial_\tau X_\mu \qquad \text{angular momentum}$$

The eom are 2D wave equations

$$\partial_\alpha \partial^\alpha X^\mu = 0$$

$$\Longrightarrow \quad X^\mu = X_L^\mu(\tau+\sigma) + X_R^\mu(\tau-\sigma)$$

subject to Virasoro conditions:

$$T_{\alpha\beta} = \begin{bmatrix} (\partial_0 X^\mu)^2 & \partial_0 X^\mu \partial_1 X^\mu \\ \partial_0 X^\mu \partial_1 X^\mu & (\partial_1 X^\mu)^2 \end{bmatrix} - \frac{1}{2}\left((\partial_0 X^\mu)^2 - (\partial_1 X^\mu)^2 \right) \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$\Rightarrow \quad \partial_0 X^\mu \partial_1 X^\mu = 0$: transverse motion only

$\quad (\partial_0 X^\mu)^2 + (\partial_1 X^\mu)^2 = 0$

$\Leftrightarrow \quad (\partial_0 X^\mu \pm \partial_1 X^\mu)^2 = 0$

$\Leftrightarrow \quad (X'^\mu_L)^2 = 0 \,, \quad (X'^\mu_R)^2 = 0$

* In addition, we must enforce boundary conditions. Several choices are allowed:

– periodic $\qquad X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma)$

$\quad \rightsquigarrow$ closed string

OMIT OPEN STRINGS

– open strings: boundary conditions must be consistent with variational principle:

$$\delta S = \int \frac{\partial \mathcal{L}}{\partial \partial_\alpha X^\mu} \, \delta \partial_\alpha X^\mu \, d\tau d\sigma$$

$$= -\int \delta X^\mu \underbrace{\partial_\alpha \frac{\partial \mathcal{L}}{\partial \partial_\alpha X^\mu}}_{\overset{\shortparallel}{0}} \, d\tau d\sigma + \int d\tau \, \delta X^\mu \frac{\partial \mathcal{L}}{\partial \partial_\sigma X^\mu}$$

$\delta X^\mu = 0$: 'Dirichlet condition' $\quad X^\mu(\tau, \sigma = 0) = x^\mu(\tau)$ fixed

$\frac{\partial \mathcal{L}}{\partial \partial_\sigma X^\mu} = 0$ ie $\partial_\sigma X^\mu = 0$: 'Neuman condition'

The only choice consistent with Lorentz invariance in TS is Neumann at both ends, $\sigma = 0$ and $\sigma = \pi$.

* For closed strings, the most general solution in conformal gauge is given by the mode expansion

$$X^\mu_R = \frac{1}{2} x^\mu + \frac{1}{2} \ell_s^2 p^\mu (\tau - \sigma) + \frac{i \ell_s}{\sqrt{2}} \sum_{k \neq 0} \frac{\alpha_k}{k} e^{-ik(\tau-\sigma)}$$

$$X^\mu_L = \frac{1}{2} x^\mu + \frac{1}{2} \ell_s^2 p^\mu (\tau + \sigma) + \frac{i \ell_s}{\sqrt{2}} \sum_{k \neq 0} \frac{\tilde{\alpha}_k}{k} e^{-ik(\tau+\sigma)}$$

subject to reality conditions $\qquad T = \frac{1}{2\pi \ell_s^2}$

$$(\alpha^\mu_k)^* = \alpha^\mu_{-k} \,, \qquad (\tilde{\alpha}^\mu_k)^* = \tilde{\alpha}^\mu_{-k}$$

and Virasoro conditions $L_m = \tilde{L}_m = 0$, $\forall m \in \mathbb{Z}$, where

$$L_m = \frac{T}{2} \int_0^{2\pi} d\sigma \left(\partial_\tau X^\mu - \partial_\sigma X^\mu\right)^2 e^{im(\tau-\sigma)} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha^\mu_{m-n} \cdot \alpha^\mu_n$$

$$\tilde{L}_m = \frac{T}{2} \int_0^{2\pi} d\sigma \left(\partial_\tau X^\mu + \partial_\sigma X^\mu\right)^2 e^{im(\tau+\sigma)} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \tilde{\alpha}_{m-n} \tilde{\alpha}_n$$

and we defined $\alpha^\mu_0 = \tilde{\alpha}^\mu_0 = \frac{\ell_s}{\sqrt{2}} p^\mu$, such that

