Non-perturbative amplitudes and instanton effects in string/M-theory

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Plan of the lectures

I. Generalities on instanton effects in field and string theories

II. Eisenstein series and instanton effects in maximally supersymmetric M/string theory

III. Exact results and instanton effects in theories with reduced supersymmetry

Goal: understand the rules of *semi-classical calculus* in string/M-theory

Means: construct examples of *exact amplitudes* using symmetries and dualities, and analyze them in a *weak coupling expansion*.

First lecture

Generalities on Instanton effects

1. Instantons in quantum mechanics
2. Instantons in quantum field theories
3. Worldsheet instantons in string theory
4. Space-time instantons in M/string theory

Collective coordinates, fermionic zero-modes, BPS saturated amplitudes, D-brane instantons, supersymmetric cycles, R^4 couplings in type IIB...

The double well potential

Coleman's Erice lecture

Consider $H = p^2/2 + V(x)$ with two classical vacua at $x = \pm a$.
In the WKB approximation:

$$E_\pm = \frac{1}{2}\hbar\omega \pm \hbar K e^{-S_0/\hbar}\ , \qquad S_0 = \int_{-a}^{a} \sqrt{2V}$$

This can be computed semi-classically by summing over *Euclidean solutions* interpolating between the two vacua at $t = \pm\infty$:

$$d\bar{x}/dt = \sqrt{2V} \qquad (E = 0)$$

$$\langle a|e^{-Ht/\hbar}|a\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega t/2} \sum_{\text{even}_n} \frac{(Ke^{-S_0/\hbar}t)^n}{n\ !} + \ldots$$

$$= \left(\frac{\omega}{\pi\hbar}\right)^{1/2} \frac{e^{-\omega t/2}}{2} \left[\exp\left(Ke^{-S_0/\hbar}t\right) - \exp\left(Ke^{-S_0/\hbar}t\right)\right]$$

K is the fluctuation determinant in the instanton background *with zero mode suppressed*:

$$K = \frac{S_0}{2\pi\hbar} \left|\frac{\det(-\partial_t^2 + \omega_2)}{\det(-\partial_t^2 + V''(\bar{x}))}\right|^{1/2}$$

The term n is the contribution from an n-instanton configuration in the *dilute gas approximation*.

Supersymmetric quantum mechanics

Consider a two-component wave function $i\hbar d\psi/dt = H\psi$,

$$\psi = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} \quad , \qquad H = \frac{1}{2}\left(p^2 + W^2(x) + \hbar\frac{dW}{dx}\sigma_3\right)$$

This is a supersymmetric system with 2 supercharges

$$\begin{array}{rcl} 2Q_1 & = & p\sigma_1 + W(x)\sigma_2 \\ 2Q_2 & = & p\sigma_2 - W(x)\sigma_1 \end{array} \quad , \qquad \{Q_i, Q_j\} = \delta_{ij} H$$

At *tree-level*, around a zero of W, the zero-point energy is $\frac{1}{2}\hbar\omega \pm \frac{1}{2}\hbar\omega$ hence SUSY is unbroken.

True SUSY ground states satisfy $Q_1\psi = 0$ ie

$$\frac{d\psi}{dx} = \frac{1}{\hbar}W(x)\sigma_3\psi(x) \ \Rightarrow \psi(x) = \left(\exp\int_0^x dy\frac{W(y)}{\hbar}\sigma_3\right)\psi_0$$

• If W has an odd number of zeros, the wave function is normalizable if $\sigma_3\psi_0 = \psi_0$: *SUSY is unbroken*.

• If W has an even number of zeros, the wave function is never normalizable for all choices of ψ_0: *SUSY is broken*.

In the instanton language: there are SUSY instantons interpolating between two vacua, annihilated by Q_1. The action of Q_2 generates a *fermionic zero-mode*, saturated when computing $\langle -a|Q_2|a\rangle \sim e^{-S}$. Then

$$\langle a|H|a\rangle = \langle a|Q_2| - a\rangle\langle -a|Q_2|a\rangle = e^{-2S} > 0$$

Instantons breaking all SUSY do not contribute.

Instantons in gauge theories

Consider bosonic YM theory,

$$S = \int d^4x \left(\frac{1}{g^2} F^2 + i\theta F\tilde{F} \right)$$

Vacua are labelled by the winding number $|n\rangle$, or equivalently by $|\theta\rangle = \sum_n e^{in\theta}|n\rangle$.

To all orders in perturbation theory, physics is independent of θ.

Extrema of the action are

$$F = \pm\tilde{F} \ , \quad S_{Cl} = \frac{8\pi^2|k|}{g^2} + ik\theta$$

Instantons induce tunnelling between vacua n and $n+k$:

$$E(\theta) = -2K \cos\theta e^{-8\pi/g^2}$$

where K is the 1-instanton contribution,

$$K = g^{-8} \int_0^\infty \frac{d\rho}{\rho^5} (\rho M)^{8\pi^2\beta_1}$$

The scale coordinate ρ comes from the breaking of conformal invariance. Other coordinates come from $SU(2) \subset G$.

Fermionic zero-modes

In the presence of chiral fermions, $\theta \sim \theta + \alpha$ through chiral rotation, hence $E(\theta) = 0$. Instantons have *fermionic collective coordinates* that need to be saturated:

$$\langle \bar{\psi}\gamma_5\psi\bar{\psi}\gamma_5\psi \rangle = e^{-8\pi/g^2 + i\theta} g^{-8} \int \frac{d\rho}{\rho^5} f(\rho M)$$

Chiral $U(1)$ is broken by non-perturbative effects (no Golstone boson)

Fermionic and bosonic zero-modes can be understood from the embedding into $N = 2$ SYM:

$$F = \pm \tilde{F} \quad \leftrightarrow \quad \delta_\epsilon \lambda = 0$$

Instantons preserve 1/2 SUSY. The other (broken) half gives fermionic zero-modes.

The no-force condition between instantons implies that a charge-n instanton has $4n$ *exact* bosonic zero-modes, and fermionic superpartners.

SUSY restricts the possible instantonic contributions.

Instantons effects in $N = 2$ SYM

Seiberg Witten 94

Consider $SU(2)$ $N = 2$ SYM. The low energy effective action can be written as a *chiral density* in $N = 2$ superspace, controlled by an *holomorphic* prepotential:

$$S = \text{Im}\left[\int_{N=2} d^4x d^4\theta \mathcal{F}(A)\right]$$

$$= \text{Im}\left[\int_{N=1} d^4x d^2\theta d^2\bar{\theta}\ A\bar{\mathcal{F}}'(A) + \int_{N=1} d^4x d^2\theta\ \mathcal{F}''(A) W^2\right]$$

By dimensional transmutation, $\mathcal{F}$ is a function of a/Λ with $\Lambda^{\beta_1} = e^{-1/g^2+i\theta}$, and is θ independent *in perturbation theory*:

$$\mathcal{F}_{pert} = \mathcal{F}_{1-loop} = \frac{i}{2\pi} a^2 \log(a/\Lambda)$$

Using holomorphy and prescribed singularities, $\mathcal{F}$ can be computed *exactly* and expanded at small coupling:

$$\mathcal{F}_{exact} = \frac{i}{2\pi} a^2 \log(a/\Lambda) + \sum_{k=1}^{\infty} c_k \left(\frac{a}{\Lambda}\right)^k a^2$$

Configurations with k-instantons have $4k$ fermionic zero-modes, 4 exact and $4k - 4$ are *lifted*. They contribute to a *four-fermion* term in the action. Numerical coefs c_k check (at small k).

BPS saturated amplitudes

SUSY restricts the contributions to particular terms in the effective action:

• At the perturbative level, non-renormalisation theorems from superspace techniques. Often, one-loop results given by *helicity supertraces*

$$F_{1-loop} = \int_{1/\Lambda}^{\infty} dt \ \mathrm{Tr}\left(h^n e^{-tH}\right)$$

and satisfy simple PDE equations:

$$\partial_a \partial_{\bar{a}} F_{1-loop} = 0$$

• At the instanton level, fermion zero modes must be saturated. For N_ψ-fermions terms in a theory with N_Q supercharges,

N_ψ/N_Q	instantons	$N=2$	$N=4$
$< 1/2$	no inst.	$m\psi\psi, V(\phi)$	$m\psi\psi, R\psi^4$
$1/2$	$1/2BPS$	$R\psi^4, g_{ij}(\phi)$	ψ^8, F^4
$3/4$	$1/4BPS$	$n.a.$	ψ^{12}, F^6

One may hope to compute such terms exactly using constraints from SUSY and dualities.

Instantons in string theory

String theory is (so far) defined as a perturbative series in $g_s = e^{\phi}$. Each term is in principle defined non-perturbatively in $\alpha' = l_s^2$ by a 2D superconformal theory.

• non-perturbative effects in α' are *worldsheet instantons*: the Euclidean worldsheet wraps a non-contractible 2-cycle: order $e^{-V/\alpha'}$. They can be exactly computed in some cases:

- on tori: T-duality
- on Calabi-Yau manifolds: mirror symmetry
- solvable models: WZW, Gepner models, ...

