Classical strings

1 - Relativistic Point Particle

* Particles in a fixed gravitational background propagate along geodesics. These trajectories can be derived from an action

$g_{\mu\nu}$: sig$(d-1, 1)$ $\oplus$ $\ominus$ 'target space metric'

$$S(x^\mu, e) = -\frac{m}{2}\int d\tau \left[\frac{\dot{x}^\mu \dot{x}^\nu}{e} g_{\mu\nu}(x) - e \right]$$

$\delta S/\delta e = 0 \longrightarrow \qquad e^2 = -\dot{x}^\mu g_{\mu\nu} \dot{x}^\nu$

$\delta S/\delta x^\mu = 0 \rightarrow \qquad \frac{d}{d\tau}\left(\frac{1}{e} g_{\mu\nu} \frac{dx^\nu}{d\tau} \right) = 0$

Plugging back in $S(x^\mu, e)$, an equivalent action is

$$S(x^\mu) = m \int d\tau \sqrt{-\dot{x}^\mu g_{\mu\nu} \dot{x}^\nu} = m \times \int ds$$

length

indeed, $ds^2 = -g_{\mu\nu} dx^\mu dx^\nu$

* Both actions are manifestly invariant under reparametrization of proper time τ. Infinitesimally,

$$\delta x^\mu(\tau) = \xi(\tau)\, \dot{x}^\mu(\tau) + O(\xi^2)$$

$$\delta e(\tau) = \partial_\tau [\xi(\tau) e(\tau)] + O(\xi^2)$$

In both cases, S varies by a total derivative.

In fact, one can view $e^2 = \gamma_{\tau\tau}$ as a worldline metric,

$$S(x^\mu, \gamma) = -\frac{m}{2}\int d\tau \sqrt{\gamma_{\tau\tau}} \left(\gamma^{\tau\tau} \partial_\tau x^\mu \partial_\tau x^\nu g_{\mu\nu}(x) - 1 \right)$$

Going back to $S(x^\mu, e)$, the momentum conjugate to x^μ is

$$p_\mu = -\frac{m}{e} g_{\mu\nu} \overset{\circ}{x}{}^\nu$$

Hamiltonian : $H = \mathcal{L} - p_\mu \overset{\circ}{x}{}^\mu = \frac{e}{2m}\left(p_\mu g^{\mu\nu} p_\nu + m^2\right)$

p_μ is constrained by the eom of e to be on the mass-shell,

$$H = 0 \Leftrightarrow p_\mu g^{\mu\nu} p_\nu + m^2 = 0 \qquad (g^{\mu\nu} = \text{inverse of } g_{\mu\nu})$$

This is a general feature of reparametrization-invariant actions.

One can fix reparametrization eg by setting $e = m$, however one must still impose $H = 0$. Another possibility is static gauge, $\tau = X^0$, particularly useful for studying non-relativistic limit.

* If the background metric is invariant under some isometry,

$\delta x^\mu = K^\mu(x)$ where K^μ satisfies the Killing equation

$$\mathcal{L}_K g = 0 \Leftrightarrow \nabla_\mu K_\nu + \nabla_\nu K_\mu = 0$$

then there is an additional conserved quantity, the momentum along K,

$$P = -K^\mu p_\mu = ? \; K_\mu \overset{\circ}{x}{}^\mu$$

For example, for Minkowski space, $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1 \ldots)$, any constant K^μ is a Killing vector, and therefore

$$P_\mu = -p_\mu = \overset{\circ}{x}{}^\mu \quad \text{is conserved}$$

Also $k_\mu = \omega_{\mu\nu} x^\rho$ is a Killing vector for any antisym $\omega_{\mu\nu}$, with conserved quantity

$$J_{\mu\nu} = x^\mu \overset{\circ}{x}{}^\nu - x^\nu \overset{\circ}{x}{}^\mu$$

2. Relativistic strings

The worldline $X^\mu(\tau)$ is replaced by a worldsheet $X^\mu(\tau,\sigma)$

* A natural generalization of the point particle action is the 'Polyakov action' [Reparam invariant both on WS and TS]

$$S[X^\mu,\gamma_{\alpha\beta}] = -\frac{T}{2}\int d^2\xi\,\sqrt{-\det\gamma_{\alpha\beta}}\left(\gamma^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu g_{\mu\nu}(X) - \Lambda\right)$$

where $\xi^0=\tau$, $\xi^1=\sigma$, $\alpha=0,1$ $\gamma^{\alpha\beta}=(\gamma_{\alpha\beta})^{-1}$ worldsheet metric

The equation of motion of $\gamma^{\alpha\beta}$ implies the 'Virasoro condition'

$$T_{\alpha\beta} \equiv g_{\mu\nu}\,\partial_\alpha X^\mu\partial_\beta X^\nu - \frac{1}{2}\gamma_{\alpha\beta}\left\{\gamma^{\rho\sigma}\partial_\rho X^\mu\partial_\sigma X_\mu - \Lambda\right\} = 0$$

$$\delta\sqrt{-\det\gamma_{\alpha\beta}} = \sqrt{-\det\gamma_{\alpha\beta}}\;\delta\ln\sqrt{-\det\gamma_{\alpha\beta}}$$
$$= \frac{1}{2}\;\;''\;\;\cdot\;\gamma_{\alpha\beta}\,\delta\gamma^{\alpha\beta}$$

$T_{\alpha\beta} = -\frac{1}{\sqrt{-\det\gamma_{\alpha\beta}}}\frac{\delta S}{\delta\gamma^{\alpha\beta}}$ is the response to a variation of the w.s metric, hence the w.s energy momentum tensor.

Contracting with $\gamma^{\alpha\beta}$, one finds $\underline{\Lambda=0}$

Further, $\gamma_{\alpha\beta} = \lambda\;\partial_\alpha X^\mu\partial_\beta X^\nu g_{\mu\nu}$ where λ is arbitrary,

Plugging back into S, one finds the Nambu-Goto action

$$S(X^\mu) = -T\int d^2\xi\,\sqrt{-\det\left(\partial_\alpha X^\mu\partial_\beta X^\nu g_{\mu\nu}\right)} = -T\times\text{area}$$
$$= -T\int d\sigma d\tau\sqrt{(\dot{X}\cdot X')^2-\dot{X}^2X'^2}$$

Thus the parameter $T\sim\frac{1}{L^2}$ is the string tension.

It is customary to define $T=\frac{1}{2\pi\alpha'}=\frac{1}{2\pi l_s^2}$, $1/l_s$ = 'string scale'

* Using reparam invariance, one can (locally at least) go to static gauge, $\tau = X^0$, $\sigma = X^1$

In flat Minkowski space,

$$\gamma_{\alpha\beta} = \begin{pmatrix} (\partial_\tau X^\mu)^2 & \partial_\tau X^\mu \partial_\sigma X^\mu \\ \partial_\tau X^\mu \partial_\sigma X^\mu & (\partial_\sigma X^\mu)^2 \end{pmatrix} = \begin{pmatrix} -1 + (\partial_\tau X^i)^2 & \partial_\tau X^i \partial_\sigma X^i \\ \partial_\tau X^i \partial_\sigma X^i & 1 + (\partial_\sigma X^i)^2 \end{pmatrix}$$

$$\det \gamma_{\alpha\beta} = -1 + (\partial_\tau X^i)^2 - (\partial_\sigma X^i)^2 + (\partial_\tau X^i)^2 (\partial_\sigma X^i)^2 - (\partial_\tau X^i \partial_\sigma X^i)^2$$

$$\sqrt{-\det \gamma_{\alpha\beta}} \sim 1 - \frac{1}{2}\left\{ (\partial_\tau X^i)^2 - (\partial_\sigma X^i)^2 \right\} + \dots$$ in non relativistic limit

$$S = T \int d\tau d\sigma \left(\underbrace{-1}_{-m\int d\tau} + \frac{1}{2} (\partial_\tau X^i)^2 - \frac{1}{2} (\partial_\sigma X^i)^2 + \dots \right)$$

$-m \int d\tau$ where $m = TL$

* The fact that | the trace of $T_{\alpha\beta}$ vanished / the parameter λ was arbitrary | reflects a key property of the string action: **Weyl invariance**

$$\gamma_{\alpha\beta} \to \gamma_{\alpha\beta}\, e^{2\phi} \quad , \quad \gamma^{\alpha\beta} \to \gamma^{\alpha\beta} e^{-2\phi}$$

$$\sqrt{-\det\gamma} \to e^{2\phi} \sqrt{-\det\gamma} \qquad \phi(\sigma,\tau) \text{ arbitrary.}$$

⚠ This would not hold for higher dim objects such as membranes

* Using this property, + reparam invariance, one can always pick $\gamma_{\alpha\beta} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ constant. 'conformal gauge'

The eom of the string are then

$$\partial_\alpha \left(g_{\mu\nu}(X)\, \partial^\alpha X^\nu \right) = 0$$

supplemented by the eom of $\gamma_{\alpha\beta}$, known as **Virasoro condition**:

$$T_{\alpha\beta} = 0$$ [analogue of $H=0$ in particle case; reflects reparam invariance on WS]

* For a background with isometry $k_\mu(x)$ there exists a conserved charge

$$Q = T \int d\sigma \; k_\mu \partial_\tau X^\mu$$

E.g. in Minkowski space,

$$P_\mu = T \int d\sigma \;\; \partial_\tau X^\mu \qquad \text{momentum}$$

$$J_{\mu\nu} = T \int d\sigma \;\; X_\mu \partial_\tau X^\nu - X_\nu \partial_\tau X_\mu \qquad \text{angular momentum}$$

The eom are 2D wave equations

$$\partial_\alpha \partial^\alpha X^\mu = 0$$

$$\longrightarrow \quad X^\mu = X_L^\mu(\tau+\sigma) + X_R^\mu(\tau-\sigma)$$

subject to Virasoro conditions:

$$T_{\alpha\beta} = \begin{bmatrix} (\partial_0 X^\mu)^2 & \partial_0 X^\mu \partial_1 X^\mu \\ \partial_0 X^\mu \partial_1 X^\mu & (\partial_1 X^\mu)^2 \end{bmatrix} - \frac{1}{2}\left((\partial_0 X^\mu)^2 - (\partial_1 X^\mu)^2 \right) \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$\Rightarrow \quad \partial_0 X^\mu \partial_1 X^\mu = 0$$

$$(\partial_0 X^\mu)^2 + (\partial_1 X^\mu)^2 = 0$$

$$\Leftrightarrow \quad (\partial_0 X^\mu \pm \partial_1 X^\mu)^2 = 0$$

$$\Leftrightarrow \quad (X_L'^\mu)^2 = 0 \,, \qquad (X_R'^\mu)^2 = 0$$

* In addition, we must enforce boundary conditions. Several choices are allowed:

— periodic $\qquad X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma)$

$\rightsquigarrow$ closed string

— open strings: boundary conditions must be consistent with variational principle:

$$\delta S = \int \frac{\partial \mathcal{L}}{\partial \partial_\alpha X^\mu} \, \delta \, \partial_\alpha X^\mu \, d\tau d\sigma$$

$$= - \int \delta X^\mu \underbrace{\partial_\alpha \frac{\partial \mathcal{L}}{\partial \partial_\alpha X^\mu}}_{\overset{!}{=} 0} d\tau d\sigma + \int d\tau \, \delta X^\mu \frac{\partial \mathcal{L}}{\partial \partial_\sigma X^\mu}$$

$\delta X^\mu = 0$: 'Dirichlet condition' $\quad X^\mu(\tau, \sigma = 0) = x^\mu(\tau)$ fixed

$\frac{\partial \mathcal{L}}{\partial \partial_\sigma X^\mu} = 0$ ie $\partial_\sigma X^\mu = 0$: 'Neuman condition'

The only choice consistent with Lorentz invariance in TS is Neumann at both ends, $\sigma = 0$ and $\sigma = \pi$.

* For closed strings, the most general solution in conformal gauge is given by the mode expansion

$$X_R^\mu = \frac{1}{2} x^\mu + \frac{1}{2} l_s^2 p^\mu (\tau - \sigma) + \frac{i l_s}{\sqrt{2}} \sum_{k \neq 0} \frac{\alpha_k}{k} e^{-ik(\tau-\sigma)}$$

$$X_L^\mu = \frac{1}{2} x^\mu + \frac{1}{2} l_s^2 p^\mu (\tau + \sigma) + \frac{i l_s}{\sqrt{2}} \sum_{k \neq 0} \frac{\tilde{\alpha}_k}{k} e^{-ik(\tau+\sigma)}$$

subject to reality conditions

$$(\alpha_k^\mu)^* = \alpha_{-k}^\mu \,, \qquad (\tilde{\alpha}_k^\mu)^* = \tilde{\alpha}_{-k}^\mu$$

and Virasoro conditions $L_m = \tilde{L}_m = 0 \,,\ \forall m \in \mathbb{Z}$, where

$$L_m = \frac{T}{2} \int_0^{2\pi} d\sigma \left(\partial_\tau X^\mu - \partial_\sigma X^\mu \right)^2 e^{im(\tau-\sigma)} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_{m-n}^\mu \alpha_n^\mu$$

$$\tilde{L}_m = \frac{T}{2} \int_0^{2\pi} d\sigma \left(\partial_\tau X^\mu + \partial_\sigma X^\mu \right)^2 e^{im(\tau+\sigma)} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \tilde{\alpha}_{m-n} \tilde{\alpha}_n$$

and we defined $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \frac{l_s}{\sqrt{2}} p^\mu$, such that

$$\partial_- X_R^\mu \equiv \frac{1}{2}(\partial_\tau - \partial_\sigma) X_R^\mu = \frac{l_s}{\sqrt{2}} \sum_{n \in \mathbb{Z}} \alpha_n^\mu e^{-in(\tau-\sigma)}$$

$$\partial_+ X_L^\mu \equiv \frac{1}{2}(\partial_\tau + \partial_\sigma) X_L^\mu = \frac{l_s}{\sqrt{2}} \sum_{n \in \mathbb{Z}} \tilde{\alpha}_n^\mu e^{-in(\tau+\sigma)}$$

The zero-frequency modes x^μ, p^μ correspond to the center of mass of the string and conserved momentum (in target space)

$$X^\mu_{CM} = \frac{1}{2\pi} \int_0^{2\pi} d\sigma\, X^\mu(\tau,\sigma) = x^\mu + l_s^2 p^\mu \tau$$

$$P^\mu = T \int_0^{2\pi} d\sigma\, \dot{X}^\mu = 2\pi T l_s^2 p^\mu = p^\mu \qquad \text{since } T = \frac{1}{2\pi l_s^2}$$

$$J_{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i \sum_{n=1}^\infty \frac{1}{n}\left(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu \right)$$
$$- i \sum_{n=1}^\infty \frac{1}{n}\left(\tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\nu - \tilde{\alpha}_{-n}^\nu \tilde{\alpha}_n^\mu \right)$$

On the other hand, the ws hamiltonian and momentum are

$$H = \frac{T}{2}\int_0^{2\pi} \mathring{X}^2 + X'^2 = L_0 + \tilde{L}_0 = \frac{\ell_s^2}{2}(p_\mu)^2 + \sum_{n=1}^{\infty}\left(\alpha_{-n}\alpha_n + \tilde{\alpha}_{-n}\tilde{\alpha}_n\right)$$

$$P = -T\int_0^{2\pi} \mathring{X}X' = L_0 - \tilde{L}_0 = \sum_{n=1}^{\infty}\left(\alpha_{-n}\alpha_n - \tilde{\alpha}_{-n}\tilde{\alpha}_n\right)$$

and must vanish by the Virasoro conditions $\Rightarrow$ mass shell + matching conditions

* The time evolution of any observable can be expressed as usual as

$$\frac{dF}{d\tau} = -\{H, F\}.$$

where $\{\cdot,\cdot\}$ is the Poisson bracket defined at equal time by

$$\{X^\mu(\sigma,\tau)\, ,\, \mathring{X}^\nu(\sigma',\tau)\} = \frac{1}{T}\eta^{\mu\nu}\delta(\sigma-\sigma')$$

In term of the modes,

$$\{x^\mu, p^\nu\} = \eta^{\mu\nu} \qquad , 0 \text{ otherwise}.$$

$$\{\alpha_m^\mu, \alpha_n^\nu\} = -i\,\delta_{m+n}\,\eta^{\mu\nu}\, m$$

$$\{\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu\} = -i\,\delta_{m+n}\,\eta^{\mu\nu}\, m$$

* In particular, the Virasoro constraints satisfy

$$\{L_m, L_n\} = -i(m-n)\,L_{m+n}$$

$$\{\tilde{L}_m, \tilde{L}_n\} = -i(m-n)\,\tilde{L}_{m+n} \qquad , \{L, \tilde{L}\} = 0$$

which is recognized as the Lie algebra of diffeomorphisms of S^1;

$$[e^{im\sigma}\partial_\sigma\, ,\, e^{in\sigma}\partial_\sigma] = -i(m-n)\,e^{i(m+n)\sigma}\partial_\sigma$$

* The existence of these conserved charges reflects the fact that the 'conformal gauge' $\gamma_{\alpha\beta} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ did not fix all the gauge symmetries Diff $\ltimes$ Weyl of the Polyakov action; indeed, defining $\xi^{\pm} = \tau \pm \sigma$, the 2D metric $ds^2 = -d\tau^2 + d\sigma^2 = -d\xi^+ d\xi^-$ transforms under any diffeo of the form $\begin{cases} \xi^+ \longrightarrow f(\xi^+) \\ \xi^- \longrightarrow g(\xi^-) \end{cases}$

into $\quad ds^2 \longrightarrow -f'(\xi^+)\, g'(\xi^-)\, d\xi^+ d\xi^-$

and the overall factor can be reabsorbed by a Weyl rescaling. Diffeo which preserve the metric up to a rescaling are called <u>conformal transformations</u>

The fact that $\{H, L\} \propto L$ ensures that if the constraints $L = 0$ are imposed at $\tau = 0$, they will continue to hold at $\tau > 0$.

(Upon quantization, we shall see that the algebra is deformed to

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1)\, \delta_{m+n}$$

where c is the 'central charge' (unique deformation consistent with Jacobi)

and the constraint $c = 0$ will determine the total dimension of target space)

* Open strings with Neumann boundary conditions

$$X^{\mu\prime}(\tau,0) = 0 \qquad \text{requires that} \quad \alpha_k^\mu = \tilde{\alpha}_k^\mu$$
$$X^{\mu\prime}(\tau,\pi) = 0$$

The general mode expansion is then (upon redefining p^μ by a factor of 2)

$$X^\mu(\tau,\sigma) = x^\mu + 2\,l_s^2\, p_\mu \tau + i\sqrt{2}\, l_s \sum_{k\neq 0} \frac{\alpha_k^\mu}{k} \cos(k\sigma)\, e^{-ik\tau}$$

such that $\qquad \frac{i}{k} \to \frac{i}{-k}$

$$X^\mu_{CM} = \frac{1}{\pi}\int_0^\pi d\sigma\, X^\mu = x^\mu$$

$$P^\mu = T\int_0^\pi d\sigma\, \dot{X}^\mu = T\pi \cdot 2l_s^2\, p_\mu = p^\mu$$

Setting $\alpha_0^\mu = \sqrt{2}\, l_s\, p^\mu$ $\qquad$ ⚠ factor of 2 difference with closed case!

the Virasoro constraints are $L_m = 0 \quad \forall n \in \mathbb{Z}$

$$L_m = \frac{T}{2}\int_0^{2\pi} d\sigma \Big((\partial_\tau X - \partial_\sigma X)^2 e^{im(\tau-\sigma)} + (\partial_\tau X + \partial_\sigma X)\, e^{im(\tau+\sigma)} \Big)$$

$$= \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{m-n}\,\alpha_n$$

In particular, the Hamiltonian (ws) is

$$H = \frac{T}{2}\int_0^\pi (\dot{X}^2 + X'^2)\, d\sigma = l_s^2 (p_\mu)^2 + \sum_{n=1}^\infty \alpha_{-n}\alpha_n = L_0$$

and $L_0 = 0$ is the mass shell condition.

Angular momentum:

$$J_{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i\sum_{n=1}^\infty \frac{1}{n}\left(\alpha_n^\mu \alpha_{-n}^\nu - \alpha_n^\nu \alpha_{-n}^\mu\right)$$

$\boxed{\text{Exercises:}}$

* Consider classical trajectory of open string in conformal gauge

$$X^0 = B\tau$$

$$X^1 = B\cos\tau\cos\sigma \qquad \alpha_1^1 = -\alpha_{-1}^1 = B/(i\sqrt{2}\ell_s)$$

$$X^2 = B\sin\tau\cos\sigma \qquad \alpha_1^2 = \alpha_{-1}^2 = B/\sqrt{2}\ell_s$$

$$X^i = 0 \qquad i>2$$

- Check that it satisfies the eom in conformal gauge + boundary condition
- Compute energy $E = p^0$ and angular momentum J, show that $|J| = 4\alpha' E^2$ ('Regge trajectory')

$$\dot{X} = \begin{vmatrix} B \\ -B\sin\tau\cos\sigma \\ B\cos\tau\cos\sigma \end{vmatrix} \qquad X' = \begin{vmatrix} 0 \\ -B\cos\tau\sin\sigma \\ -B\sin\tau\sin\sigma \end{vmatrix} = 0 \text{ at } \sigma = 0,\pi$$

$$\dot{X}^2 = -1 + \cos^2\sigma \qquad X'^2 = \sin^2\sigma \qquad \dot{X}X' = 0$$

$$E_0 = B/(2\ell_s^2) \quad ; \quad x^\mu = 0$$

$$J^{12} = -i\cdot\left(\alpha_1^2\alpha_{-1}^2 - \alpha_1^2\alpha_{-1}^1\right) = -i\cdot\frac{B^2}{2\ell_s^2}\cdot 2i = \frac{B^2}{\ell_s^2}$$

$$\longrightarrow |J| = 4\ell_s^2 E^2$$

* Compute mode expansion for open strings with D-D boundary conditions

$$X^I = x^I + w^I\sigma - \sqrt{2}\ell_s \sum_{n\in\mathbb{Z}} \frac{\alpha_n^\mu}{n} e^{-in\tau}\sin n\sigma$$

* Show that endpoint of NN strings propagate at speed of light

$$\underset{\text{at } \sigma=0,\pi}{X' = 0} \Rightarrow (\dot{X})^2 \underset{\text{Virasoro}}{=} (X')^2 = 0$$