$$\partial_- X^\mu_R \equiv \frac{1}{2}(\partial_\tau - \partial_\sigma) X^\mu_R = \frac{\ell_s}{\sqrt{2}} \sum_{n \in \mathbb{Z}} \alpha^\mu_n e^{-in(\tau-\sigma)}$$

$$\partial_+ X^\mu_L \equiv \frac{1}{2}(\partial_\tau + \partial_\sigma) X^\mu_L = \frac{\ell_s}{\sqrt{2}} \sum_{n \in \mathbb{Z}} \tilde{\alpha}^\mu_n e^{-in(\tau+\sigma)}$$

The zero-frequency modes x^μ, p^μ correspond to the center of mass of the string and conserved momentum (in target space)

$$X^\mu_{CM} = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \, X^\mu(\tau,\sigma) = x^\mu + \ell_s^2 p^\mu \tau$$

$$P^\mu = T \int_0^{2\pi} d\sigma \, \dot{X}^\mu = 2\pi T \ell_s^2 p^\mu = p^\mu \qquad \text{since } T = \frac{1}{2\pi \ell_s^2}$$

$$J_{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i \sum_{n=1}^\infty \frac{1}{n}\left(\alpha^\mu_{-n} \alpha^\nu_n - \alpha^\nu_{-n} \alpha^\mu_n\right)$$

$$- i \sum_{n=1}^\infty \frac{1}{n}\left(\tilde{\alpha}^\mu_{-n} \tilde{\alpha}^\nu_n - \tilde{\alpha}^\nu_{-n} \tilde{\alpha}^\mu_n\right)$$

On the other hand, the ws hamiltonian and momentum are

$$H = \frac{T}{2}\int_0^{2\pi} d\sigma \left(\dot{X}^2 + X'^2\right) = L_0 + \tilde{L}_0 = \frac{\ell_s^2}{2}(p_\mu)^2 + \sum_{n=1}^{\infty}\left(\alpha_{-n}\alpha_n + \tilde{\alpha}_{-n}\tilde{\alpha}_n\right)$$

$$P = -T\int_0^{2\pi} d\sigma \left(\dot{X} X'\right) = L_0 - \tilde{L}_0 = \sum_{n=1}^{\infty}\left(\alpha_{-n}\alpha_n - \tilde{\alpha}_{-n}\tilde{\alpha}_n\right)$$

and must vanish by the Virasoro conditions $\Rightarrow$ mass shell + matching conditions

* The time evolution of any observable can be expressed as usual as

$$\frac{dF}{d\tau} = -\{H, F\}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket defined at equal time by

$$\{X^\mu(\sigma,\tau), \dot{X}^\nu(\sigma',\tau)\} = \frac{1}{T}\eta^{\mu\nu}\delta(\sigma-\sigma')$$

In term of the modes,

$$\{x^\mu, p^\nu\} = \eta^{\mu\nu} \qquad , \text{ 0 otherwise.}$$

$$\{\alpha_m^\mu, \alpha_n^\nu\} = -i\,\delta_{m+n}\,\eta^{\mu\nu}\,m$$

$$\{\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu\} = -i\,\delta_{m+n}\,\eta^{\mu\nu}\,m$$

* In particular, the Virasoro constraints satisfy

$$\{L_m, L_n\} = -i(m-n)L_{m+n} \qquad , \{L, \tilde{L}\} = 0$$

$$\{\tilde{L}_m, \tilde{L}_n\} = -i(m-n)\tilde{L}_{m+n}$$

which is recognized as the Lie algebra of diffeomorphisms of S^1;

$$[e^{im\sigma}\partial_\sigma, e^{in\sigma}\partial_\sigma] = -i(m-n)e^{i(m+n)}\partial_\sigma$$

* The existence of these conserved charges reflects the fact that the 'conformal gauge' $\gamma_{\alpha\beta} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ did not completely fix all the gauge symmetries Diff $\ltimes$ Weyl of the Polyakov action; indeed, defining $\xi^\pm = \tau \pm \sigma$,

the 2D metric $ds^2 = -d\tau^2 + d\sigma^2 = -d\xi^+ d\xi^-$

transforms under any diffeo of the form $\begin{cases} \xi^+ \longrightarrow f(\xi^+) \\ \xi^- \longrightarrow g(\xi^-) \end{cases}$

into $ds^2 \longrightarrow -f'(\xi^+)\, g'(\xi^-)\, d\xi^+ d\xi^-$

and the overall factor can be reabsorbed by a Weyl rescaling.