• non-perturbative effects in g_s are *space-time instantons*. String theory does not specify their contributions yet. Classical BPS configurations are known:

- Euclidean D-branes: order e^{-1/g_s}
- Neveu-Schwarz 5-brane, Kaluza-Klein monopoles: order e^{-1/g_s^2}

Dualities interchange these effects:

$$g_s \to 1/g_s \ , \alpha' \to \alpha' g_s \quad (Het - II)$$

Worldsheet instantons

Consider a string amplitude at one-loop on T^n:

$$A_n = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \Phi(\tau, \bar\tau) \sum \exp\left(-\frac{|m^i + n^i \tau|^2}{\tau_2} + 2\pi i m^i B_{ij} n^j \right)$$

For BPS-saturated amplitudes, the modular form Φ simplifies: (quasi)holomorphic, or constant:

$$A_1 = \frac{\pi^2}{3} \left(R + \frac{1}{R} \right)$$

$$A_2 = -\ln T_2 |\eta(T)|^2 - \ln U_2 |\eta(U)|^2$$

where η is the Dedekind function,

$$\eta(T) = q^{-1/24} \prod_{n>1} (1 - q^n) \ , \qquad q = e^{2\pi i T}$$

Rk: A_n *is T-duality invariant*

At large volume, $A_n \sim V_n$, but for $n > 1$ there are exponentially suppressed corrections:

$$A_2 = T_2 - \log T_2 - \ln U_2 |\eta(U)|^2 \sum_N \mu(N) \left(e^{2\pi i N T} + e^{-2\pi i N T} \right)$$

coming from worldsheet instantons wrapping N times T^2.

The summation measure can be read off:

$$\mu(N) = \sum_{n|N} \frac{1}{n}$$

From worldsheet to spacetime instantons

Heterotic/T^6- type IIA/$_3 \times T^2$ duality:

The R^2 coupling is 1/2-BPS saturated ($N_Q = 16$, $N_\psi = 8$). It is given exactly at one-loop on the type II side:

$$f_{R^2} = -\log T_2 |\eta(T)|^2$$

It is T-duality invariant and quasi-harmonic:

$$T_2^2 \partial_T \partial_{\bar{T}} = 1$$

Under duality, the volume of type II torus maps to heterotic coupling:

$$T_{IIA} = B + i\frac{V_2}{l_s^2} \equiv S_H = a + i\frac{V_6}{g_H^2 l_s^6}$$

T-duality maps to non-pertubative S-duality on het side.

From the het point of view, R^2 couplings come from non perturbative effects

$$f_{R^2} = \frac{V_6}{g_H^2 l_s^6} - \log S_2 - \sum_n \mu(n) \left(e^{2\pi i N S} + e^{-2\pi i N S} \right)$$

The np effects come from NS5-brane instantons and anti-instantons wrapping T^6, with action and measure

$$S_{cl} = \frac{|N| V_6}{g_H^2 l_s^6} \pm iNa \ , \quad \mu(N) = \sum_{n|N} \frac{1}{n}$$

R^4 *couplings in type IIB*

$D = 10$ type IIB is conjectured to be invariant under $Sl(2, Z)$ self-duality

$$\tau \to \frac{a\tau + b}{c\tau + d} , \quad ad - bc = 1 , \quad \tau = a + i/g_s$$

coming from the geometric modular group of M/T^2.

R^4 couplings are 1/2-BPS saturated. Their tensor structure is fixed by SUSY:

$$S = \int d^{10}x\sqrt{-g}f\,(t_8 t_8 + \epsilon_8\epsilon_8)\,R^4$$

A tree-level and one-loop computation gives

$$f_{R^4} = \frac{2\zeta(3)}{g_s^2 l_s^2} + \frac{2\pi^2}{3l_s^2}$$

SUSY requires that f be an eigenmode of the Laplacian,

$$\tau_2^2\partial_\tau\partial_{\bar\tau}f = \frac{3}{4}f$$

Higher perturbative corrections have to be independent of the RR scalar τ_1, hence cannot occur.

There exist a duality invariant function with the same leading behavior and compatible with SUSY:

$$f_s = \sum_{(m,n)\neq(0,0)} \left(\frac{\tau_2}{|m + n\tau|^2}\right)^s , \quad s = 3/2$$

D-instanton effects in type IIB

At small coupling, this reproduces the tree-level and one-loop terms plus a sum of non-perturbative effects:

$$
\begin{aligned}
f &= 2\zeta(3)e^{-3\phi/2} + \frac{2\pi^2}{3}e^{\phi/2} + 0 \\
&+ 4\pi e^{-\phi/2}\hat{\sum_m}\hat{\sum_n}\left|\frac{m}{n}\right| K_1\left(2\pi e^{-\phi}|mn|\right) e^{2\pi i m n a} \\
&\sim 4\pi\hat{\sum_N}\mu(N)e^{-2\pi N(e^{-\phi}\pm ia)}\left(1+O(e^{\phi})\right)
\end{aligned}
$$

The np. effects come from D-instantons, with action and measure

$$
S_{cl} = \frac{|n|}{g_s} + ina \ , \quad \mu(N) = \sqrt{N}\sum_{n|N}\frac{1}{n^2}
$$

The instanton measure comes from the fluctuation determinant in the background of the instanton. Fluctuations are governed by the IKKT matrix model for $U(N)$ matrices:

$$
\int dX^i d\psi^\alpha \exp\left(\mathrm{Tr}\sum [X^i,X^j]^2 + \psi\Gamma_i[X^i,\psi]\right) = \sqrt{N}\sum_{n|N}\frac{1}{n^2}
$$

Moore,Nekrasov,Shatashvili;Vanhove,Kostov

Poisson resummation trick

Consider the weak coupling expansion $\tau_2 \to \infty$ of the $Sl(2,Z)$ Eisenstein series

$$f_s = \sum_{(m,n)\neq 0} \left[\frac{\tau_2}{|m+n\tau|^2}\right]^s$$

Represent $1/M^s$ by a Gamma function integral:

$$f_s = \frac{\pi^s}{\Gamma(s)} \sum_{(m,n)\neq 0} \int \frac{dt}{t^{1+s}} \exp\left(-\pi \frac{|m+n\tau|^2}{t\tau_2}\right)$$

The leading term comes when $n=0$: $f_s = 2\zeta(2s)\tau_2^s + f_s'$. Then, all m give nearly the same, so Poisson resum on m:

$$f_s' = \frac{\pi^s}{\Gamma(s)} \sum_{\tilde{m}, n\neq 0} \int \frac{dt}{t^{1/2+s}} \exp\left(-\pi\tau_2\left(\tilde{m}^2 t + \frac{n^2}{t}\right) + 2\pi i \tilde{m} n \tau_1\right)$$

Extract the $\tilde{m}=0$ contribution. Then the t integral is a Bessel function $K_{s-1/2}$.

$$f_s = 2\zeta(2s)\tau_2^s + 2\sqrt{\pi}\tau_2^{1-s}\frac{\Gamma(s-1/2)}{\Gamma(s)}\zeta(2s-1)$$

$$+\frac{\pi^s\sqrt{\tau_2}}{\Gamma(s)} \sum \left|\frac{\tilde{m}}{n}\right|^{s-1/2} K_{s-1/2}(2\pi\tau_2|\tilde{m}n|) e^{2\pi i \tilde{m} n \tau_1}$$

Saddle point approximation

At large τ_2 the t-integral has a saddle point behaviour:

$$\int \frac{dt}{t^{1/2+s}} \exp\left[-\pi\tau_2\left(\tilde{m}^2 t + \frac{n^2}{t}\right)\right] \sim \frac{1}{2\sqrt{\tau_2}} \frac{\tilde{m}^{s-1}}{n^s} e^{-2\pi\tau_2|\tilde{m}n|}$$

The action of the saddle point depends only on $N = \tilde{m}n$:

$$f_s'' \sim \frac{\pi^s\sqrt{\tau_2}}{2\Gamma(s)} \sum_N N^{s-1} \left(\sum_{n|N} \frac{1}{n^{2s-1}}\right) e^{-2\pi\tau_2|\tilde{m}n| + 2\pi i\tau_1 mn}$$

For $s = 3/2$, we find the summation measure for D-instantons.

There are perturbative corrections around the saddle point:

$$K_\nu(z) = \sqrt{\pi} 2z e^{-z} \left(1 + \sum_{k=1}^{\infty} \frac{1}{(2z)^k\ k!} \frac{\Gamma(\nu+k+1/2)}{\Gamma(\nu-k+1/2)}\right)$$

For ν half-integer, corrections truncate !

ex: show that $f_1 = -\pi \log(\tau_2|\eta(\tau)|^4)$ and recover instanton expansion of Dedekind function.

Eisenstein series of $Sl(2,Z)$

$$f_s = \sum_{(m,n)\neq 0} \left[\frac{\tau_2}{|m+n\tau|^2} \right]^s$$

f_s is a function of $\tau \in U(1)\backslash Sl(2,R)$, invariant under the modular group $Sl(2,Z)$.

It is an eigenmode of the Laplacian on $U(1)\backslash Sl(2,R)$,

$$\tau_2^2 \partial_\tau \partial_{\bar\tau} f_s = s(s-1) f_s$$

with asymptotic behavior

$$f_s(\tau,\bar\tau) = 2\zeta(2s)\tau_2^s + a(s)\tau_2^{1-s} + \mathcal{O}(e^{-\tau_2})$$

In type IIB, $\tau = a + i/g_s$, so the perturbative terms should differ by even numbers of τ_2:

$$s = 1 - s + 2n \ \Rightarrow s = n + 1/2$$

Higher derivative terms $R^4 H^{4g-4}$ are given by $f_{g+1/2}$

There also exists *cusp forms* with no perturbative terms, at discrete values on the axis $s = 1/2 + iR$: ruled out by SUSY.

