Quantizing BPS Black Holes

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Main references

- Ooguri Verlinde Vafa [hep-th/0502211]
- Gunaydin, Neitzke, BP and Waldron [hep-th/0512296]
- Work in progress with same + Rocek and Vandoren, to appear
- Lecture notes: BP [hep-th/0607227]
- Early reference: Ferrara Kallosh Gibbons [hep-th/9702103]

Introduction

- String theory successfully accounts for the leading order Bekenstein-Hawking entropy of BPS black holes in many cases:
  1. 5D black holes in Type II / $T^4 \times S^1$ or $K3 \times S^1$, using D1-D5-P system, possibly rotating;
  2. 4D black holes in Type II / T^6 or $K3 \times T^2$, using D2-D6-NS5 or D1-D5-P-KKM system;
  3. 4D black holes with zero D6-brane charge in Type II / CY_3, using M5-branes in M / $CY_3 \times S_1$ (including first subleading correction)

Strominger Vafa; Johnson Khuri Myers; Maldacena Strominger Witten

- The modern understanding relies on AdS/CFT in the near horizon geometry $AdS_3 \times S^3 \times CY_2$, or $AdS_3 \times S^2 \times CY_3$. The dual gauge theory is a "black string SCFT", states can be counted via the Ramanujan-Hardy-Cardy formula.

AdS_2/CFT_1 and channel duality

- In general however, the near-horizon geometry of a BPS black hole is $AdS_2 \times M$, whose holographic description has remained obscure: some superconformal quantum mechanics at one or two boundaries of AdS_2.
- A possible strategy is to try and get at the spectrum of the SQM by channel duality, as in usual open/closed string duality:

$$\mathrm{Tr} e^{-\pi t H_{open}} = \langle B | e^{-\frac{\pi}{t} H_{closed}} | B \rangle$$

  Here, H_{closed} is the Hamiltonian for string theory in AdS_2 in radial quantization. The real interest is in H_{open}.
- This is hardly doable in practice, except if one truncates to spherically symmetric SUGRA modes, and restrict to the BPS sector. It is far from clear whether this truncation is justifiable.

Topological amplitude and black hole wave function I

- Recently, OVV suggested that the OSV conjecture

$$\Omega(p^I, q_I) \sim \int d\phi^I \, |\Psi_{top}(p^I + i\phi^I)|^2 \, e^{\phi^I q_I}$$

can be interpreted just in this way (with $H_{closed} = H_{open} = 0$):

$$\Omega(p, q) = \langle \Psi_{p,q} | \Psi_{p,q} \rangle$$

where

$$|\Psi_{p,q}\rangle = e^{(q_I \Xi^I + p^I \tilde{\Xi}_I)} |\Psi_{top}\rangle = e^{\frac{1}{2} q_I \phi^I} \Psi_{top}(p^I + i\phi^I)$$

Here $\Psi_{p,q}(\chi) = \langle \Psi_{top} | \chi \rangle$ is the topological amplitude in the real polarization, and $[\Xi^I, \tilde{\Xi}_J] = 2i\delta^I_J$ is the Heisenberg algebra acting on the Hilbert space of the topological string amplitude.

Topological amplitude and black hole wave function II

- If correct, this proposal would answer a long standing question: “What is the physical system whose “preferred” wavefunction is the topological amplitude ? ”.
- I will not review OVV’s heuristic arguments here: one of the goals of this talk will be to provide a rigorous treatment of radial quantization.

Topological amplitude and black hole wave function I

- Suffice it so say that, in terms of “large phase space variables”, OVV gave some evidence that the black hole wave function, in Kähler polarization, is given by

$$\Psi_{p,q}(X^I) = e^{-\frac{i\pi}{2} W_{p,q}(X)} , \quad W_{p,q} = q_I X^I - p^I F_I(X)$$

- With this “natural” normalization, its squared norm

$$\int dX^I d\bar{X}^I \exp\left[-\frac{\pi}{4} \left(K(X,\bar{X}) + 2i W_{p,q}(X) - 2i \bar{W}_{p,q}(\bar{X}) \right) \right]$$

agrees with $\Omega(p,q)$ in the saddle point approximation.

Preliminary comments

- The idea of mini-superspace radial quantization of black holes was in fact much studied by the gr-qc community, yielding as yet little insight on the nature of black hole micro-states.