Diffeo which preserve the metric up to a rescaling are called <u>conformal transformations</u>

The fact that $\{H, L\} \propto L$ ensures that if the constraints $L = 0$ are imposed at $\tau = 0$, they will continue to hold at $\tau > 0$.

(Upon quantization, we shall see that the algebra is deformed to

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1)\, \delta_{m+n}$$

where c is the 'central charge' (unique deformation consistent with Jacobi)

and the constraint $c = 0$ will determine the total dimension of target space)

Dec 7 / 2012 :

Summary of last course :

* bosonic closed string is described by embedding coordinates

$$X^\mu(\sigma,\tau) \; : \; \underbrace{S_1 \times \mathbb{R}}_{\text{worldsheet}} \longrightarrow \underbrace{\mathcal{M}}_{\text{target space}} \qquad \mu = 0 .. D-1$$

and worldsheet metric $\gamma_{\alpha\beta}(\sigma,\tau)$ $\qquad \alpha = 0,1$

with action
$$S_P = -\frac{T}{2}\int d\sigma\, d\tau \sqrt{-\det\gamma}\; \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X)\, \gamma^{\alpha\beta}$$

invariant under Diff $\ltimes$ Weyl

$$\text{Weyl}: \quad \gamma_{\alpha\beta} \to e^\phi \gamma_{\alpha\beta} \qquad \phi(\sigma,\tau) \text{ arbitrary.}$$

* Eom of $\gamma_{\alpha\beta}$ impose Virasoro constraints

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu} - \frac{1}{2}\gamma_{\alpha\beta}\left(\partial_\rho X^\mu \partial^\rho X_\mu\right) = 0$$

This determines $\gamma_{\alpha\beta} = e^\phi \underbrace{\partial_\alpha X^\mu \partial_\beta X_\mu}_{\text{induced metric } G_{\alpha\beta}}$

Substituting back gives the Nambu Goto action

$$S_{NG} = -T\int d\sigma\, d\tau \sqrt{-\det G_{\alpha\beta}}$$

Eom of X^μ gives $\quad \partial_\alpha \sqrt{-\det\gamma}\; \gamma^{\alpha\beta} g_{\mu\nu}(X)\, \partial_\beta X^\nu = 0$

* Using Diff $\ltimes$ Weyl, one may locally choose $\gamma_{\alpha\beta} = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$

If in addition $g_{\mu\nu}(X) = \eta_{\mu\nu}$ constant, X^μ become free fields in 1+1 :

subject to
$$\begin{cases} \partial_\alpha \partial^\alpha X^\mu = 0 \\ T_{\alpha\beta} = 0 \end{cases}$$

* X^μ can therefore be decomposed into left + right movers

$$X^\mu(\sigma,\tau) = X^\mu_L(\tau+\sigma) + X^\mu_R(\tau-\sigma)$$

$$(X'_L)^2 = (X'_R)^2 = 0$$

For closed strings, $X^\mu(\sigma+2\pi,\tau) = X^\mu(\sigma,\tau)$

so X_L, X_R have the mode expansion

$$X^\mu_R = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2\, p^\mu(\tau-\sigma) + \frac{i l_s}{\sqrt{2}} \sum_{k\neq 0} \frac{\alpha_k}{k} e^{-ik(\tau-\sigma)}$$

$$X^\mu_L = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2\, p^\mu(\tau+\sigma) + \frac{i l_s}{\sqrt{2}} \sum_{k\neq 0} \frac{\tilde{\alpha}_k}{k} e^{-ik(\tau+\sigma)}$$

where $T = \frac{1}{2\pi l_s^2}$,

$$(\alpha^\mu_k)^* = \alpha^\mu_{-k}, \quad (\tilde{\alpha}^\mu_k)^* = \tilde{\alpha}^\mu_{-k}$$

The constraints $X'^2_L = X'^2_R = 0$ can similarly be decomposed in Fourier modes

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha^\mu_{m-n}\alpha^\mu_n \qquad \alpha^\mu_0 = \tilde{\alpha}^\mu_0 = \frac{l_s}{\sqrt{2}} p^\mu$$

$$\tilde{L}_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} \tilde{\alpha}^\mu_{m-n}\tilde{\alpha}^\mu_n$$