Generalized Eisenstein series

The Eisenstein series are sums over tensions of (p,q) strings:

$$f_s = \sum_{(m,n)\neq 0} \left[\alpha'_{(m,n)}\right]^{-2s} = \zeta(s) \sum_{p\wedge q=1} \left[\alpha'_{(p,q)}\right]^{-2s}$$

where $\alpha'_{(p,q)} = \sqrt{|p+q\tau|^2/\tau_2}$.

More generally, we can construct duality invariant functions by summing over orbits of BPS states,

$$\mathcal{E}^{G(Z)}_{s;R} = \sum_{m\in\Gamma} \left[\mathcal{M}^2(m)\right]^{-s} \delta(m\wedge m)$$

The delta-functions restricts to the orbit of 1/2-BPS states, and is necessary to have an eigenmode of the Laplacian

$$\Delta\mathcal{E}^{G(Z)}_{s;R} = as(s-b)\mathcal{E}^{G(Z)}_{s;R}$$

For $G = Sl(2,R)$ and $R = \mathbf{2}$, this is f_s (no constraint in that case).

Second lecture

Eisenstein series and instanton effects in maximally supersymmetric M/string theory

1. Algebraics of T and U-duality in M/T^d

2. Generalized Eisenstein series

3. T-duality invariant thresholds in ST/T^d

4. Exact R^4 couplings in M/T^d

M-theory and U-duality

- Any Lorentz invariant theory compactified on a torus T^d has a discrete global symmetry, the *mapping class group* of the torus :

$$S_{IJ} \;:\; \gamma_I \leftrightarrow \gamma_J \;,\quad R_I \leftrightarrow R_J$$

$$\Gamma_{IJ} \;:\; \gamma_I \to \gamma_I + \gamma_J \;,\quad A_J^{(I)} \to A_J^{(I)} + 1$$

where $ds^2 = R_I^2 (dx^I + A_J^{(I)} dx^J)^2 + dx^J g_{JK} dx^K$

$$S_{IJ} \bowtie \Gamma_{IJ} = Sl(d, Z)$$

- Perturbative type IIA theory on a torus T^d has in addition a discrete global symmetry valid to all orders, namely (double) T-duality :

$$T_{ij} \;:\; (R_i, R_j, g_s) \to \left(\frac{l_s^2}{R_j}, \frac{l_s^2}{R_i}, \frac{g_s l_s^2}{R_i R_j} \right)$$

$$\Delta_{ij} \;:\; B_{ij} \to B_{ij} + 1$$

$$Sl(d, Z) \bowtie T_{ij} \bowtie \Delta_{ij} = SO(d, d, Z)$$

- M-theory on a torus T^d should exhibit both Lorentz invariant on T^d and T-duality invariance on T^{d-1} :

$$Sl(d, Z) \bowtie SO(d-1, d-1, Z) = E_{d(d)}(Z)$$

This defines *a* discrete subgroup of Cremmer–Julia hidden symmetry group of 11D SUGRA on T^d, which has to remain valid at the quantum level.

T-duality Weyl group

- The action of the Weyl group can be represented on a weight space :

$$\mathcal{T} = g_s^{x^0} R_1^{x^1} R_2^{x^2} \dots R_d^{x^d} \to \lambda = x^0 e_0 + x^1 e_1 + \dots + x^d e_d$$

- The Weyl generators T and S_i act as *orthogonal reflections* $\rho_{\alpha_0}, \rho_{\alpha_i}, i = 1 \dots d-1$

$$\lambda \to \rho_\alpha(\lambda) = \lambda - 2\frac{\alpha \cdot \lambda}{\alpha \cdot \alpha}\alpha$$

w.r.t. the signature $(- + + \dots)$ metric

$$\mathrm{d}s^2 = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^i)^2 + \mathrm{d}x^0(dx^1 + \dots + \mathrm{d}x^d)$$

- The Planck length $V/g_s^2 l_s^8$ is invariant, so the action restricts to the spacelike hyperplane $\delta \cdot x = x^0 = 0$ normal to

$$\delta = e_1 + \dots + e_d - 2e_0$$

- The Coxeter group is characterized by its Cartan matrix or Dynkin diagram

$$\begin{array}{llllll} T & & & & & \\ & \diagdown & & & & \\ & & S_2 & - & S_3 & - \cdots - \quad S_{d-1} \\ & \diagup & & & & \\ S_1 & & & & & \end{array}$$

This is the Dynkin diagram of $SO(d, d)$.

T-duality Weyl multiplets

- The representation of the finite group $\mathcal{W}(SO(d,d))$ can be obtained from the fundamental weights dual to the simple roots :

$$\begin{tikzcd} \frac{1}{g} \arrow[dr, dash] & & & & \\ & \frac{R_1 R_2}{g_s^2} \arrow[r, dash] & \frac{R_1 R_2 R_3}{g_s^2} \arrow[r, dash] & \cdots \arrow[r, dash] & \frac{1}{R_d} \\ \frac{R_1}{g} \arrow[ur, dash] & & & & \end{tikzcd}$$

- Depending on the power of the invariant string length we include, this gives the T-duality Weyl multiplets of particle masses, or string tensions.

- The Kaluza-Klein modes on the torus together with the string winding states transform as a *vector* of $SO(d,d)$, with highest weight $\mathcal{M} = 1/R_{d-1}$.

- The D0-branes transform as a *spinor* of $SO(d,d)$ with highest weight $\mathcal{M} = 1/gl_s$, just as the RR gauge fields :

$$\mathcal{R} \leftrightarrow \mathrm{d}x^i \cdot \mathcal{R} + \mathrm{d}x^i \wedge \mathcal{R} \; , \;\; \mathcal{R} = \sum_p \mathcal{R}_p$$

which generates a Clifford algebra. The same is true for the D2-brane tension, with highest weight $\mathcal{T}_2 = 1/gl_s^3$.

- The D-strings obtained by wrapping a D2-brane on a circle transform as a spinor of opposite chirality and highest weight $\mathcal{T}_1 = R_1/gl_s^3$.

U-duality Weyl and Borel generators

- In M-theory variables, the T-duality $(R_i, R_j) \to (1/R_j, 1/R_i)$ reads

$$T_{ij} : R_i \to \frac{l_p^3}{R_j R_s}\ , \quad R_j \to \frac{l_p^3}{R_s R_i}\ , \quad R_s \to \frac{l_p^3}{R_i R_j}\ , \quad l_p^3 \to \frac{l_p^6}{R_i R_j R_s}$$

- By a $Sl(d, Z)$ M-theory Lorentz boost, it can be conjugated to

$$T_{IJK} : R_I \to \frac{l_p^3}{R_J R_K}\ , \quad l_p^3 \to \frac{l_p^6}{R_I R_J R_K}$$

  which involves a set of *three* directions. A minimal set of Weyl transformations can be chosen as $T = T_{123}$ and $S_I \ : R_I \leftrightarrow R_{I+1}$.

- The Borel generators include as usual the Dehn twists $\gamma_I \to \gamma_I + \gamma_J$ in $Sl(d, Z)$, as well as the $B_{ij} = \mathcal{C}_{sij}$ shifts in $SO(d-1, d-1, Z)$. These can be conjugated into

$$C_{IJK} : \mathcal{C}_{IJK} \to \mathcal{C}_{IJK} + 1$$

  and we also need, for $d \geq 6$ ($d \geq 8$),

$$E_{IJKLMN} \ : \mathcal{E}_{IJKLMN} \to \mathcal{E}_{IJKLMN} + 1$$

$$K_{I;JKLMNPQR} \ : \mathcal{K}_{I;JKLMNPQR} \to \mathcal{K}_{I;JKLMNPQR} + 1$$

U-duality Weyl group

- The action of the Weyl group can again be represented on a weight space :

  $$\mathcal{T} = l_p^{3x^0} R_1^{x^1} R_2^{x^2} \dots R_d^{x^d} \to \lambda = x^0 e_0 + x^1 e_1 + \dots + x^d e_d$$

- The Weyl generators T and S_i act as *orthogonal reflections*

  $$\lambda \to \rho_\alpha(\lambda) = \lambda - 2\frac{\alpha \cdot \lambda}{\alpha \cdot \alpha}\alpha$$

  for the signature $(- + + \dots)$ metric

  $$\mathrm{d}s^2 = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^i)^2$$

- The Planck length V/l_p^9 being invariant, the action restricts to the spacelike hyperplane $\delta \cdot x = 0$ normal to

  $$\delta = e_1 + \dots + e_d - 3e_0 \ ,$$

  except when $d = 9$ *where* δ *is null*.

- The Coxeter group is characterized by its Cartan matrix or Dynkin diagram

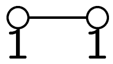
  $$\begin{array}{ccccccccc} & & & & T & & & & \\ & & & & | & & & & \\ S_1 & - & S_2 & - & S_3 & - & S_4 & - \cdots - & S_{d-1} \end{array}$$

  This is the Dynkin diagram of E_d, or the *extended* Dynkin diagram of $\hat{E}_8 = E_9$ for $d = 9$.