Cavaglia de Alfaro Filippov; Kuchar; Thiemann Kastrup; Breitenlohner Hellmann

- One novelty here is that one works in a SUSY context, for which the "mini-superspace" truncation to spherically symmetric geometries, and omission of D-term interactions, has some chance of being exact.
- Further interest possibly arises from the relation between black hole attractor equations and SUSY vacua in flux compactifications.

Outline

Outline

Stationary solutions and KK* reduction I

- Stationary solutions in 4D can be parameterized in the form

$$ds_4^2 = -e^{2U}(dt+\omega)^2 + e^{-2U} ds_3^2 \ , \quad A_4^I = \zeta^I dt + A_3^I$$

where $ds_3, U, \omega, A_3^I, \zeta^I$ and the 4D scalars $z^i \in \mathcal{M}_4$ are independent of time. The D=3+1 theory reduces to a field theory in three Euclidean dimensions.

- In contrast to the usual KK ansatz,

$$ds_4^2 = e^{2U}(dy+\omega)^2 + e^{-2U} ds_{2,1}^2 \ , \quad A_4^I = \zeta^I dy + A_3^I$$

where the fields are independent of y, we reduce along a time-like direction.

Stationary solutions and KK* reduction II

- For the usual KK reduction to 2+1D, the one-forms (A_3^I, ω) can be dualized into pseudo-scalars $(\tilde\zeta_I, a)$, where a is the twist (or NUT) potential. The 4D Einstein-Maxwell equations reduce to 3D gravity + scalars living in a Riemannian space

$$\mathcal{M}_3 = \frac{Sl(2)}{U(1)}|_{U,a} \times \mathcal{M}_4 \times \mathbb{R}^{2n_v+2}|_{\zeta^I,\tilde\zeta_I}$$

- The KK* reduction is simply related to the KK reduction by letting $(\zeta^I, \tilde\zeta_I) \to i(\zeta^I, \tilde\zeta_I)$. As a result, the scalar fields live in a pseudo–Riemannian space $\mathcal{M}_3^*$, with non-positive definite signature.

Breitenlohner Gibbons Maison; Hull Julia

Stationary solutions and KK* reduction III

- $\mathcal{M}_3^*$ always has $2n_V + 4$ isometries corresponding to the shifts of $\zeta^{\cdot}\tilde{\zeta}_I, a, U$, satisfying the graded Heisenberg algebra

$$
\begin{aligned}
[p^I, q_J] &= 2\delta^I_J\, k \\
\left[m, p^I\right] = p^I \ , \quad [m, q_I] &= q_I \ , \quad [m, k] = 2k
\end{aligned}
$$

- The notation anticipates the identification of the corresponding conserved charges with the electric and magnetic charges q_I and p_I, NUT charge k and ADM mass m.

Spherically symmetric BH and geodesics I

- Now, restrict to spherically symmetric solutions, with spatial slices

$$ds_3^2 = N^2(\rho) d\rho^2 + r^2(\rho) d\Omega_2^2$$

- The sigma-model action becomes, up to a total derivative (g_{ij} is the metric on $\mathcal{M}_3^*$):

$$S = \int d\rho \left[\frac{N}{2} + \frac{1}{2N} \left(\dot{r}^2 - r^2 g_{ij} \dot{\phi}^i \dot{\phi}^j \right) \right]$$

- This is the Lagrangian for the geodesic motion of a fiducial particle with unit mass on the (hyperbolic) cone $\mathbb{R}^+ \times \mathcal{M}_3^*$. Invariance under reparameterizations of ρ is achieved thanks to the ein-bein N.

Spherically symmetric BH and geodesics II

- The equation of motion of N imposes the Hamiltonian constraint,or Wheeler-DeWitt equation

$$H_{WDW} = (p_r)^2 - \frac{1}{r^2} g^{ij} p_i p_j - 1 \equiv 0$$

- The gauge choice $N = r^2$ allows to separate the problem into radial motion along r, and geodesic motion on $\mathcal{M}_3^*$:

$$g^{ij} p_i p_j = C^2 \,, \quad (p_r)^2 - \frac{C^2}{r^2} - 1 \equiv 0 \quad \Rightarrow \quad r = \frac{C}{\sinh C\rho} \,,$$

  Thus, the problem reduces to affinely parameterized geodesic motion on the three-dimensional moduli space $\mathcal{M}_3^*$.