$H = L_0 + \tilde{L}_0$ is the worldsheet Hamiltonian, $P = L_0 - \tilde{L}_0$ is momentum

* The oscillators and constraints satisfy Poisson algebra

$$\{x^\mu, p^\nu\} = \eta^{\mu\nu}$$

$$\{\alpha^\mu_m, \alpha^\nu_n\} = \{\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n\} = -i\delta_{m+n}\, m\, \eta^{\mu\nu}$$

$$\{L_m, L_n\} = -i(m-n)L_{m+n}$$

$$\{\tilde{L}_m, \tilde{L}_n\} = -i(m-n)\tilde{L}_{m+n}$$

* The constraints L_m, $\tilde{L}_m$ are the generators of conformal transformations, ie subgroup of Diff ⋉ Weyl which preserve the conformal gauge $ds^2 = -d\tau^2 + d\sigma^2$

$$\xi^+ \to f(\xi^+) \qquad \text{if} \quad \xi^\pm = \tau \pm \sigma$$
$$\xi^- \to g(\xi^-)$$

* The canonical quantization proceeds by replacing $\{,\} \to \frac{1}{i}[\,,\,]$

$$[x^\mu, p^\nu] = i\,\eta^{\mu\nu}$$
$$[\alpha_m^\mu, \alpha_n^\nu] = m\,\eta^{\mu\nu}\,\delta_{m+n} = [\hat{\alpha}_m^\mu, \hat{\alpha}_n^\nu] \,,$$

- represent $\alpha^\mu_{m>0}$ as annihilation operators in a Fock space
  $\alpha^\mu_{m<0}$ creation
- restrict to states annihilated by positive frequency part of the constraints:

$$L_{m>0}\,|\text{phys}\rangle = \tilde{L}_{m>0}\,|\text{phys}\rangle = 0 \qquad \forall m>0$$
$$(L_0 - a)\,|\text{phys}\rangle = (\tilde{L}_0 - a)\,|\text{phys}\rangle = 0$$

a = normal ordering ambiguity

modulo "null states", ie physical states of the form

$$\sum_{m>0}(a_m L_m + \hat{a}_m \tilde{L}_m)\,|\text{anything}\rangle$$

- This however requires that the gauge sym be non anomalous. It turns out that this is only possible if $D=26$, $a=1$. The spectrum then contains
  - a tachyon $|0,\tilde{0},p^\mu\rangle$, $\quad l_s^2 m^2 = -4$
  - massless graviton + dilaton + $B_{\mu\nu}$: $\alpha^\mu_{-1}\hat{\alpha}^\nu_{-1}$, $\quad m^2=0$

The proof will await CFT techniques ... Today we'll use LC quantization.

1. Light-cone gauge quantization

$\eta_{\mu\nu} = \begin{pmatrix} & -1/2 & \\ -1/2 & & \\ & & \mathbb{1} \end{pmatrix}$

$A_+ = -\frac{A^-}{2}, \quad A_- = -\frac{A^+}{2}$

1.1. In field theory:

* Consider a massive free scalar field in D-dimensions:
the usual mode expansion is $\qquad [a(\vec{k}), a^\dagger(\vec{k}')] = (2\pi)^{D-1} 2\omega_{\vec{k}} \delta^{D-1}(\vec{k}-\vec{k}')$

$$\phi(x^\mu) = \int \frac{d^{D-1}k}{(2\pi)^{D-1} 2\omega_{\vec{k}}} \left[a(\vec{k}) e^{-ik_\mu x^\mu} + a^\dagger(\vec{k}) e^{ik_\mu x^\mu} \right]$$

where $\quad k_0 = +\sqrt{\vec{k}^2 + m^2}$, such that $\quad (k_0)^2 - (k_i)^2 = m^2$

Choosing light-cone coordinates $\quad x^\pm = x^0 \pm x^1$

$$k_\pm = \frac{1}{2}(k_0 \pm k_1), \quad k^\pm = k^0 \pm k^1$$

the mass-shell condition can be solved instead for k^-:

$$k^+ k^- = m^2 + (k_i)^2 \quad \Rightarrow \quad k^- = \frac{k_i^2 + m^2}{k^+}$$