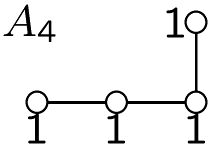
Group disintegrations

$E_2 = A_1$

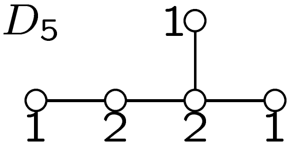
$E_3 = A_2 \oplus A_1$ $\circ$



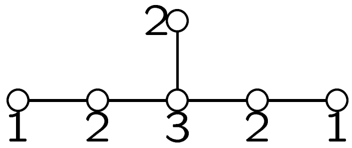
$E_4 = A_4$



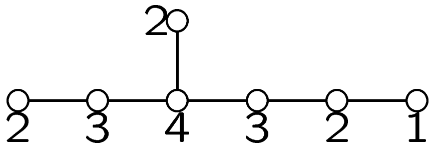
$E_5 = D_5$



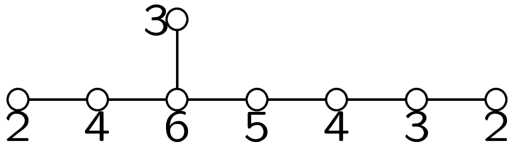
E_6



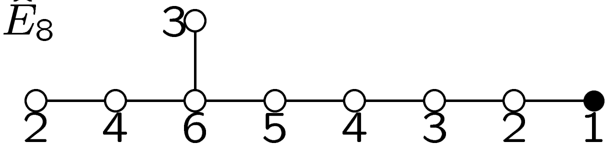
E_7



E_8



$E_9 = \hat{E}_8$



Towards the boundary of moduli space

- The boundaries of the moduli space correspond to a scaling limit of the dilatonic scalars R_I, l_M parametrising the Cartan "torus". Modulo the action of the Weyl group, we can choose them to lie in a *Weyl chamber* such that $\alpha_+ \cdot \lambda > 0$:

$$R_1 < R_2 < \cdots < R_d \ , \qquad R_1 R_2 R_3 > l_M^3$$

- The 11D SUGRA description is valid provided all radii are larger than the Planck length :

$$11D \ SUGRA \ : \qquad l_M < R_1 < R_2 < \dots$$

- If one of the radii is smaller, then we may have a type IIA description with weak coupling $g_s^2 = (R_1/l_M)^3$, provided all radii are larger than the string length $l_s^2 = l_M^3/R_1$:

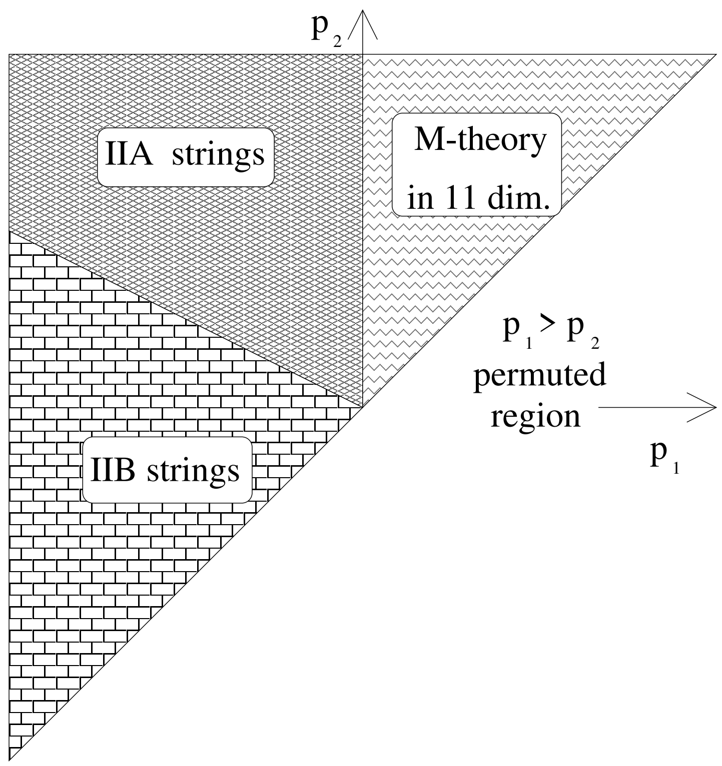
$$R_1 < l_M \ , \qquad R_1 R_2^2 > l_M^3$$

- If one of the radii is smaller than the above string length, we may instead try a type IIB description with weak coupling $g_s = R_1/R_2$, same string length $l_s^2 = l_M^3/R_1$ and 10-th radius $R_B = l_M^3/(R_1 R_2)$. The IIB radii are smaller than the string length provided

$$R_1 R_i^2 > l_M^3 \ , \qquad R_1 R_2^2 < l_M^3$$

  The first relation is automatically satisfied if $R_1 R_2 R_3 > l_M^3$, and the second implies $R_1 < l_M$.

- The 11D SUGRA, type IIA and type IIB therefore cover the complete asymptotic region of the moduli space.



- For $d > 9, 10$, the metric is now Lorentzian, and the above Weyl chamber generates the *future light cone* only. There is a truly strongly coupled space-like region, excluded by the holographic principle.

Banks,Fischler,Motl

U-duality Weyl multiplets

- The representation of the finite group $\mathcal{W}(E_{d(d)})$ ($d < 9$) can be obtained from the fundamental weights dual to the simple roots :

$$\begin{array}{ccccccccccc} & & & & \frac{1}{l_p^3} & & & & & & \\ & & & & | & & & & & & \\ \frac{R_1}{l_p^3} & - & \frac{R_1 R_2}{l_p^6} & - & \frac{R_1 R_2 R_3}{l_p^9} & - & \frac{R_1 R_2 R_3 R_4}{l_p^9} & - & \cdots & - & \frac{1}{R_d} \end{array}$$

- The *particle multiplet* with highest weight $\mathcal{M} = 1/R_{d-1}$ transforms as the representation on the rightmost node.

- The *string multiplet* with highest weight $\mathcal{T}_1 = R_1/l_p^3$ transforms as the representation on the leftmost node.

- The *membrane multiplet* with highest weight $\mathcal{T}_2 = 1/l_p^3$ appears on the T-duality node.

- The other nodes do not correspond to other simple multiplets, but to Weyl submultiplets.

- For $d = 9$, the Weyl orbit becomes infinite. The level and degree are given by

$$k = \delta \cdot \lambda = x^1 + \cdots + x^9 + 3x^0$$

$$d = \kappa \cdot \lambda = x^9 - k/2$$

and correspond to the scaling dimension and the number of 9-th directions.

Borel generators and spectral flow

Weyl generators are analogous to $\tau \to -1/\tau$ in $Sl(2, Z)$. In addition, there exists further symmetries analogous to $\tau \to \tau + 1$:

$$ds^2 = R^2(dx^0 + A_i dx^i)^2 + g_{ij} dx^i dx^j$$

The *Dehn twist* $A_i \to A_i + 1$ leaves the torus invariant, but acts on cohomology as $dx^0 \to dx^0 + dx^i$.

Changing $A_i \to A_i + 1$ continuously generates a *spectral flow* on momentum eigenstates, taking $p_i \to p_i + 1$.

More generally, $dx^i \to dx^i + dx^j$ for $i < j$ generate an Heisenberg-type infinite group, and together with $dx^i \leftrightarrow dx^j$ generates the *mapping class group* $Sl(d, Z)$.

The Borel generators can be identified as the *positive roots* $e_i - e_j$.

They bring Weyl multiplets together into representations of the full group $Sl(d, Z)$.

n-dimensional reps form a lattice $Z^n \subset R^n$ of integral charge vectors, i.e. for momentum states:

$$\mathcal{M}^2 = p_i g^{ij} p_j \ , \qquad p \in Z^n$$

Vielbein and homogeneous spaces

Constant metrics on T^n can be parameterized by a vielbein

$$g = e^t e \ , \qquad e \in Gl(n,R)$$

The metric is invariant under $SO(n,R)$ rotations acting on the left $e \to \Omega e$, so the parameter space is

$$\frac{Gl(n,R)}{SO(n,R)} = R^+ \times \frac{Sl(n,R)}{SO(n,R)}$$

Any real matrice can be decomposed uniquely as rotation $\times$ upper triangular matrix. More precisely,

$$e = \Omega \cdot \begin{pmatrix} R_1 & & & \\ & R_2 & & \\ & & \ddots & \\ & & & R_n \end{pmatrix} \cdot \begin{pmatrix} 1 & A_1^1 & A_1^2 & A_1^n \\ & 1 & A_2^1 & \\ & & \ddots & \ddots \\ & & & 1 \end{pmatrix}$$

The gauge symmetry can be fixed by choosing $\Omega = 1$.

The mapping class group $Sl(n,Z)$ acts by multiplication on the right by integer valued matrices, $e \to eg$.

Dehn twists are preserve the upper triangular gauge and leave radii invariant, but action of Weyl generators needs to be compensated by $SO(n,R)$ rotations on the left.

The mass formula is simply $\mathcal{M}^2 = w^t(e^t e)w$ for 1-cycles, or $\mathcal{M}^2 = p^t(e^{-t}e^{-1})p$ for momenta.