Spherically symmetric BH and geodesics III

- It turns out that $C = 2T_H S_{BH}$ is the extremality parameter: extremal (in particular BPS) black holes correspond to light-like geodesics on $\mathcal{M}_3^*$. Since $r = 1/\rho$, the 3D spatial slices are flat.
- Other gauges are also possible: e.g. $N = e^U$, where ρ becomes the radial geodesic distance to the horizon.
- For the purpose of defining observables such as the horizon area, $A_H = 4\pi e^{-2U} r^2|_{U\to-\infty}$ and ADM mass $M_{ADM} = r(e^{2U} - 1)|_{U\to 0}$, it may convenient to leave the gauge unfixed.

Isometries and conserved charges

- The isometries of $\mathcal{M}_3$ imply conserved Noether charges, whose Poisson bracket reflect the Lie algebra of the isometries:

$$
\begin{aligned}
[p^I, q_J] &= 2\delta^I_J\, k \\
\left[m, p^I\right] = p^I, \quad [m, q_I] &= q_I\,, \quad [m,k] = 2k
\end{aligned}
$$

- If $k \neq 0$, the off-diagonal term in the 4D metric

$$
ds_4^2 = -e^{2U}(dt + k\cos\theta d\phi)^2 + e^{-2U}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]
$$

  implies the existence of closed time-like curves around ϕ direction, near $\theta = 0$.
- Bona fide 4D black holes arise in the "classical limit" $k \to 0$. Keeping $k \neq 0$ will allow us to greatly extend the symmetry.

Attractor flow in $N = 2$ supergravity

- Consider $N = 2$ SUGRA coupled to n_V abelian vector multiplets [*hypers go along for the ride*]: the vector multiplet scalars z^i take values in a special Kähler manifold $\mathcal{M}_4$. For type IIA on $X = CY_3$, z^i parameterize the complexified Kähler structure of X.
- After reduction to 3 dimensions, the vector multiplet scalars take value in a quaternionic-Kähler space $\mathcal{M}_3$, known as the $c - map$ of the special Kähler space $\mathcal{M}_4$.
- Under T-duality along the 4th direction, this becomes the hypermultiplet space for type IIB compactified on X at tree-level.

c-map and c^*-map

- The explicit metric reads

$$ds^2 = 2(dU)^2 + g_{i\bar{j}}(z,\bar{z})dz^i dz^{\bar{j}} + \frac{1}{2}e^{-4U}\left(da + \zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right)^2$$
$$-e^{-2U}\Big[(\mathrm{Im}\mathcal{N})_{IJ} d\zeta^I d\zeta^J + (\mathrm{Im}\mathcal{N}^{-1})^{IJ}\left(d\tilde{\zeta}_I + (\mathrm{Re}\mathcal{N})_{IK} d\zeta^K\right)\left(d\tilde{\zeta}_J + (\right.$$

Ferrara Sabharwal; de Wit Van Proyen Vanderseypen

- The manifold $\mathcal{M}_3^*$ obtained by analytic continuation $(\zeta^I, \tilde{\zeta}_I) \to i(\zeta^I, \tilde{\zeta}_I)$ is sometimes called “para-quaternionic-Kahler manifold”; it has indefinite signature $(2n_V + 2, 2n_V + 2)$

Cortes Mayer Mohaupt Saueressig

Conserved charges and black hole potential

- The Heisenberg isometries

$$\zeta^I \to \zeta^I + \epsilon^I\,, \quad \tilde\zeta_I \to \tilde\zeta_I + \tilde\epsilon_I\,, \quad a \to a - \epsilon^I \tilde\zeta_I + \tilde\epsilon_I \zeta^I$$

  yield conserved charges p^I, q_I, k.
- Setting $k = 0$ for simplicity, one arrives at the Hamiltonian,

$$H = \frac{1}{2}\left[\dot U^2 + \frac{1}{4}\dot z^i g_{i\bar j}\dot z^{\bar j} - e^{2U} V_{BH}\right] \equiv C^2$$

  where V_{BH} is the "black hole potential",

$$V_{BH} = -\frac{1}{2}(q_I - \mathcal{N}_{IJ}p^J)[1/\mathrm{Im}(\mathcal{N})]^{IK}(q_K - \bar{\mathcal{N}}_{KL}p^L) - \frac{1}{2}p^I[\mathrm{Im}(\mathcal{N})]_{IJ}p^J$$