Thus we can expand just as well [except for mode with $k^+ = 0$]

$$\phi(x^\mu) = \int_{k^+>0} \frac{d^{D-2}k_i \, dk^+}{k^+} \left[a(\vec{k}_i, k^+) e^{-ik_\mu x^\mu} + a^\dagger(\vec{k}_i, k^+) e^{ik_\mu x^\mu} \right]$$

with commutator

$$[a(\vec{k}_i, k^+), a^\dagger(\vec{k}_i', k^{+\prime})] = 4\pi k^+ (2\pi)^{D-2} \delta^{D-2}(k_i - k_i') \delta(k^+ - k^{+\prime})$$

* This scheme becomes more useful in the presence of [gauge symmetries; constraints]

<u>ex 1</u> Maxwell field $A_\mu(x)$,

e.o.m $\quad \partial_\mu F^{\mu\nu} = 0$,

gauge sym $\quad \delta A_\mu = \partial_\mu \varepsilon$

In momentum space, light-cone coordinates:

$$k^2 A^\mu - k^\mu (k \cdot A) = 0, \qquad \delta A^\mu = i k^\mu$$

In light-cone coordinates, the gauge sym becomes

$$\delta A^+ = i k^+ \varepsilon$$
$$\delta A^- = i k^- \varepsilon$$
$$\delta A^i = i k^i \varepsilon$$

We can choose $\varepsilon = A^+/(ik^+)$ to reach the <u>light-cone gauge $A^+ = 0$</u>

The + component of e.o.m (for $k^+ \neq 0$)

then implies the transversality condition $k_\mu A^\mu = 0$

ie $$-k^0 A^0 + k^1 A^1 + k^i A^i = 0$$

$$-\frac{1}{2}\left[(k^0+k^1)(A^0-A^1) + (k^0-k^1)(A^0+A^1)\right] + k^i A^i = 0$$

ie $$-\frac{1}{2}\left(k^+ A^- + k^- A^+\right) + k^i A^i = 0$$

Since $A^+ = 0$, this can be used to solve for

$$A^- = 2 k^i A^i / k^+$$

The remaining components of eom imply $k^2 = 0$ ie $k^- = k_i^2 / k^+$

Thus the mode expansion is in terms of D-2 unconstrained, transverse oscillators

$$A^i(x^\mu) = \int_{k^+>0} d^{D-2}k_i \, \frac{dk^+}{k^+} \left[a^i(\vec{k}_i, k^+)\, e^{-ik_\mu x^\mu} + a^{i\dagger}\, e^{ik_\mu x^\mu} \right]$$

$$A^+ = 0 \,; \quad A^- = 2 \int d^{D-2}k_i \, \frac{dk^+}{(k^+)^2} \, k_i \left[a_i\, e^{()} + a_i^\dagger\, e^{-()} \right]$$

<u>Rk</u> for a massive spin 1 field, the mode expansion involves D-1 oscillators, as required by the fact that the little group of a massive particle in (1, D-1) dimensions is $SO(D-1)$; in contrast, for a massless particle it is $SO(D-2) \ltimes \mathbb{R}^\#$

ex 2 Graviton $h^{\mu\nu}$, symmetric $h \equiv \eta_{\mu\nu} h^{\mu\nu}$

eom $k^2 h^{\mu\nu} - k_\rho \left(k^\mu h^{\nu\rho} + k^\nu h^{\mu\rho} \right) + k^\mu k^\nu h = 0$

gauge invariance $\delta h^{\mu\nu} = i\left(k^\mu \varepsilon^\nu + k^\nu \varepsilon^\mu \right)$

In light cone coordinates,

$$\delta h^{++} = 2i\, k^+ \varepsilon^+$$
$$\delta h^{+-} = i\left(k^+ \varepsilon^- + k^- \varepsilon^+ \right)$$
$$\delta h^{+i} = i\left(k^+ \varepsilon^i + k^i \varepsilon^+ \right)$$

one can always choose $\varepsilon^+, \varepsilon^-, \varepsilon^i$ to reach the light-cone gauge

$$\underline{h^{++} = h^{+-} = h^{+i} = 0} \qquad (k^+ \neq 0)$$

The $++$ component of eom then implies $h = 0$

ie $h^{ij}\delta_{ij} = 0$

The $+\nu$ component gives $k^+ \left(k_\rho h^{\nu\rho} \right) = 0$

$$k_- h^{i-} + k_j h^{ij} = 0 \qquad \rightarrow \qquad h^{i-} = \frac{1}{2k^+} k_j h^{ij}$$
$$k_- h^{--} + k_j h^{-j} = 0 \qquad \qquad h^{--} = \frac{1}{2k^+} k_j h^{-j}$$

and the ij components give $k^2 = 0$

hence $k^- = (k^i)^2 / k^+$

Thus the mode expansion involves $\frac{(D-2)(D-1)}{2} - 1$ harmonic oscillators $a^{ij}(k^+, k^i)$, transforming as a symmetric traceless rep of $SO(D-2)$.

ex 3 (exercise) : show that a 2-form $B_{\mu\nu}$ with

eom ; $\partial_\mu H^{\mu\nu\rho} = 0$, $H_{\mu\nu\rho} = \partial_{[\mu} B_{\rho\sigma]} + \text{circ}$

gauge invariance ; $\delta B_{\mu\nu} = \partial_\mu \varepsilon_\nu - \partial_\nu \varepsilon_\mu$

light cone gauge : $B^{+-} = B^{+i} = 0$ gives $\frac{(D-2)(D-3)}{2}$ oscillators in antisym rep of $SO(D-2)$

1.2. Bosonic closed strings in light-cone gauge

Conformal transformations $\xi^{+\prime} = f(\xi^+)$ $\qquad \xi^{\pm} = \tau \pm \sigma$

$\qquad\qquad\qquad\qquad\qquad \xi^{-\prime} = g(\xi^-)$

Can be used to map τ or σ to an arbitrary solution of $\partial_+\partial_- = 0$:

$$2\tau' = f(\tau+\sigma) + g(\tau-\sigma)$$
$$2\sigma' = f(\tau-\sigma) - g(\tau-\sigma)$$

Since $\partial_+\partial_- X^+ = 0$, we can choose $\boxed{X^+ = x^+ + \ell_s^2 p^+ \tau}$

ie set all oscillators $\alpha_k^+,\ \tilde{\alpha}_k^+ = 0$ for $k \neq 0$

The Virasoro constraints $\qquad$ [Recall $\alpha_0^\mu = \frac{\ell_s}{\sqrt{2}} p^\mu = \tilde{\alpha}_0^\mu$]

$$L_m = \frac{1}{2} \sum_{m\in\mathbb{Z}} \alpha_{m-n}^\mu \alpha_n^\mu$$

$$= -\frac{1}{2} \sum_{m\in\mathbb{Z}} \alpha_{m-n}^- \alpha_n^+ + \frac{1}{2} \sum_{m\in\mathbb{Z}} \alpha_{m-n}^i \alpha_n^i$$

$$= -\frac{\ell_s}{2\sqrt{2}} p^+ \alpha_m^- + \frac{1}{2} \sum_{m\in\mathbb{Z}} \alpha_{m-n}^i \alpha_n^i$$

allow to express α_m^- in terms of the transverse oscillators:

$$\alpha_m^- = \frac{\sqrt{2}}{\ell_s p^+} \sum_{m\in\mathbb{Z}} \alpha_{m-n}^i \alpha_n^i$$

equiv: $\partial_+ X^- = \frac{1}{\ell_s^2 p^+} \partial_+ X^i \partial_+ X^i$

$\qquad \partial_- X^- = \frac{1}{\ell_s^2 p^+} \partial_- X^i \partial_- X^i$

For $m = 0$, one obtains the mass-shell condition:

$$p^- = \frac{2}{\ell_s^2 p^+}\left[\sum_{m=1}^{\infty} \left(\alpha_{-n}^i \alpha_n^i + \alpha_n^i \alpha_{-n}^i\right) + \frac{\ell_s^2}{2} p_i^2\right]$$

Comparing with $p^- = \frac{p_i^2 + M^2}{p^+}$ we read off the mass

$$M^2 = \frac{2}{\ell_s^2} \sum_{m=1}^{\infty} \left(\alpha_{-n}^i \alpha_n^i + \alpha_n^i \alpha_{-n}^i\right)$$

To evaluate the mass of a Fock state $\prod_j \alpha_{-n_j}^{i_j} |0\rangle$,

one needs to normal-order M^2:

$$M^2 = \frac{4}{\ell_s^2}\left(\sum_{m=1}^{\infty} \alpha_{-n}^{i}\alpha_{n}^{i} + \underbrace{\frac{D-2}{2}\sum_{m=1}^{\infty} m}\right)$$

Casimir energy of D-2 massless bosons on cylinder

The sum $\sum_{m=1}^{\infty} m$ is divergent. One way to treat it

is to use zeta function regularization, $\zeta(s) = \sum_{m=1}^{\infty}\frac{1}{m^s}$

and analytically continue to $s=-1$: $\zeta(-1) = -\frac{1}{12}$

[This can be seen from the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\, \zeta(1-s) \quad]$$

thus

$$\boxed{M^2 = \frac{4}{\ell_s^2}\left(\sum_{m=1}^{\infty} \alpha_{-m}^{i}\alpha_{m}^{i} - \frac{D-2}{24}\right)}$$

$$\sum_{n=1}^{\infty} n e^{-n\varepsilon} = -\frac{\partial}{\partial\varepsilon}\frac{1}{1-e^{-\varepsilon}} = \frac{1}{\varepsilon^2} - \frac{1}{12} + \dots$$

(Alternatively, one can introduced an undetermined constant a such that $M^2 = \frac{4}{\ell_s^2}\left(\sum_{m=1}^{\infty}\alpha_{-n}^{i}\alpha_{n}^{i} - a\right)$ and ultimately fix both D and a by requiring Lorentz invariance)

Of course, the same considerations on the left-moving side require

$$M^2 = \frac{4}{\ell_s^2}\left(\sum_{m=1}^{\infty}\tilde{\alpha}_{-m}^{i}\tilde{\alpha}_{m}^{i} - \frac{D-2}{24}\right)$$

hence the 'level matching condition' $\boxed{\sum_{m=1}^{\infty}\alpha_{-m}^{i}\alpha_{m}^{i} = \sum_{m=1}^{\infty}\tilde{\alpha}_{-m}^{i}\tilde{\alpha}_{m}^{i}}$

(In the case of open strings, there is only one set of modes, so no matching conditions, but $\alpha_0^\mu = \sqrt{2}\,\ell_s p^\mu$ rather than $\ell_s p^\mu/\sqrt{2}$,)

hence

$$M^2_{open} = \frac{1}{\ell_s^2}\left(\sum_{m=1}^{\infty}\tilde{\alpha}_{-m}^{i}\alpha_{m}^{i} - \frac{D-2}{24}\right)$$

* let us now consider the low-lying modes of the closed bosonic string:

- ground state $|0\rangle$: $m^2 = -4a/\ell_s^2$

- first excited state (satisfying level matching):

$$\alpha^i_{-1} \tilde{\alpha}^{\tilde{i}}_{-1} |0\rangle : \quad m^2 = -4(a-1)/\ell_s^2 \qquad : (D-2)^2 \text{ states}$$
$$i, \tilde{i} \quad 1 \ldots D-2$$

splits into
- symmetric traceless tensor under $SO(D-2)$
- antisym tensor
- scalar

In order for this mode to furnish a rep of the Lorentz group $SO(1, D-1)$, it is necessary that $m^2 = 0$, ie $\underline{a = 1}$ (since they do not form a rep of the little group $SO(D-1)$ for massive particles.) $\Rightarrow$ $\boxed{D = 26}$

→ . the symmetric traceless tensor describes a massless spin 2 field in $D=26$ dim ; $\delta h_{\mu\nu} = k_\mu \varepsilon_\nu + k_\nu \varepsilon_\mu$

, the antisymm tensor describes a 2-form field $B_{\mu\nu}$ (massless) with gauge invariance $\delta B_{\mu\nu} = k_\mu \varepsilon_\nu - k_\nu \varepsilon_\mu$

, the massless scalar mode is known as the dilaton

, Alas, the ground state $|0\rangle$ describes a tachyonic scalar, with $m^2 = -4/\ell_s^2 < 0$ which indicates that the bosonic closed string is unstable

. second excited level:

$$\alpha^i_{-2}\ \tilde{\alpha}^{\tilde{i}}_{-2}\ |0\rangle \qquad \left[D-2+\frac{(D-2)(D-1)}{2}\right]^2 \text{ states}$$

$$\alpha^i_{-1}\alpha^j_{-1}\ \tilde{\alpha}^{\tilde{i}}_{-2}\ |0\rangle \qquad \text{mass} \quad m^2 = -4\,(a-2)/\ell_s^2$$

$$\alpha^i_{-2}\ \tilde{\alpha}^{\tilde{i}}_{-1}\ \tilde{\alpha}^{\tilde{j}}_{-1}\ |0\rangle \qquad = 4/\ell_s^2$$

$$\alpha^i_{-1}\ \alpha^j_{-1}\ \tilde{\alpha}^{\tilde{i}}_{-1}\ \tilde{\alpha}^{\tilde{j}}_{-1}\ |0\rangle$$

Unlike the first excited level, these states can be fit into a representation of $SO(D-1)$:

$$f^{(MN)(\tilde{M}\tilde{N})} \qquad \left[\frac{D(D-1)}{2}-1\right]^2 \sim \quad f^{0i;\tilde{i}\tilde{j}} \quad f^{0i,0\tilde{i}} \quad f^{ij,\tilde{i}\tilde{j}} \quad f^{ij,0\tilde{i}}$$

Sym. traceless over (MN), $(\tilde{M},\tilde{N})$

. The number of states grows very quickly with the level e.g. on the left-moving side, at level N, the number of states $c(N)$ is given by

$$\sum_{N=0}^{\infty} c(N)\, q^N = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{24}}$$

$$c(N) \sim \exp\left(4\pi\sqrt{N}\right)$$

Tensoring the left and right moving side, the total number of states is of order $\exp\left(8\pi\sqrt{N}\right) = \exp\left(8\pi\sqrt{\frac{\ell_s^2 M^2}{4}}\right)$

As a result the canonical partition function $\sum \Omega(M)\, e^{-M/T}$ diverges at $T > T_H = \frac{1}{4\pi\ell_s}$: Hagedorn temperature

* The dimension D and 'intercept' a can be fixed more conceptually by requiring that the Lorentz symmetry be non anomalous.

Classically, the Lorentz generators (for the open string, say) are

$$J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha_n^\mu \alpha_{-n}^\nu - \alpha_n^\nu \alpha_{-n}^\mu \right)$$

In light cone gauge, the candidate operators are, for $\mu\nu = -i$.

$$J^{-i} = x^- p^i - \frac{1}{2}\left(x^i p^- + p^- x^i \right) - i \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha_{-n}^- \alpha_n^i - \alpha_{-n}^i \alpha_n^- \right)$$

a long computation shows

$$\left[J^{-i}, J^{-j} \right] = \frac{1}{\ell_s^2 (p^+)^2} \sum_{m=1}^{\infty} \left(\alpha_{-m}^i \alpha_m^j - \alpha_{-m}^j \alpha_m^i \right) \times \left\{ m \left(1 - \frac{D-2}{24} \right) + \frac{1}{m} \left(\frac{1}{24}(D-2) - a \right) \right\}$$

Which vanishes iff $a = \frac{D-2}{24} = 1$

(cf Zwiebach chap 12.5)

String theory – Exercises 1.

* Discuss the non-relativistic limit of Nambu-Goto action

$$S[X^\mu(\tau,\sigma)] = -T \int d^2\xi \, \sqrt{-\det \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}}$$

in 'static gauge' $\xi^0 \equiv \tau = X^0$, $\xi^1 \equiv \sigma = X^1$

and show that T is the string tension, ie mass by unit length.

* Discuss the light-cone quantization of an antisymmetric tensor $B_{\mu\nu}$ in $(1, D-1)$ dimensions, subject to equation of motion

$\partial^\mu H_{\mu\nu\rho} = 0$ where $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu}$

and gauge invariance $\delta B_{\mu\nu} = \partial_\mu \varepsilon_\nu - \partial_\nu \varepsilon_\mu$.

* Explain why the number of level N states in the Fock space generated by oscillators α^i_m, $i = 1 \dots D-2$, $m \in \mathbb{Z}$

is the coefficient $c(N)$ in
$$\sum_{N=0}^{\infty} c(N)\, q^N = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{D-2}}$$

Using the mathematical fact that the Dedekind η function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n) \quad , \quad q \equiv e^{2\pi i \tau}$$

Satisfies $\eta(-1/\tau) = \sqrt{-i\tau}\, \eta(\tau)$, show that the coefficients $c(N)$ grow asymptotically as $\exp\left(2\pi \sqrt{\frac{D-2}{6} N}\right)$