Moduli space and Iwasawa gauge

For compactification of string or M-theory with high SUSY, the moduli space is still homogeneous, but includes gauge backgrounds:

$$e \in K \backslash G(R)$$

where $G(R)$ is a *non-compact* real group and K its maximal compact subgroup:

$$(g, B) \in \frac{SO(n,n,R)}{SO(n) \times SO(n)} \quad \text{(string/}T^n\text{)}$$

$$(g, C, E, \dots) \in \frac{E_{n+1}}{K_{n+1}} \quad (M/T^{n+1})$$

Any element g has the Iwasawa decomposition

$$e = k \cdot a \cdot n \in K \cdot A \cdot N$$

with k compact, a abelian and N nilpotent:

- k can be gauged to 1
- a is a diag matrix of "dilatonic scalars"
- n is the set of Borel moduli = gauge fields

Mass/tension formula is $\mathcal{M}^2 = m^t \cdot R(e)^t R(e) \cdot m$

Mass formula for windings/momenta

For square tori with $B=0$, the mass of the Weyl multiplet in the vector rep are

$$1/R_1\ ,\ 1/R_2\ ,\ \dots 1/R_d\ ,\ R_1\ ,\ R_2\ ,\ R_d$$

In the Iwasawa gauge, the $SO(n,n)$ vielbein reads

$$\mathcal{V}=\left(\begin{array}{cc|cc} R_1 & & & \\ & R_2 & & \\ \hline & & 1/R_1 & \\ & & & 1/R_2 \end{array}\right)\cdot\left(\begin{array}{cc|cc} 1 & A^1_2 & B_{11} & B_{12} \\ & 1 & B_{21} & B_{22} \\ \hline & & 1 & -A^1_2 \\ & & & 1 \end{array}\right)$$

The mass formula in the vector representation is

$$\mathcal{M}^2=(m_i+B_{ij}m^j)g^{ik}(m_k+B_{kl}m^l)+m^i g_{ij} m^j$$

where m_i are momenta and m^i windings.

Half-BPS states require $m_i m^i=0$.

D-brane T-duality multiplets

- Similarly, the D-brane states with mass

$$1/g_s l_s \ , \ \ R_i R_j / g_s l_s^3 \ , \ \ R_i R_j R_k R_l / g_s l_s^5 \dots$$

turn into a lattice of integer charges

$$m = \{ m \ , \ \ m^{ij} \ , \ \ m^{ijkl} \dots \}$$

transforming as a spinor of $SO(d,d,Z)$:

$$T_{ij} \ \ : m \to dx^i \wedge dx^j \wedge m + (dx^i \wedge dx^j) \cdot m$$

$$\Delta_{ij} \ \ : m \to m + dx^i \wedge dx^j \wedge m$$

- The 1/2-BPS mass is given by

$$\begin{aligned}
\mathcal{M}^2 &= \tilde{m}^2 + \frac{1}{2}(\tilde{m}^{ij})^2 + \frac{1}{4!}(\tilde{m}^{ijkl})^2 + \dots \\
\tilde{m} &= m + \frac{1}{2} m^{ij} B_{ij} + \frac{1}{8} m^{ijkl} B_{ij} B_{kl} + \dots \\
\tilde{m}^{ij} &= m^{ij} + \frac{1}{2} m^{klij} B_{kl} + \dots \\
\tilde{m}^{ijkl} &= m^{ijkl} + \dots
\end{aligned}$$

where the $\tilde{m}$ are the (partially) dressed charges $\mathcal{V}{\cdot}m$, subject to the conditions

$$k^{ijkl} \equiv m^{[ij} m^{kl]} + m \ m^{ijkl} = 0 \ , \dots$$

- When $n^{ijkl} \neq 0$, the BPS state is at best 1/4-BPS, and its mass as an extra contribution :

$$\mathcal{M}^2 = \tilde{m}^2 + \frac{1}{2}\left(\tilde{m}^{ij}\right)^2 + \frac{1}{4!}\left(\tilde{m}^{ijkl}\right)^2 + \sqrt{\frac{1}{4!}\left(\tilde{k}^{ijkl}\right)^2}$$

D-brane and Born-Infeld mass

- The D-brane charges can be understood from the Born-Infeld action, by considering a D-brane configuration wrapped on T^d, in static gauge :

$$X^i = N^i_\alpha \sigma^\alpha \ , \ \ i = 1 \dots d \ , \ \ \alpha = 1 \dots p$$

- The wrapping numbers N^i_α and world-volume gauge field $F_{\alpha\beta}$ can be changed by a world-volume diffeomorphism, which leaves invariant

$$m^{ijkl} = \epsilon^{\alpha\beta\gamma\delta} N^i_\alpha N^j_\beta N^k_\gamma N^l_\delta$$

$$m^{ij} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} N^i_\alpha N^j_\beta F_{\gamma\delta} \quad , \quad m = \frac{1}{8} \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}$$

which automatically satisfies $n^{ijkl} = 0$.

- The evaluation of the Born-Infeld action

$$S_{\mathrm{BI}} = \frac{1}{l_s^{p+1}} \int \mathrm{d}^{p+1} \xi e^{-\phi} \sqrt{\hat{g} + \hat{B} + \alpha' F}$$

on this configuration precisely gives the 1/2-BPS mass.

BP, Kiritsis

- 1/4-BPS configurations can be obtained by considering 1/2-BPS excitations on the world-volume.

Dijkgraaf, Verlinde[2] 96

Hidden symmetry groups of M/T^d

D	d	$G_d = E_{d(d)}(R)$	H_d
10	1	1	1
9	2	$Sl(2,R)$	$U(1)$
8	3	$Sl(3,R) \times Sl(2,R)$	$SO(3) \times U(1)$
7	4	$Sl(5,R)$	$SO(5)$
6	5	$SO(5,5,R)$	$SO(5) \times SO(5)$
5	6	$E_{6(6)}$	$USp(8)$
4	7	$E_{7(7)}$	$SU(8)$
3	8	$E_{8(8)}$	$SO(16)$

Central charges and (dualized) gauge fields fit in reps of K and G_d: transform *linearly* under G_d :

d	$E_{d(d)}$	Scalars	1-f.	2-f.
1	1	1	$\mathbf{1}$	$\mathbf{1}$
2	$Sl(2,R)$	3-1	$\mathbf{3}$	$\mathbf{2}$
3	$Sl(3,R) \times Sl(2,R)$	11-4	$(\mathbf{3},\mathbf{2})$	$(\mathbf{3},\mathbf{1})$
4	$Sl(5,R)$	24-10	$\mathbf{10}$	$\mathbf{5}$
5	$SO(5,5,R)$	45-20	$\mathbf{16}$	$\mathbf{10}$
6	$E_{6(6)}$	78-36	$\mathbf{27}$	$\bar{\mathbf{27}}$
7	$E_{7(7)}$	133-63	$\mathbf{56}$	·
8	$E_{8(8)}$	248-120	·	·

The particle multiplet

In $d = 8$: a $\mathbf{248}$ of $E_{8(8)}$.

mass $\mathcal{M}$	$Sl(8)$ irrep	charge
$\frac{1}{R_I}$	$\mathbf{8}$	m_1
$\frac{R_I R_J}{l_p^3}$	$\mathbf{28}$	m^2
$\frac{R_I R_J R_K R_L R_M}{l_p^6}$	$\mathbf{56}$	m^5
$\frac{R_I^2 R_J R_K R_L R_M R_N R_P}{l_p^9}$, $8\ \frac{V_R}{l_p^9}$	$\mathbf{1{+}63}$	$m^{1;7}$
$\frac{R_I^2 R_J^2 R_K^2 R_L R_M R_N R_P R_Q}{l_p^{12}}$	$\mathbf{56}$	$m^{3;8}$
$\frac{R_I^2 R_J^2 R_K^2 R_L^2 R_M^2 R_N^2 R_P R_Q}{l_p^{15}}$	$\mathbf{28}$	$m^{6;8}$
$\frac{R_I^3 R_J^2 R_K^2 R_L^2 R_M^2 R_N^2 R_P^2 R_Q^2}{l_p^{18}}$	$\mathbf{8}$	$m^{1;8;8}$

In lower d, drop the charges with two many antisymmetric indices :

D	d	U-duality group	irrep	$Sl(d)$ content
10	1	1	$\mathbf{1}$	$\mathbf{1}$
9	2	$Sl(2,Z)$	$\mathbf{3}$	$\mathbf{2} + \mathbf{1}$
8	3	$Sl(3,Z) \times Sl(2,Z)$	$(\mathbf{3},\mathbf{2})$	$\mathbf{3} + \mathbf{3}$
7	4	$Sl(5,Z)$	$\mathbf{10}$	$\mathbf{4} + \mathbf{6}$
6	5	$SO(5,5,Z)$	$\mathbf{16}$	$\mathbf{5} + \mathbf{10} + \mathbf{1}$
5	6	$E_{6(6)}(Z)$	$\mathbf{27}$	$\mathbf{6} + \mathbf{15} + \mathbf{6}$
4	7	$E_{7(7)}(Z)$	$\mathbf{56}$	$\mathbf{7} + \mathbf{21} + \mathbf{21} + \mathbf{7}$
3	8	$E_{8(8)}(Z)$	$\mathbf{248}$	$2(\mathbf{8} + \mathbf{28} + \mathbf{56}) + \mathbf{63} + \mathbf{1}$

The string multiplet

In $d = 8$: a $\mathbf{3875}$ of $E_{8(8)}$. In $d = 7$, a $\mathbf{133}$ of $E_{7(7)}$:

tension $\mathcal{T}$ mass	$Sl(7)$ irrep	charge
$\frac{R_I}{l_p^3}$	$\mathbf{7}$	n^1
$\frac{R_I R_J R_K R_L}{l_p^6}$	$\mathbf{35}$	n^4
$\frac{R_I^2 R_J R_K R_L R_M R_N}{l_p^9}$, $7\frac{V_R}{l_p^9}$	$\mathbf{1{+}48}$	$n^{1;6}$
$\frac{R_I^2 R_J^2 R_K^2 R_L R_M R_N R_P}{l_p^{12}}$	$\mathbf{35}$	$n^{3;7}$
$\frac{R_I^2 R_J^2 R_K^2 R_L^2 R_M^2 R_N^2 R_P}{l_p^{15}}$	$\mathbf{7}$	$n^{6;7}$

and on lower dimensional tori :