Conserved charges and black hole potential I

- In terms of the central charge $Z = e^{K/2}(q_I X^I - p^I F_I)$, this is rewritten as

$$V_{BH} = |Z|^2 + |D_i Z|^2 = |Z|^2 + \partial_i |Z|\, g^{i\bar{j}} \partial_{\bar{j}} |Z|$$

- Supersymmetric solutions are obtained by cancelling each term separately, leading to the celebrated attractor flow equations:

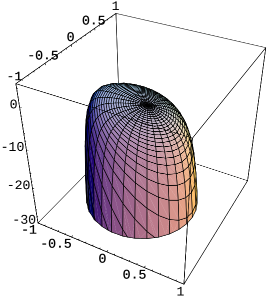
$$\frac{dU}{d\rho} = -e^U |Z| \, , \quad \frac{dz^i}{d\rho} = -2 e^U g_{i\bar{j}} \partial_{\bar{j}} |Z| \quad \Rightarrow \quad \frac{dz^i}{dU} = -g^{i\bar{j}} \partial_{\bar{j}} \log |Z|^2$$

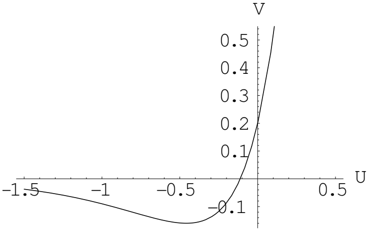
The 4D moduli are attracted towards the horizon to the value $z^*_{p,q}$ minimizing $m_{BPS} = |Z| m_P$ at fixed values of the charges. If $|Z_*| \neq 0$, this is an $AdS_2 \times S_2$ throat, with $S_{BH} = \pi |Z_*|^2$.

Gradient flow vs. potential flow

- The actual potential $-e^{2U} V_{BH}$ has in fact a local maximum at $z^*_{p,q}$. BPS trajectories are fine-tuned to reach the top of the potential with 0 velocity.

$$\partial_i \partial_{\bar{j}} V_{BH}|_{z_{p,q}} = 2 g_{i\bar{j}} V_{BH}$$





Attractor flow and SUSY geodesic motion I

- The above Bogomolny-type argument does not fix the phase in the second attractor equation, and does not guarantee that the solution is supersymmetric.
- The correct procedure is to reduce the full $D = 4$ SUGRA including fermions, and look at BPS solutions of the resulting SUSY mechanics. A short cut is to restrict the $D = 3 + 1$ sigma-model on a quaternionic-Kähler space to $D = 0 + 1$.
- In order to express the fermionic variation, recall that a quaternionic-Kähler space has restricted holonomy $Sp(2) \times Sp(2n_V + 2)$; it admits a covariantly constant quaternionic vielbein $V^{\alpha A}$ ($\alpha = 1, 2$, $A = 1, ..2n_V + 2$), which provides the metric together with three almost complex structures ($a = 1..3$):

$$ds^2 = \epsilon_{\alpha\beta}\rho_{AB} V^{\alpha A} \otimes V^{\beta B} \ , \quad \omega^a = \epsilon_{\alpha\gamma}\sigma^{a|\gamma}_{\beta}\rho_{AB} V^{\alpha A} \wedge V^{\beta B}$$

Attractor flow and SUSY geodesic motion II

- The fermionic variation reads

$$\delta\psi^A = V_i^{\alpha A}\dot{\phi}^i \epsilon_\alpha + O(\psi^2)$$

- BPS geodesics are obtained when the quaternionic viel-bein obtains a null eigenvector:

$$V^{\alpha A}\epsilon_\alpha = \begin{pmatrix} u & v \\ e^i & E^i \\ -\bar{E}^{\bar{i}} & \bar{e}^{\bar{i}} \\ -\bar{v} & \bar{u} \end{pmatrix} \cdot \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} = 0$$

Attractor flow and SUSY geodesic motion III

- Expressing the components of V in terms of p^I, q_I, k,

$$
\begin{aligned}
u &= -\frac{i}{2} e^{K/2+U} X^I \left[q_I - 2k\tilde{\zeta}_I - \mathcal{N}_{IJ}(p^J + 2k\zeta^J) \right] \\
v &= -dU + \frac{i}{2} e^{2U} k \\
e^A &= e^A_i \, dz^i \\
E^A &= -\frac{i}{2} e^U e^{Ai} g^{i\bar{j}} \bar{f}^I_{\bar{j}} \left[q_I - 2k\tilde{\zeta}_I - \mathcal{N}_{IJ}(p^J + 2k\zeta^J) \right]
\end{aligned}
$$

we recover the attractor flow equations, generalized to non-zero NUT charge k:

$$
-\frac{dU}{d\rho} + \frac{i}{2} e^{2U} k = -\frac{i}{2} e^{i\theta} e^U Z \,, \qquad \frac{dz^i}{d\rho} = -i e^{i\theta} \frac{|Z|}{Z} e^U g^{i\bar{j}} \partial_{\bar{j}} |Z|
$$

where the phase $\epsilon_2/\epsilon_1 = e^{i\theta}$ is chosen to maintain the reality of U.

Outline

Improved SUSY mechanics - HKC and twistors I

- This SUSY mechanics is rather unusual, insofar as the SUSY comes from a triplet of non-integrable complex structures.
- It is possible to remedy this problem by adding 4 real scalar degrees of freedom, extending the QK manifold to a real cone over an S^3 bundle over the QK space:

$$\mathbb{R}^+ \times S^3 \to HKC \to QK$$

  This is equivalent to the original model after gauging the $SU(2)$ and dilation symmetries. By cancelling the $Sp(2)$ holonomy on QK with the $SU(2)$ holonomy on S^3, one obtains the Hyperkähler cone (HKC), with three integrable complex structures.
- This construction is very natural in the framework of conformal supergravity.

De Wit Rocek Vandoren

The twistor space

- The relevant information is captured by an intermediate space, the twistor space Z, a Kähler quotient of HKC by $U(1) \subset SU(2)$:

$$S^2 \to Z \to QK$$

  which admits one canonical complex structure; in contrast to HKC, the action of $SU(2)$ is no longer isometric.
- Explicitly, the Kähler-Einstein metric on Z reads

$$ds^2_Z = ds^2_{QK} + \frac{1}{(1+z\bar{z})^2} |dz - A_+ + iA_3 z - A_- z^2|^2$$

  where $z, \bar{z}$ are the stereographic coordinates on S^2, and $A_\pm = (A_1 \pm iA_2)/2, A_3$ is the $SU(2)$ connection on the base. Its complex structure is

$$J = \frac{z+\bar{z}}{1+z\bar{z}} J^1 + \frac{i(z-\bar{z})}{1+z\bar{z}} J^2 + \frac{1-z\bar{z}}{1+z\bar{z}} J^3 + i\left(z \otimes \partial_z - \bar{z} \otimes \bar{\partial}_z\right)$$

Twistor space and HKC for the c-map

- In general, the metric on the HKC, and consequently on Z, is controlled by the Hyperkähler potentiel χ.
- In the presence of triholomorphic isometries, it may be obtained by Legendre transform

$$\langle \chi(v^L, \bar{v}^L, w_L + \bar{w}_L) + x^L(w_L + \bar{w}_L) \rangle = \mathcal{L}(v^L, \bar{v}^L, x^L)$$

from a tensor potential $\mathcal{L}$ satisfying some 2nd order PDE's.

- In favorable cases, the solution is given by a contour integral

$$\mathcal{L}(v^L, \bar{v}^L, x^L) = \oint \frac{d\zeta}{2\pi i \zeta} G(\eta^L(\zeta), \zeta) \, , \quad \eta^L = \frac{v^L}{\zeta} + x^L - \bar{v}^L \zeta$$

- The potentiel G controlling the c-map is a function of $n_V + 2$ variables, proportional to the prepotential $F(X^I)$ on the Special Kähler base:

$$G(\eta^L) = \frac{F(\eta^I)}{\eta^\sharp} = \frac{C_{ABC}\eta^A \eta^B \eta^C}{\eta^0 \eta^\sharp} + \dots$$

Rocek Vafa Vandoren

The twistor transform

- For later purposes, it will be useful to express the complex coordinates $\xi^I, \tilde{\xi}_I, \alpha$ on Z in terms of the coordinates $U, z^i, \bar{z}^I, \zeta^I, \tilde{\zeta}_I, a$ on the base, and $z, \bar{z}$ on the fiber:

$$\begin{aligned} \xi^I &= \zeta^I + i\, e^{U+\mathcal{K}(X)/2} \left(z\bar{X}^I + z^{-1} X^I \right) \\ \tilde{\xi}_I &= \tilde{\zeta}_I - i\, e^{U+\mathcal{K}(X)/2} \left(z\, \bar{F}_I + z^{-1}\, F_I \right) \\ \alpha &= a + \zeta^I \tilde{\xi}_I - \tilde{\zeta}_I \xi^I \end{aligned}$$

- A key feature is that $(\xi^I, \tilde{\xi}_I, \alpha)$ are holomorphic functions of the fiber coordinate z: the fiber is a rational curve. Starting from a holomorphic function Φ on Z, we can produce a conformally harmonic function Ψ on QK:

$$\Psi(U, z^i, \bar{z}^I, \zeta^I, \tilde{\zeta}_I, a) = e^{2U} \oint \frac{dz}{2\pi i z}\, \Phi\left[\xi^I(z), \tilde{\xi}^I(z), \alpha(z) \right]$$

Attractor flow and twistor variables I

- The requirement of SUSY on Z is that the momentum be holomorphic in the canonical complex structure on Z, or in one of the the complex structures on HKC.
- BPS geodesics, or BPS black holes, correspond to holomorphic curves $\xi^I(\rho), \tilde{\xi}_I(\rho), \alpha(\rho)$ at constant $\bar{\xi}^I, \bar{\tilde{\xi}}_I, \bar{\alpha}$ (and with vanishing $SU(2)$ momenta) $\Rightarrow$ *completely integrable*.
- The twistor variable z (now in the Poincaré disk, $z\bar{z} < 0$) encodes the projectivized Killing spinor $z = \epsilon_2/\epsilon_1$:

$$dz - A_+ + i\,A_3 z - A_- z^2 = 0 \quad \Rightarrow d\alpha + Q + ke^{2U} = 0$$

  where α is the phase of z. In fact, the 4 real variables of the HKC can be interpreted as the unprojectivized Killing spinor (ϵ_1, ϵ_2).
- A degenerate possibility is that the momentum be tri-holomorphic on HKC: "super BPS trajectories"...

Outline

From geodesic motion to wave functions

- We have seen that generic spherically symmetric black holes are in one-to-one correspondence with parameterized geodesics on the (Wick rotated) three-dimensional moduli space $\mathcal{M}_3^*$.
- There is a standard prescription to quantize geodesic motion: replace the classical trajectories by wave functions in $L_2(\mathcal{M}_3^*)$, satisfying the Klein-Gordon equation

$$\Delta\Psi(U, z^i, \bar{z}^{\bar{i}}, \zeta^I, \tilde{\zeta}_I, a) = C^2\Psi$$

where Δ is the Laplace-Beltrami operator on $\mathcal{M}_3^*$.
- Equivalently, we may consider the space of $\mathbb{R}^+ \times SU(2)$ invariant functions on HKC, or $SU(2)$-invariant functions on Z.
- Before discussing any of the subtleties associated with SUSY, let us make some general comments about the physical meaning of the wave function.

Physical interpretation of the wave function

- As in quantum cosmology, the wave function is independent of the "time" variable ρ, and some other variable should be chosen as a "clock". A natural choice is U, which goes from $-\infty$ at the horizon to 0 at spatial infinity.
- Observables are defined at a fixed value of U. One might –*wrongly*– expect the wave function to become more and more peaked around the attractor values of the moduli as $U \to -\infty \ldots$
- The natural inner product is obtained by using the Klein-Gordon inner product (or Wronskian) at fixed values of U. Unfortunately, it is famously known NOT to be positive definite.
- A possible way out is "third quantization", where the wave function Ψ becomes itself an operator... this may describe the possible black hole fragmentation near the horizon...

- Now we restrict to the BPS Hilbert space. In the framework of geodesic motion on the QK base, SUSY requires

$$\exists\epsilon / \begin{pmatrix} u & v \\ e^i & E^i \\ -\bar{E}^{\bar{i}} & \bar{e}^{\bar{i}} \\ -\bar{v} & \bar{u} \end{pmatrix} \cdot \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} = 0 \quad \Leftrightarrow \quad \begin{pmatrix} u\bar{u} + v\bar{v} & = & 0 \\ uE^i - e^i v & = & 0 \\ u\bar{e}^{\bar{j}} + \bar{E}^{\bar{j}} v & = & 0 \end{pmatrix}$$

- Quantum mechanically, these conditions become 2nd order differential operators which have to annihilate the wave function Ψ. In particular, the *conformal Laplacian*

$$\left(\Delta_{QK} - \frac{1}{2(4n_V + 2)} R \right) \Psi = 0$$

The BPS Hilbert space (first pass) II

- In the framework of geodesic motion on the twistor space, BPS geodesics have purely holomorphic momenta:

$$p_{\bar{L}} = 0 \quad \Rightarrow \quad i \frac{\partial}{\partial \bar{\xi}^{\bar{L}}} \Psi = 0$$

  Thus, the BPS Hilbert space corresponds to holomorphic functions on the twistor space, modulo the action of $SU(2)$.

- The equivalence between the two approaches is the consequence of the Penrose transform (a quaternionic generalization of the usual Penrose-Ward transform on S^4)

$$\Psi(U, z^i, \bar{z}^I, \zeta^I, \tilde{\zeta}_I, a) = e^{2U} \oint \frac{dz}{2\pi i z} \, \Phi \left[\xi^I(z), \tilde{\xi}^I(z), \alpha(z) \right]$$

The BPS Hilbert space (second pass)

- More correctly, one needs to take into account the fermionic degrees of freedom. In the usual SQM, the fermions ψ^μ become Dirac matrices. The wave function satisfies $i\gamma^\mu\nabla_\mu + m = 0$.
- Equivalently, one can treat ψ^μ as a differential dx^μ, and $\gamma^\mu\partial_\mu$ as an exterior derivative. The Hilbert space at $m = 0$ is the de Rham complex, while the BPS Hilbert space is the de Rham cohomology.
- For SQM on a Kähler manifold, ψ^μ splits into ψ^i and $\psi^{\bar{i}}$. The Hilbert space becomes the Dolbeault complex (with its Lefschetz $SU(2)$ action)

Quaternionic cohomology)

- For SQM on a quaternionic-Kähler manifold, ψ^μ splits as $\psi^{A\alpha} \in E \otimes H$, where $E \sim \mathbb{R}^{2n}$, $H \sim \mathbb{R}^2$. The relevant complex is:

$$0 \;\to\; triv. \stackrel{R}{\Longrightarrow} \Lambda^2(E^*) \;\to\; \Lambda^3(E^*) \times H^* \;\to \dots$$
$$\to \Lambda^m(E^*) \times \Sigma^{m-2}(H^*) \;\to\; \dots \;\to\; 0$$

with arrows

$$Q = \psi^A \epsilon^\alpha \nabla_{A\alpha} \; , \quad R = \epsilon^{\alpha\beta} \psi^A \psi^B \left[\nabla_{A\alpha} \nabla_{B\beta} + \frac{4}{4(n+2)} R_{A\alpha;B\beta} \right]$$

Here $\psi^A = \epsilon_\alpha \psi^{A\alpha}$, and ϵ_α keeps track of the H index, as if it was the HKC fiber...

Baston; Baston Eastwood

The true BPS Hilbert space

- The twistor transform identifies the cohomology of this complex with the sheaf cohomology $H^1(Z, \mathcal{O}(-2))$ on Z. We conjecture that this is the correct Hilbert space for BPS black holes.

Gunaydin Neitzke BP Rocek Vandoren Waldron, in progress

- This is analogous to the usual Penrose-Ward transform

$$\mathrm{Harm}(\mathbb{R}^4) = H^1(CP_3, \mathcal{O}(-2))$$

  Versions for other $O(-k)$ yield other higher-spin fields.

- On general grounds, because the SQM can be lifted to $1+5$ dimensions, there should exist a $SO(5)$ Lefschetz-type action...

The BPS Black Hole Wave-Function I

- Ignore fermionic subtleties, and go back to the simple-minded twistor transform

$$\Psi(U, z^i, \bar{z}^I, \zeta^I, \tilde{\zeta}_I, a) = e^{2U} \oint \frac{dz}{2\pi i z} \, \Phi \left[\xi^I(z), \tilde{\xi}^I(z), \alpha(z) \right]$$

- Consider a black hole with $k = 0$: p^I and q_I can be diagonalized simultaneously, and completely determine (up to normalization) the wave function as a coherent state on Z:

$$\begin{aligned} \Phi &= \exp \left[i (p^I \tilde{\xi}_I - q_I \xi^I) \right] \\ &= \exp \left[i (p^I \tilde{\zeta}_I - q_I \zeta^I) + i e^{U + K(X)/2} (z \bar{W}_{p,q}(\bar{X}) + z^{-1} W_{p,q}(X)) \right] \end{aligned}$$

The BPS Black Hole Wave-Function II

- The integral over z is of Bessel type, leading to

$$\Psi = J_0\left(2e^U|Z_{p,q}|\right)\; e^{i(p^I\tilde{\zeta}_I - q_I\zeta^I)}$$

  This is peaked around the classical attractor points, with slowly damped, increasingly faster oscillations away from them.

- We could have reached this result 36 mins ago, by naively quantizing the attractor flow:

$$\left\{\begin{array}{rcl} p_U & = & -e^U|Z| \\ p_{\bar{z}^{\bar{i}}} & = & -2e^U\partial_{\bar{i}}|Z| \end{array}\right\} \quad \Rightarrow \Psi \sim \exp\left[2ie^U|Z|\right]$$

Black-hology vs. cosmology

- Contrary perhaps to expectations, the wave flattens out towards the horizon ! This is because of the large fine-tuning needed to produce a BPS solution.
- Continuing to quantum cosmology, the wave function becomes exponentially peaked at late times, which is gratifying.
- So far, we haven't checked that $\langle \Psi | \Psi \rangle \sim \exp(S_{BH})$. The normalization can always be adjusted so this is true.
- Our formalism allows to define quantum mechanical observables, compute rms fluctuations, etc.

Where is the topological string ?

- Before integrating along the fiber, we found that $\Psi_{p,q} \sim \exp[i e^{U+K/2}(z\bar{W} + z^{-1} W)]$, in “rough” agreement with OVV’s answer $\Psi_{p,q} \sim \exp(W)$. The precise relation to the “large phase space variables” is unclear at present.
- In order to compare to the more familiar real-polarized topological amplitude $\Psi_{\rm top} \sim e^F$, one should find real Darboux coordinates on Z.
- We find it unlikely that Ψ_{top} can be identified as a black hole wave function: it naturally depends on $n_V + 1$ variables, while Ψ_{BH} depends on $2n_V + 3$ variables.
- In fact, consideration of the holomorphic anomaly eqs in symmetric theories hints at one-parameter generalization of the topological string, best viewed as a tri-holomorphic function on HKC . . .

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Outline

Summary

- Stationary black holes in 4D are in 1-1 correspondence with geodesics on the 3D moduli space. In extended SUGRA, BPS black holes correspond to geodesics with momenta in a non-generic orbit, e.g. holomorphic geodesics for $N = 2$.
- While the phase space of generic geodesics is $T^*(QK)$, of dimension $8n + 8$, the phase space of BPS geodesics is the twistor space $Z(QK)$, of dimension $4n + 6$, with its canonical symplectic form.
- The BPS Hilbert space is the Kähler quantization of Z, roughly the space of holomorphic functions on Z: $2n + 3$ variables, considerably smaller than the dimension $4n + 4$ of $\mathcal{H} = L_2(QK)$. $\mathcal{H}_{BPS}$ is embedded inside $\mathcal{H}$ via the twistor transform.
- For given electric and magnetic charges ($k = 0$), there is a unique state in $\mathcal{H}_{BPS}$, up to normalization. Its wave function is peaked at the attractor values, but flattens out near the horizon. No evidence yet that it is related to $\Psi_{\rm top}$.

Outlook

- Higher derivative corrections remain to be incorporated: higher derivative scalar interactions on QK space.
- Multi-centered configurations can be described by certain harmonic maps from $\mathbb{R}^3$ to QK: does that correspond to “second quantization”, i.e. including vertices ?
- For $N \geq 4$, this suggests that the 3D U-duality group controls the BH spectrum: can one obtain the exact degeneracies as Fourier coefs of some “BPS automorphic forms” ? Improve on DVV.
- The equivalence between BH attractor flow and geodesic flow on QK is a reflection of mirror symmetry. Can this be used to compute instanton corrections on hypermultiplet moduli space ?