D	d	U-duality group	irrep	$Sl(d)$ content
10	1	1	$\mathbf{1}$	$\mathbf{1}$
9	2	$Sl(2, Z)$	$\mathbf{2}$	$\mathbf{2}$
8	3	$Sl(3, Z) \times Sl(2, Z)$	$(\mathbf{3}, \mathbf{1})$	$\mathbf{3}$
7	4	$Sl(5, Z)$	$\mathbf{5}$	$\mathbf{4} + \mathbf{1}$
6	5	$SO(5, 5, Z)$	$\mathbf{10}$	$\mathbf{5} + \mathbf{5}$
5	6	$E_{6(6)}(Z)$	$\overline{\mathbf{27}}$	$\mathbf{6} + \mathbf{15} + \mathbf{6}$
4	7	$E_{7(7)}(Z)$	$\mathbf{133}$	$\mathbf{7} + \mathbf{35} + \mathbf{49} + \mathbf{35} + \mathbf{7}$
3	8	$E_{8(8)}(Z)$	$\mathbf{3875}$	$\mathbf{8} + \mathbf{70} + \ldots$

A particle multiplet on T^d decomposes on T^{d-1} into a singlet $1/R_d$, a particle multiplet, a string multiplet, and extra states in $d \geq 6$.

U-duality invariant mass formula

The U-duality invariant mass formula on tori with arbitrary gauge backgrounds can again be obtained by considering the *non-commuting* $\mathcal{C}_3, \mathcal{E}_6, \mathcal{K}_{1;8}$ flows ; in $d = 7$:

$$\mathcal{M}^2 = (\tilde{m}_1)^2 + \frac{1}{2!\ l_p^6}\left(\tilde{m}^2\right)^2 + \frac{1}{5!\ l_p^{12}}\left(\tilde{m}^5\right)^2 + \frac{1}{7!\ l_p^{18}}\left(\tilde{m}^{1;7}\right)^2$$

with dressed charges given by

$$\begin{aligned}
\tilde{m}_1 &= m_1 + \frac{1}{2}\mathcal{C}_3 m^2 + (\mathcal{C}_3\mathcal{C}_3 + \mathcal{E}_6)\, m^5 + (\mathcal{C}_3\mathcal{C}_3\mathcal{C}_3 + \mathcal{C}_3\mathcal{E}_6)\, m^{1;7} \\
\tilde{m}^2 &= m^2 + \mathcal{C}_3 m^5 + (\mathcal{C}_3\mathcal{C}_3 + \mathcal{E}_6)\, m^{1;6} \\
\tilde{m}^5 &= m^5 + \mathcal{C}_3 m^{1;7} \\
\tilde{m}^{1;7} &= m^{1;7}
\end{aligned}$$

subject to the half-BPS conditions

$$\begin{aligned}
k^1 &= m_1 m^2 \equiv 0 \\
k^4 &= m_1 m^5 + m^2 m^2 \equiv 0 \\
k^{1;6} &= m_1 m^{1;7} + m^2 m^5 \equiv 0 \\
k^{3;7} &= m^2 m^{1;7} + m^5 m^5 \equiv 0 \\
k^{6;7} &= m^5 m^{1;7} \equiv 0
\end{aligned}$$

which transform as a *string multiplet*. When these composite charges do not vanish, the state is at most 1/4-BPS, in which case its mass formula is given by

$$\mathcal{M}^2 = \mathcal{M}_0^2(m) + \sqrt{[\mathcal{T}(k)]^2}$$

where $\mathcal{M}_0(m)$ and $\mathcal{T}(k)$ are given by the half-BPS mass and tension formulae

Generalized Eisenstein series

For an arithmetic group $G(Z)$, representation $\mathcal{R}$ and weight s, define

$$\mathcal{E}^{G(Z)}_{\boldsymbol{\mathcal{R}};s}(g) = \sum_{m\in\Lambda_{\mathcal{R}}\backslash\{0\}} \delta(m\wedge m)\ \left[m\cdot\mathcal{R}^t\mathcal{R}(g)\cdot m\right]^{-s}$$

This is invariant under $G(Z)$, and an eigenmode of the Laplacian on $K\backslash G$,

$$\Delta_{K\backslash G}\ \mathcal{E}^{G(Z)}_{\boldsymbol{\mathcal{R}}_\lambda;s} = s(\lambda,\rho-s\lambda)\ \mathcal{E}^{G(Z)}_{\boldsymbol{\mathcal{R}}_\lambda;s}\ ,$$

where λ is the highest weight associated to rep $\mathcal{R}$, and ρ the Weyl vector.

The delta function projects on m such that $\mathcal{R}\otimes_s\mathcal{R}$ is irreducible: half-BPS condition.

Rk1: mathematicians define Eisenstein series as

$$\mathcal{E}^{G(Z)}_{\{w_i\}}(g) = \sum_{h\in G(Z)/N}\prod_{i=1}^{r} a_i(gh)^{-w_i}\ ,$$

This agrees up to an s dependent factor for $w=s\lambda_{\mathcal{R}}$, but does not require to solve for $G(Z)/N$.

Rk2: two reps may generate the same function due to the constraint.

One-loop modular integral

We want to compute the integral in a T-duality invariant way,

$$A_n = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} V \sum \text{exp}\left(-\frac{|m^i + n^i \tau|^2}{\tau_2} + 2\pi i m^i B_{ij} n^j \right)$$

The integral can be computed by decomposing according to the modular orbits of (m_i, n_i):

• the contribution of zero orbit $(, n)$ is the volume of the fundamental domain,

$$A_{\text{zero}} = \frac{2\pi^2}{3} V_d$$

• the degenerate orbit can be unfolded over the strip $-1/2 \leq \tau_1 \leq \tau_1$, $\tau_2 > 0$ at the expense of restricting the sum to $(m_i, 0)$:

$$A_{\text{deg}} = 2V_d \ \mathcal{E}^{Sl(d,Z)}_{\boldsymbol{d}; s=1}$$

• the non-degenerate orbit can be unfolded to (twice) the upper half plane $\tau_2 > 0$ at the expense of restricting the sum to $Sl(2, Z)$ orbits;

$$A_{\text{nondeg}} \sim \mathcal{O}(e^{-1/\alpha'})$$

Laplacians and T-duality Eisenstein series

In addition to the large volume behavior, we also know that the one-loop integral is an eigenmode of the Laplacian on $[SO(n) \times SO(n)]\backslash SO(n,n)$

$$\Delta_{SO(d,d)} I_d = \frac{d(2-d)}{4} I_d$$

Surprisingly it is also an eigenmode of a non-duality invariant Laplacian,

$$\Box_d = \Delta_{Sl(d)} + \frac{2-d}{8d}\left(g_{ij}\frac{\partial}{\partial g_{ij}}\right)^2 \ , \qquad \Box_d I_d = \frac{d(2-d)}{8} I_d$$

This is also the case of the T-duality invariant Eisenstein series

$$I_d = 2\frac{\Gamma(\frac{d}{2}-1)}{\pi^{\frac{d}{2}-2}}\ \mathcal{E}^{SO(d,d,Z)}_{\mathrm{V};s=\frac{d}{2}-1} = 2\ \mathcal{E}^{SO(d,d,Z)}_{\mathrm{S};s=1} 2\ \mathcal{E}^{SO(d,d,Z)}_{\mathrm{C};s=1}$$

For low values $d = 1, 2$, this is rather

$$I_d = 2\ \mathcal{E}^{SO(d,d,Z)}_{\mathrm{S};s=1} + 2\ \mathcal{E}^{SO(d,d,Z)}_{\mathrm{C};s=1}$$

The property under $\Box_d$ is a very non-trivial check.

For $d = 4$, we see $SO(4,4)$ triality, vector=spinor=conj. spinor.

Genus g modular integral

Consider the BPS g-loop integral

$$I_d^g = \int_{\mathcal{M}_g} d\mu \; Z_{d,d}^g(g_{ij}, B_{ij}; \tau)$$

where $Z_{d,d}^g$ is the genus g lattice sum, invariant under $Sp(g, Z) \times SO(d, d, Z)$.

The period matrix takes value in

$$\tau \in U(g) \backslash Sp(g, R) / Sp(g, Z)$$

The genus g lattice partition function is annihilated by

$$\left[\Delta_{SO(d,d)} - \Delta_{Sp(g)} - \frac{dg(g+1-d)}{4} \right] I_d^g = 0$$

so the integral is an eigenmode of the T-duality Laplacian. Comparing to Eisenstein series, one finds

$$I_d^g \propto \; \mathcal{E}_{\mathrm{S};s=g}^{SO(d,d,Z)} + \; \mathcal{E}_{\mathrm{C};s=g}^{SO(d,d,Z)}$$

Numerical constant could be fixed from the volume of the moduli space of genus g curves.

R^4 couplings in M/T^{d+1}

The $t_8 t_8 R^4$ couplings in type II/T^d can be computed at tree-level and one-loop:

$$f_{R_4} = 2\zeta(3)\frac{V}{g_s^2} + I_d + \dots$$

Using the T-duality expressions,

$$f_{R^4} = \frac{V_d}{g_s^2}\ \mathcal{E}^{SO(d,d,Z)}_{\mathbf{1};s=3/2} +\ \mathcal{E}^{SO(d,d,Z)}_{\mathrm{S};s=1} + \dots$$

SUSY requires to have an eigenmode of the U-duality Laplacian,

$$\Delta_{E_{d+1(d+1)}} f_{R^4} = \frac{3(d+1)(2-d)}{2(8-d)} f_{R^4}$$

A solution can be found in terms of U-duality invariant Eisenstein series,

$$f_{R^4} = \frac{V_{d+1}}{l_M^9}\ \mathcal{E}^{E_{d+1(d+1)}(Z)}_{\boldsymbol{string};s=3/2}$$

$$= \frac{\Gamma(d/2-1)}{\pi^{d/2-2}}\ \mathcal{E}^{E_{d+1(d+1)}(Z)}_{\boldsymbol{particle};s=d/2-1} = \frac{V_{d+1}}{l_M^9}\ \mathcal{E}^{E_{d+1(d+1)}(Z)}_{\boldsymbol{membrane};s=1}$$

For $d = 1, 2$, these give different $t_8 t_8 \pm \epsilon_8 \epsilon_8$ tensor structures.

Euclidean D-brane instantons

Expanding at weak coupling using the Poisson resummation trick, this ansatz reproduces the tree-level and one-loop terms, and predicts non-perturbative contributions

$$f_{\text{non-pert.}} = 4\pi V e^{-\phi} \sum_{m\neq 0} \sum_{\gamma\in\Lambda^*} \frac{|m|}{\int_\gamma \sqrt{\det(\hat{G}+\hat{B}+F)}}$$

$$K_1\left(\frac{2\pi|m|}{g_s}\int_\gamma \sqrt{\det(\hat{G}+\hat{B}+F)}\right) e^{2\pi i m \int_\gamma \hat{\mathcal{R}}}$$

These are identified as Euclidean D-branes wrapped on subtori γ of T^d, with Born-Infeld action

$$S = \frac{1}{g_s}|m| \int \sqrt{\det(\hat{G}+\hat{B}+F)} + im\int \hat{\mathcal{R}}$$

For $d > 4$, the constraint complicates the analysis of weak coupling. It seems to predict further e^{-1/g_s^2} corrections due to the branching

$$\text{string} = \text{singlet} + \text{spinor} + \text{new reps as } d > 4$$

$R^4 H^{4g-4}$ couplings

$R^4 H^{4g-4}$ can be written as chiral densities

$$\int d^n x d^{16}\theta \; W^{4g} \; , \quad W = H + \theta^4 R + \dots$$

They have been argued to be given at tree-level and g-loop only in type II/T^d and related to topological amplitudes of $N=4$ strings.

H are 3-form field strength transforming in the string rep. The same holds for $R^4 F^{4g-4}$, where F transforms in the particle rep.

Using the result for the g-loop BPS amplitude, one can construct a possible non-perturbative completion,

$$I = \frac{V_{d+1}}{l_M^9} \int d^{10-d}x\sqrt{-\gamma} \,\widehat{\sum_m} \delta(m \wedge m) \; \frac{R^4 \; (m \cdot H)^{4g-4}}{(m \cdot M(\boldsymbol{string}) \cdot m)^{3g-\frac{3}{2}}}$$

$$I = \int d^{10-d}x\sqrt{-\gamma} \,\widehat{\sum_m} \delta(m \wedge m) \; \frac{R^4 \; (m \cdot F)^{4g-4}}{(m \cdot M(\boldsymbol{particle}) \cdot m)^{4g-5+\frac{d}{2}}} \; ,$$

For $g=1$, this does reproduce the R^4 couplings.

The one-loop membrane

Instead of expanding at weak type II coupling, one may expand at large volume in M-theory. For $d < 5$, one finds membrane instantons:

$$f_{R^4} \sim \frac{\pi^2 l_M^6}{3} + \pi \sum \frac{l_M^9}{\sqrt{(m^3)^2}}$$

$$+\pi l_M^6 \sum \left[\frac{l_M^6}{(n^3)^2} \right]^{1/2} \mu(n^3) \exp\left(-\frac{2\pi}{l_M^3} \sqrt{(n^3)^2} + 2\pi i n^3 C_3 \right)$$

One should be able to derive R^4 couplings from first principles in M-theory.

Since only toroidal membrane instantons appear, and the amplitude is BPS saturated, one expects

$$A = \int_{SO(3)\backslash GL(3)/Sl(3,Z)} d\gamma \sum_{m_a^i}$$

$$\exp\left(-\pi (m_a^i \gamma_{ij}^{ab} m_b^j - 1) \sqrt{\gamma} + 2\pi i m_a^i m_b^j m_c^k \epsilon^{abc} C_{ijk} \right)$$

where we integrate over *constant* worldvolume metrics γ in the fundamental domain and sum over integer winding numbers m_a^i , $a = 1, 2, 3$.

Upon orbit decomposition under $Sl(3, Z)$, one should reproduce the perturbative terms plus membrane instantons.

Third lecture

Exact results and instanton effects
theories with reduced supersymmetry

1. Heterotic-type II triality and F^4 couplings
2. U-duality in 3D theories with 16 SUSY
3. Instanton corrections to hypermultiplet geometry
4. 1/4-BPS amplitudes in type II/tori

Heterotic $Spin(32)/Z_2$ *- Type I duality*

- The two ten-dimensional theories have the same gauge group and effective action, upon identifying

$$g_I = \frac{1}{g_{HO}} \;, \qquad l_I = \sqrt{g_{HO}} l_H$$

This duality maps KK states to KK states, the fundamental heterotic string to the type I D-string, the heterotic 5-brane to the type I D5-brane. Non-BPS heterotic states in the spinor conjugacy class map to Non-BPS type I D0-particles. The non-BPS type I string has no known representation on the heterotic side.

- Using the Horava-Witten picture of the heterotic string $E_8 \times E_8$, one can "derive" this duality by considering M-theory on $I(R_{I'}) \times S_1(R_1)$ in two different ways :

$$HE/S_1(R_1) \left\{ \begin{array}{l} R_I = g_{\mathrm{HE}} l_{\mathrm{H}} \\ l_M^3 = g_{\mathrm{HE}} l_{\mathrm{H}}^3 \end{array} \right. = I'/I(R_I') \left\{ \begin{array}{l} l_M^3 = g_{I'} l_I^3 \\ R_1 = g_{I'} l_I \end{array} \right.$$

Applying a T-duality on each side, we find heterotic $SO(32)$ and type I with inverse coupling compactified on a common circle :

$$g_{HO} = \frac{g_{\mathrm{HE}} l_{\mathrm{H}}}{R_1} = \frac{R_{I'}}{R_H} = \frac{R_{I'}}{g_{I'} l_I} = \frac{1}{g_I}$$

$$R_{HO} = \frac{l_{\mathrm{H}}^2}{R_1} = \frac{l_M^3}{R_{I'} R_H} = \frac{l_I^2}{R_{I'}} = R_{I'}$$

The 10D duality is obtained by decompactification.

Heterotic $E_8 \times E_8$ - Type II/K_3

- The heterotic-type II duality is usually presented in 6 dimensions, where the two theories have the same effective action, upon identifying

$$g_{I6} = \frac{1}{g_{\mathrm{HE6}}} \ , \qquad l_I = g_{\mathrm{HE6}} l_H$$

This duality maps het KK and winding states to type II D-branes wrapped on K_3, the het NS5-brane wrapped on T^4 to the fundamental type IIA string, the type II NS5 wrapped on K_3 to the fundamental het string.

- In particular, by identifying the volume of K_3 as the radius (square) of heterotic circle,

$$\frac{R_1^2}{l_H^2} = \frac{V_{K3}}{l_{\mathrm{II}}^4}$$

we see that type II D0-branes are mapped to Het KK states along R_1, so the duality decompactifies to a Het/T^3-M/K_3 7D duality.

- This duality can in fact be derived in the Horava-Witten picture by considering M-theory on $I(R_I) \times T^3(R_1, R_2, R_3)$ and performing a double T-duality on T^3 :

$$\frac{R_1 R_2 R_3 R_4}{g_{\mathrm{HE}}^2 l_{\mathrm{H}}^4} = \frac{R_1 R_2 R_3 R_4}{R_I l_M^3} \to \frac{R_4 l_M^3}{R_I R_1 R_2 R_3} = \frac{g_{\mathrm{II}}^2 l_{\mathrm{II}}^4}{R_1 R_2 R_3 R_I}$$

- This identifies the $E_8 \times E_8$ heterotic string with M-theory compactified on K_3 at an $E_8 \times E_8$ singularity :

$$K_3 = I(R_I) \times T^3(\tilde{R}_2, \tilde{R}_3, \tilde{R}_4)$$

and gives the complete mapping of the dilatonic $SO(4,4)$ moduli:

$$l_{\rm II} = \frac{g_{\sf HE} l_{\sf H}^3}{\sqrt{R_1 R_2 R_3 R_4}}\ , \qquad g_{\rm II} = \frac{\sqrt{R_1^3 R_2 R_3 R_4}}{g_{\sf HE} l_{\sf H}^3}\ ,$$

$$R_I = g_{\sf HE} l_{\sf H}\ , \qquad \tilde{R}_i = \frac{g_{\sf HE} l_{\sf H}^3}{R_j R_k}\ ,\ i,j,k = 2,3,4$$

- This is nothing but a SO(4,4) triality transforming the vector of heterotic KK and windings into the spinor of type II D-branes :

$$\frac{R_1}{l_{\sf H}} \to \sqrt{\frac{R_1 R_2 R_3 R_4}{l_{\rm II}^4}}\ , \qquad \frac{R_2}{l_{\sf H}} \to \sqrt{\frac{R_1 R_2}{R_3 R_4}}$$

Aspinwall;Polchinski;Dine,Shirman

which can in fact be extended to the full $SO(4,4) \subset SO(4,20)$ moduli :

$$M(vect, g_{ij}, B_{ij}) = M(spin, V g^{ij}, \epsilon^{ijkl} B_{kl})$$

Heterotic-type I-type II triality

The four theories with $N = 16$ supersymmetries are thus unified in the M-theory setting as

$$\begin{tikzcd}
\begin{array}{c}\text{Type I } SO(32) \text{ on} \\ T^4(\tilde{R}_I, R_2, R_3, R_4)\end{array} & & \begin{array}{c}\text{Het } SO(32) \text{ on} \\ T^4(\tilde{R}_1, R_2, R_3, R_4)\end{array} \\
& \begin{array}{c}\text{M-theory on} \\ S_1(R_1) \times I(R_I) \times T^3(R_2, R_3, R_4)\end{array} \arrow[ul, "T_{1I}"] \arrow[ur, "T_{I1}"'] \arrow[dl, "T_{234}"'] \arrow[dr] & \\
\begin{array}{c}\text{Type IIA on } K_3 = \\ I(R_I) \times T^3(\tilde{R}_2, \tilde{R}_3, \tilde{R}_4)\end{array} & & \begin{array}{c}\text{Het } E_8 \times E_8 \text{ on} \\ T^4(R_1, R_2, R_3, R_4)\end{array}
\end{tikzcd}$$

Monopole moduli space and instantons

The dimension of the moduli space of charge n monopoles in $SU(2)$ is $4n-4$: three positions and 1 phase for each monopole, modulo overall translations and gauge rotations:

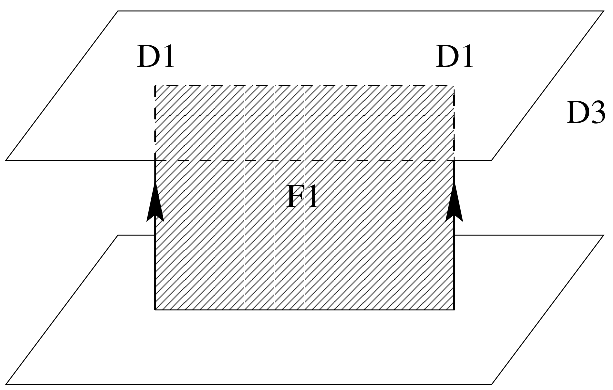
$$\mathcal{M} \sim (R^3 \times S^1)^{n-1}/S_n$$

On general grounds, the moduli space carries an hyper-kähler metric which controls the low-energy dynamics of monopoles.

At large distance, monopoles interact via exchange of massless photon and scalars. For $n=2$, the metric on $\mathcal{R}^3 \times S^1$ is corrected to Taub-Nut.

At smaller distance, there are corrections $O(e^{-M_W r})$ coming from exchange of massive W-boson and Higgs; equivalently, Euclidean worldlines of massive W stretched between 2 monopoles.

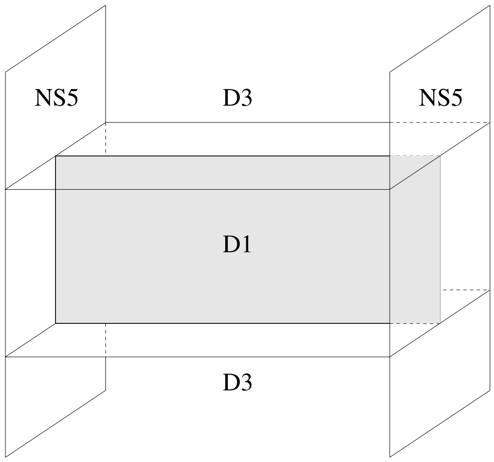
In the D1-D3 brane picture, these are worldsheet instantons of open F-strings ending of the D1-strings:



Monopoles and Hypermultiplet moduli spaces

The same hyperkähler manifold can be argued to describe the Coulomb branch of 3D gauge theories with 8 SUSY.

Each VM has 3 scalars and 1 $U(1)$ gauge field which can be dualized in a periodic scalar σ: the VM moduli space has the same structure as the HM (up to exchange of $SU(2)_{L,R}$ R-symmetry groups).



W-boson instanton are now interpreted as usual 3D instantons (ie 4D monopoles reduced to 3D).

It also describes the hypermultiplet moduli space of Heterotic on K_3 at an ALE_{n-1} singularity: the moduli space is given at tree-level, and W-boson instantons correspond to worldsheet instantons wrapping the blown up sphere.

Atiyah-Hitchin revisited

For $n = 2$, the moduli space of 2 $SU(2)$ monopoles can be computed completely using $SO(3)$ isometry:

$$ds^2 = (abc)^2 dt^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2$$

where σ_i are 1-forms in the Euler parameterization,

$$\begin{aligned}
\sigma_1 &= \sin\psi d\theta - \cos\psi \sin\theta d\phi, \\
\sigma_2 &= -\cos\psi d\theta - \sin\psi \sin\theta d\phi, \\
\sigma_3 &= d\psi + \cos\theta d\phi
\end{aligned}$$

The curvature is self-dual iff

$$\frac{a'}{abc} = \frac{b^2 + c^2 - a^2}{2bc} - \lambda \ , \ \text{etc} \qquad (1)$$

Define $w_1 = bc, w_2 = ca, w_3 = ab$ and put $\lambda = 1$:

$$d(w_1 + w_2)/dt = -2w_1 w_2 \ , \ \text{etc}$$

This is solved in terms of modular forms by

$$w_1 = \frac{2}{\pi}\frac{\vartheta_2'}{\vartheta_2} \ , \ w_2 = \frac{2}{\pi}\frac{\vartheta_3'}{\vartheta_3} \ , \ w_3 = \frac{2}{\pi}\frac{\vartheta_4'}{\vartheta_4}$$

with modular parameter $\tau = it = i\pi/r$.

The q-expansion of modular forms gives the instanton expansion of the exact metric: Taub-NUT + e^{-r} corrections. This identifies $r(t)$.

We have 8 SUSY, so instanton corrections are best read off on the four fermion terms.

Instanton expansion of Atiyah-Hitchin

Go to an orthonormal basis $e_0 = abcdt, e_i = a_i \sigma_i$, and complex coordinates $dw_+ = e_1 + ie_2$. The Riemann tensor splits into 5 components

$$
\begin{aligned}
R_{(0)} &= \frac{4}{(r-2)^3} - \sum_{n=1}^{\infty} \frac{P_{2n}(r)}{(r-2)^{n+3}} q^{2n} \\
R_{(1)} &= 0 \\
R_{(\pm 2)} &= \sum_{n=0}^{\infty} \frac{P_{2n+1}(r)}{(r-2)^{n+2}} q^{2n+1} e^{\pm i\sigma}
\end{aligned}
$$

$P_n(r)$ is a polynomial of order $3([\frac{n}{2}]+1)$ in r and $q = e^{-r}$

$$P_1(r) = 8(-3 + 9r - 6r^2 + r^3)$$

$P_2(r) = 32(6 - 54r + 123r^2 - 124r^3 + 66r^4 - 18r^5 + 2r^6)$

(i) Only 1 perturbative term, occurring in $R_{(0)}$: $U(1)_R$ conserved in perturbation theory

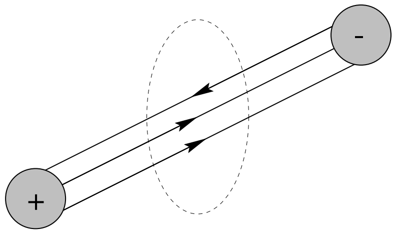
(ii) Arbitrary powers of $q = e^{-r}$, but only 1 power of $e^{i\sigma}$: bound states of n instantons and $\bar{n}$ anti-instantons, with overall topological charge $n - \bar{n} = 0$ for $R_{(0)}$ and ± 1 for $R_{(\pm 2)}$

(iii) $R_{(1)} = 0$: no half-instanton.

(iv) coefficients are integral: topological origin ?

Instanton-anti-instanton bound states

The semi-classical effects can be interpreted in terms of W-boson/anti-W-boson stretched between the 2 monopoles:



The non-vanishing of this contribution does not contradict supersymmetry:

$$A = \int_{\mathcal{M}_{(n,\bar{n})}} \frac{\mathrm{Pf}'(D_F)}{\mathrm{Pf}'(\nabla^2_{)}} e^{-S_{cl}+i\sigma(n-\bar{n})}$$

In the D1-D3 picture, these effects correspond to bound states of fundamental open strings with opposite orientation ending on the D1/D3. In the Het/ALE picture, these are worldsheet instantons with opposite orientation.

More generally, bound states of instantons and anti-instantons are expected to correct hypermultiplet metrics in field/string theory. For the n monopole problem, one expects contributions from multistring configurations: