Strings in homogeneous wave backgrounds and null holography

Boris Pioline

LPTHE, Paris

w/ E. Kiritsis `hep-th/0204004`

transparencies available from
`http://www.lpthe.jussieu.fr/ pioline/academic.html`

Introduction

String theory on pp-wave backgrounds and Penrose limits have recently been the focus of much interest:

Blau, Figueroa O'Farrill, Hull, Papadopoulos 0110242
Metsaev, Tseytlin 0202109
Berenstein Maldacena Nastase 0202021

One motivation is to deeper understand AdS/CFT:

- Despite RR fluxes, string theory on the light-cone is (in principle) exactly solvable in this regime

- Bulk string modes can be identified 1-1 with (nearly BPS) gauge-invariant observables:

$$
\begin{aligned}
a_n^{I\dagger}|0\rangle &= \sum_l \mathrm{Tr}\ [Z^l \cdot \phi^I \cdot Z^{J-l}] e^{2\pi i n l/J} \\
a_n^{i\dagger}|0\rangle &= \sum_l \mathrm{Tr}\ [Z^l \cdot \partial_i Z \cdot Z^{J-l}] e^{2\pi i n l/J}
\end{aligned}
$$

- It corresponds to a new stringy regime of the large N gauge theory, rather different from the 't Hooft limit,

$$\mu \alpha' p_+ = J/\sqrt{g_{YM}^2 N}\ , \quad g_{eff} = J^2/N$$

This allows for detailed checks of the AdS/CFT correspondence that are so far impractical in the $AdS_5 \times S^5$ case.

Introduction

Another reason to be interested in this set-up is that it allows for an holographic description of a space "close" to flat Minkowski:

- Curvature invariants are all zero, the effective curvature $\rho = \mu\alpha' p_+$ depends on the light-cone momentum. At small p_+ one recovers flat space.
- The spectrum of the light-cone Hamiltonian matches on smoothly to that of flat Minkowski space:

$$p_- = \sum_{n=-\infty}^{+\infty} \sqrt{\mu^2 + \frac{n^2}{\alpha' p_+}} = \mu \sum_{n=-\infty}^{+\infty} \sqrt{1 + \frac{g_{YM} N n^2}{J^2}}$$

- The anomalous dimensions can still be computed even though the 't Hooft coupling is large.

More generally, we are blessed with a new background with a well-defined holographic dual, it is interesting to compare with more familiar holographic set-ups and extract background independent features of holography.

Introduction

Despite these attractive features, this new ppwave/CFT holographic duality has some disturbing shortcomings:

- At finite J, there are lots of other gauge-invariant operators that remember that the theory lived on $AdS \times S$.

- The symmetries of the bulk are not manifest on the dual side. E.g. the discrete symmetry exchanging AdS and S transverse coordinates is very non-trivial.

- Even in the $J \gg 1$ limit, the dual is still a local CFT, with symmetry $SO(2,4) \times SO(4)$. This does not match the symmetry of the bulk, which is a Wigner contraction of both factors in $SO(2,4) \times SO(6)$.

- It does not come with a prescription for computing boundary correlators from bulk S-matrix elements. Rather, one should compute correlators in usual AdS and then take the limit.

It would be desirable to have an holographic description of the pp-wave backgrounds that makes the symmetries manifest. It could also teach us about holography in Minkowski or other spaces.

Summary of results

Here, we attempt constructing an holographic correspondence for pp-wave backgrounds, without prejudices about the possible AdS origin of that background (although taking inspiration from AdS experience).

- We identify a patch of the geometry analogous to the Poincaré patch in AdS, i.e. conformal to Minkowski. The analogue of the radial direction is found to be the null direction x^+, i.e. the light cone time. The holographic dual leaves on a slice at fixed x^+.

- We find a realization of the bulk isometries as symmetry acting on the boundary fields: the conformal group is replaced by an extended Heisenberg algebra. Ward identities restrict the form of 2 and 3-point functions.

- We evaluate the 2 and 3 point correlation functions for a minimally coupled massive scalar field using a boundary to bulk correspondence.

- Boundary data at fixed $x^+ \to -\infty$ are unsufficient to determine the full evolution: one also needs a complementary screen at $x^- = -\infty$. This provides the missing "normalizable" modes controlling the vacuum.

The nature of the HFT living on the boundary will be left unanswered.

Outline

1. Geometry of homogeneous pp-waves
2. RNS strings in NS homogeneous pp-waves
3. The “Poincaré” patch of pp-wave backgrounds
4. Holography at work
5. Summary / outlook

1. Geometry of pp-waves
The Nappi-Witten background

The simplest example of pp-wave background can be obtained as the group manifold (WZW model) of the solvable (non-semi-simple) group $E_2^c := H_4$:

$$[P_1, P_2] = K \ , \quad [J, P_i] = \epsilon_{ij} P_j \ , \quad [K, \cdot] = 0$$

Any group element can be parameterized as

$$g = \exp(uJ + vK) \cdot \exp(a_1 P_1 + a_2 P_2)$$

- H_4 possesses an invariant non-degenerate bilinear form, giving rise to a metric invariant under $H_{4L} \times H_{4R}$:

$$\begin{aligned} ds^2 &= 2dudv + da_1^2 + da_2^2 + (a_2 da_1 - a_1 da_2) du \\ H &= du \wedge da_{12} \end{aligned}$$

- If u was a spacelike compact direction, that would be an (exact) constant magnetic background for the KK gauge field $g_{\mu u}$. The electric charge is the momentum p_+ along v. Classical trajectories are Larmor orbits in the transverse plane (a_1, a_2).

- Going to the rotating frame $a_1 + ia^2 = e^{iu/2}(x_1 + ix_2)$ yields the familiar metric

$$\begin{aligned} ds^2 &= 2dudv + dx_1^2 + dx_2^2 - \frac{1}{4}\mu^2(x_1^2 + x_2^2)\ du^2 \\ H &= du \wedge dx_{12} \end{aligned}$$

- This is an exact conformal background with $c = 4$ and 16 SUSY. It can be completed by flat R^6 to $c = 10$.

Nappi-Witten as the Penrose limit of NS5

Let us consider the Penrose limit of the near horizon geometry of the NS5-brane, for a geodesic spinning along the sphere S^3:

$$
\begin{aligned}
ds^2 &= -dt^2 + dr^2 + N^2(d\psi^2 \cos^2\theta + d\theta^2 + \sin^2\theta d\phi^2) + (dx^i)^2 \\
g_s^2 &= e^{-2r/(l_s\sqrt{N})}
\end{aligned}
$$

Define

$$
\begin{array}{rcl} t + N\psi &=& u\sqrt{2}, \\ t - N\psi &=& v\sqrt{2}/\alpha^2 \end{array} , \quad \theta = \alpha\rho \; , \quad x^i = \alpha y_i \; , \quad r = \alpha y_6
$$

In the limit $\alpha \to \infty$:

$$
\begin{aligned}
ds^2 &= -dt^2 + N^2 d\psi^2 (1 - \frac{1}{6}\alpha^2\rho^2) + \alpha^2(dr^2 + r^2 d\phi^2) + \alpha^2(dy^i)^2 \\
&= \alpha^2 \left[2dudv - \frac{1}{12} r^2 du^2 + N^2(dr^2 + r^2 d\phi^2) + (dy^i)^2 \right]
\end{aligned}
$$

Furthermore, the dilaton gradient disappears in this limit. This is just the Nappi-Witten geometry !

Gomis, Ooguri

[Had we considered a geodesic along the radial direction, we would have found a flat geometry with a null dilaton profile, $\phi = \phi(u)$. This is the simplest wave background in string theory !]

Other avatars of Nappi-Witten

The WZW construction of the NW can be trivially generalized to any even dimension by considering the Heisenberg algebra $[P_i, Q^j] = \delta^i_j K$ extended by J rotating positions and momenta :

$$\begin{aligned} ds^2 &= 2dudv + \sum dx_i^2 - \frac{1}{4}\mu^2 x_i^2 \, du^2 \\ H &= du \wedge (dx_{12} + dx_{34} + \dots) \end{aligned}$$

This gives a CFT realization of all Lorentzian maximally symmetric (Cahen-Wallach) spaces, supported by NS flux.

- For $N = 6$, we get the Penrose limit of the NS5-F1 near horizon geometry. The D1-D5 system is obtained by trading the H_{NS} flux for an H_{RR}.

- For $N = 10$, we get the same geometry as that of the Penrose limit of $AdS_5 \times S_5$, but with H_{NS} flux instead of RR 5-form $F = du(dx_{1234} + dx_{5678})$. This has 16 SUSY instead of 32.

While all these backgrounds are in principle solvable, those with NS flux allow for a covariant quantization of the NSR string, whereas those with RR flux can only be considered in the Green-Schwarz light-cone string. General features of holography are expected to be the same.

Geometry of pp-wave backgrounds

- As any group manifolds, pp-wave spaces are invariant under the left and right action of the group H. This is most easily computed using the (non-unitary) matrix representation

  $$g = \begin{pmatrix} 1 & a_1 \sin u - a_2 \cos u & a_1 \cos u + a_2 \sin u & 2v \\ 0 & \cos u & -\sin u & a_1 \\ 0 & \sin u & \cos u & a_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

  Closed null curves are avoided by taking the universal covering in the u coordinate (as in $Sl(2)$ vs AdS_3).

- Killing vectors are best expressed in a complex basis, $z = x_1 + ix_2$, $P^{\pm} = P_1 \pm iP_2$:

  $$\begin{aligned} K_L &= K_R = \partial_v, \\ J_L &= \partial_u + (x_1\partial_2 - x_2\partial_1)\ , \quad J_R = \partial_u + (x_1\partial_2 - x_2\partial_1)\ , \\ P_L^+ &= e^{iu/2}(4\partial_{\bar{z}} - iz\partial_v)\ , \quad P_L^- = e^{-iu/2}(4\partial_z + i\bar{z}\partial_v) \\ P_R^+ &= e^{-iu/2}(4\partial_{\bar{z}} + iz\partial_v)\ , \quad P_R^- = e^{iu/2}(4\partial_z - i\bar{z}\partial_v) \end{aligned}$$

  These are as numerous as in $R \times S^3$.

Geodesics in homogeneous wave backgrounds

- Geodesics are generated by one-parameter subgroups of H, $g(\tau) = U \exp(\tau h) V$. For those passing through the origin at $\tau = 0$ ($UV = 1$):

$$
\begin{aligned}
u(\tau) &= 2p_+\tau \\
v(\tau) &= \left(2p_- + \frac{p_1^2 + p_2^2}{p_+}\right)\tau - \frac{\sin 2p_+\tau}{2p_+^2}(p_1^2 + p_2^2) \\
z(\tau) &= 2\sin(p_+\tau)\ (p_1 + ip_2)/p_+
\end{aligned}
$$

  All geodesics come back to (minus) their original position in transverse space after $u \to u + 2\pi$. The net motion along v depends on the sign of $p^2 = 2p_+p_- + p_i^2$.

- The geodesic (Lorentzian) distance between 2 points is

$$
\begin{aligned}
r^2 &= 2(u-u')(v-v') + \\
&+ \frac{u-u'}{2\sin\left(\frac{u-u'}{2}\right)}\left[\left(x_i^2 + x_i'^2\right)\cos\left(\frac{u-u'}{2}\right) - 2x_i x_i'\right]
\end{aligned}
$$

  As $u' = u + 2\pi$ the distance goes to infinity unless $x_i = -x_i'$.

Classical dispersion relation

- Free scalar fields propagating on NW satisfy $(\Delta - m^2)\phi = 0$ where

$$\Delta = 2\partial_u\partial_v + \frac{1}{4}x_i^2\partial_v^2 + \partial_i^2 + \Delta_\perp$$

- Acting on a plane wave $e^{i(p_- u + p_+ v)}$, this is the harmonic oscillator with frequency $|p_+|/4$. The ground state is

$$\phi(u, v, z) = \exp\left[i(p_+ v + p_- u + p_\perp x_\perp) - \frac{1}{4}|p_+|z\bar{z}\right]$$

- Other states in the same multiplet are obtained by acting n_L times (resp. n_R) with the creation operator P_L^- (resp. P_R^+). The dispersion relation is

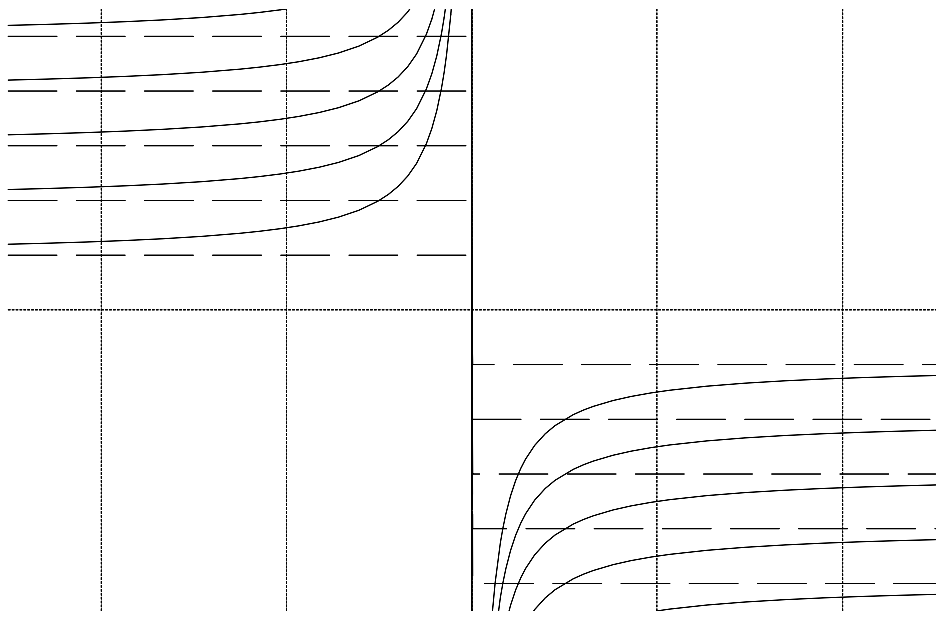
$$2p_+p_- + |p_+|(n_L + n_R + 1) + p_\perp^2 + m^2 = 0 \ ,$$

  while the helicity in transverse place is $h = \epsilon(p_+)(n_L - n_R)$.

- For $p_+ = 0$ and $m^2 = 0$, the solution is an arbitrary profile in u, it is a mode travelling along the wave. There can also be tachyonic modes with $p_+ = 0$ and p_i non vanishing.

The dispersion relation, pictured

$$2p_+p_- + |p_+|(n_L + n_R + 1) + p_\perp^2 + m^2 = 0$$



- String theory on group manifolds can be solved by standard WZW techniques. Here the situation is even simpler, since the chiral currents can be rewritten in terms of free bosons:

  $$K = \partial u \ , \quad J = \partial v \ , \quad P_L^+ = e^{-iu}\partial w \ , \quad P_L^- = e^{iu}\partial\bar{w} \ ,$$

  where $P_L^{\pm} = (P_1 \pm iP_2)/\sqrt{2}$. The scalars $(u, v, w, \bar{w})$ have free boson propagators,

  $$\langle x^\mu(z) x^\nu(z') \rangle \ = \ \eta^{\mu\nu} \log(z - z')$$

  with $\eta_{\mu\nu} dx^\mu dx^\nu = 2du dv + dw d\bar{w}$.

- For $0 < p_+ < 1$, chiral primaries can be written as

  $$R_{p_+,p_-} = \exp\left[i(p_- u + p_+ v)\right] H_{p_+}$$

  where H_{p_+} is a twist field modifying the periodicity of the transverse coordinate w by an (irrational) angle p_+,

  $$w(e^{2\pi i} z) = e^{-2\pi i p_+} w(z) \ , \ \bar{w}(e^{2\pi i} z) = e^{2\pi i p_+} \bar{w}(z).$$

  Its conf. dimension is $\Delta = p_+ p_- + \frac{1}{2} p_+ (1 - p_+)$. The quadratic contribution in p_+ cancels in the superstring case.

- For $p_+ = 1$, the effect of the twists disappears, and one expect a new zero mode. States with $p_+ > 1$ can be obtained by spectral flow from those in $0 < p_+ < 1$.

Spectral flow

- The current algebra is invariant under the spectral flow

$$\begin{aligned} J_m &\to J_m \ , \quad K_m \to K_m + iw\delta_{m,0} \ , \quad P^{\pm}_m \to P^{\pm}_{m\pm w} \\ L_0 &\to L_0 + iw\,(K_0 - J_0) + \frac{1}{2}w(1-w)\delta_{m,0} \end{aligned}$$

as long as $m \in Z$. Highest weight conditions however change, and one gets different representations, with $p_+ \to p_+ + w$.

- At the classical level, the spectral flow acts as

$$g(\tau,\sigma) \to e^{w_L(\tau+\sigma)J} g(\tau,\sigma) e^{w_R(\tau-\sigma)J}$$

This amounts to transforming the string embedding coordinates,

$$\begin{aligned} u &\to u + (w_L + w_R)\tau + (w_L - w_R)\sigma \\ v &\to v \\ x_1 + ix_2 &\to e^{\frac{i}{2}(w_L+w_R)\sigma + \frac{i}{2}(w_L-w_R)\tau}(x_1 + ix_2) \end{aligned}$$

For non-compact u, one should impose $w_L = w_R$.

- Starting from a $p_+ \neq 0$ geodesic, one obtains a string winding w times in the harmonic potential, and oscillating with period $2\pi/p_+$.

Spectral flow and long strings

- Starting from a $p_+ = 0$ spacelike geodesic instead, one obtains after symmetric spectral flow

$$\begin{aligned} u(\tau,\sigma) &= 2w\tau \\ v(\tau,\sigma) &= 2p_-\tau \\ z(\tau,\sigma) &= e^{iw\sigma}\left[(x_1+ix_2)+\tau(p_1+ip_2)\right] \end{aligned}$$

- The Virasoro conditions $L_0 + \bar{L}_0 = 2h, L_0 - \bar{L}_0 = 0$ determine the momenta p_i in terms of the initial position x_i. The resulting state

$$z(\tau,\sigma) = e^{iw\sigma}(x_1+ix_2)\left[1 \pm 2\tau\sqrt{\frac{h-2p_-w}{x_1^2+x_2^2}}\right]$$

describes a closed string of winding number w around the bottom of the potential, contracting or expanding linearly with time. Its light-cone energy is $p_+ = 2w$.

- At the critical value $p_- = h/(2w)$, this becomes a static string, whose radius can stretch in the transverse directions at no cost in energy. Its light-cone momentum p_+ is fixed to be integer. This string is the analogue of the long string appearing in Euclidean AdS_3

String spectrum on NW

Putting together the two chiral halfs of the string, and choosing a physical gauge in which all light-cone oscillators are eliminated, one gets

$$
\begin{aligned}
L_0 &= p_+ p_-^L + p_+^L n_L + \frac{1}{2} p_+ (1 - p_+) + p_\perp^2 + N_L = 1 \\
\bar{L}_0 &= p_+ p_-^R + p_+^R n_R + \frac{1}{2} p_+ (1 - p_+) + p_\perp^2 + N_R = 1
\end{aligned}
$$

where $p_-^L = p_- - h/2$, $p_-^R = p_- + h/2$, and, independently on the left and on the right,

$$
\begin{aligned}
N &= \sum_{k=1}^{\infty} k N_k + \sum_{\bar{k}=1}^{\infty} \bar{k} N_{\bar{k}} \\
n &= \sum_{k=0} N_k - \sum_{\bar{k}=1} \bar{N}_{\bar{k}}
\end{aligned}
$$

Taking the difference gives the matching condition,

$$
p_+ (h - \mathrm{sgn}(p_+)(n_L - n_R)) = N_L - N_R
$$

which implies $N_L = N_R$ and $h = \mathrm{sgn}(p_+)(n_L - n_R)$. The sum gives the mass formula for the bosonic string,

$$
2p_+ p_- + |p_+|(n_L + n_R) + |p_+|(1 - |p_+|) + p_\perp^2 + N_L + N_R - 2 = 0 \, ,
$$

Here p_+ runs from 0 to ∞, with $|p_+| > 1$ states obtained by spectral flow.

2. Holography for pp-wave backgrounds ?

A necessary condition for an holographic correspondance (for a conservative mind brought up to AdS/CFT) is the existence of a well-defined Cauchy problem.

- Evolution along the light-cone direction v does not seem consistent: classical Zitterbewegung, ill-defined evolution for $x_1^2 + x_2^2 = 0$.

- The transverse directions are effectively compact of radius $r = 1/\sqrt{\mu p_+}$: there is no infinity in this direction.

  See however Leigh Okuyama Rozali

- The light-cone direction u passes this requirement: globally defined on geodesics, consistent evolution (except for $p_+ = 0$ states, as in standard light-cone quantization).

Another crucial ingredient (for an ultra-conservative mind) is the relation between holographic flow in the bulk. This is particularly obvious in the Poincaré patch of AdS:

$$ds^2 = (-dt^2 + dx_i^2 + dz^2)/z^2$$

Motion along z generates scaling transformations of the holographic dual. $z = 0$ corresponds to the UV of the gauge theory, while $z \to \infty$ is the horizon of the Poincaré patch, or the IR regime of the holographic dual.

The "Poincaré" patch

- It is well known that the homogeneous pp-wave background is conformally flat. The conformal factor $1/\sin^2[(u-u_0)/2]$ is non-singular on any interval in u of length 2π.

- The change of coordinates mapping the slice $u \in [0, 2\pi]$ of NW conformally to Minkowski is

$$\begin{aligned} x^+ &= -\cot(u/2)\ , \quad v = x^- - \frac{1}{2}(y_1^2 + y_2^2)\sin u \\ x_i &= 2y_i \sin(u/2) \end{aligned}$$

  In these coordinates, the metric and flux read

$$\begin{aligned} ds^2 &= \frac{4}{1+x^{+2}}(dx^+dx^- + dy_1^2 + dy_2^2) \\ H &= \frac{1}{(1+x^{+2})^2} dx^+ dy_1 dy_2 \end{aligned}$$

- The motion along x^+ rescales the metric of a fixed x^+ slice. The conformal factor goes to 0 at $x^+ = \pm\infty$, which corresponds to the horizon of the patch. The conformal factor is always < 1, so that the UV is effectively cut-off.

- We propose that there should exist *a holographic dual to string theory in a Neveu-Schwarz pp-wave background of type H_D, with degrees of freedom living on the degenerate space described by the (x^-, y_i) coordinates at $x^+ \to -\infty$, together with the other flat transverse coordinates.*

compare to Das, Gomez, Rey

Comments on the proposal

- Evolution along a null direction involves a first order differential equation. At fixed m^2, only one "non-normalizable" mode propagates, in contrast to AdS/dS.

- The Cauchy problem is not well defined unless one specifies boundary conditions on the other side of the wedge, $x^- \to -\infty$. These are the missing modes, they encode the vacuum structure of the theory.

- The two regions at $x^+ = \pm\infty$ are asymptotically flat. The holographic correspondence relates S-matrix elements between $x^+ = \pm\infty$ to correlators of a putative dual gauge theory at fixed x^+.

- Translation along x^+ is not Killing vector. Correspondingly, the RG flow involves not only of a rescaling of the boundary coordinates, but also a additional action: the dual is no more a CFT.

Geometry of the conformal patch

The Killing vectors of NW can be expressed in terms of the conformal coordinates:

$$
\begin{aligned}
K &= \partial_- \\
J_L - J_R &= y_1\partial_2 - y_2\partial_1 \\
J_L + J_R &= (1 + x^{+2})\partial_+ - (y_1^2 + y_2^2)\partial_- + x^+(y_1\partial_1 + y_2\partial_2) \\
P_{iL} - P_{iR} &= \epsilon_{ij}\partial_j \\
P_{iL} + P_{iR} &= 2y_i\partial_- - x^+\partial_i
\end{aligned}
$$

- There is now translational invariance in the transverse directions, generated by the Killing vectors $P_{iL} - P_{iR}$.

- Geodesic motion is correspondingly unbounded in these directions

- The holographic generator $2P_- = J_L + J_R$ still contains a rescaling part proportional to p_+, but also a quadratic part.

Generalized conformal wave backgrounds

The pp-wave background in conformal coordinates can be generalized at little cost to the D-dimensional conformal wave background

$$ds^2 = \frac{1}{f^2(x^+)}(dx^+dx^- + \sum dy_i^2) \ , \qquad H = \sqrt{f''/f^5}\ dx^+ \wedge \omega$$

This solves the eom for any convex profile, $f'' > 0$. The homogeneous pp-wave arises for $f = \sqrt{1 + {x^+}^2}/2$

- For $f \sim x^+$, this can be transformed into flat space by $(x^+, x^-, y^i) \to (-1/x^+, x^- + y_i^2/x^+, y^i/x^+)$: the $x^+ \to \pm\infty$ regions are asymptotically flat.

- The geodesic motion in the transverse directions is free:

$$\begin{aligned} y_i(\tau) &= y_i + \frac{p_i}{p_+}(x^+(\tau) - x^+) \\ x^-(\tau) &= x^- - \frac{p_i^2}{p_+^2}(x^+ - x^+) + \frac{p_-}{p_+}\int_{x^+_0}^{x^+(\tau)} \frac{du}{f^2(u)} \end{aligned}$$

- The wave equation can be integrated for any profile:

$$\phi = f^{\frac{D-2}{2}}(x^+) \exp\left(i\left(p_+x^- + p_iy_i - \frac{p_T^2}{4p_+}x^+\right) - i\frac{m^2}{4p_+}\int_{-\infty}^{x^+} \frac{du}{f^2(u)}\right)$$

- The only effect is a phase shift, $\Delta\varphi = \pi m^2/(\mu p_+)$ for NW: varying refraction index.

4. Holography at work
Boundary-to-bulk prop. and boundary sym.

Since the wave equation is first order in ∂_+, it can be integrated with arbitrary initial conditions at x_0^+:

$$\phi_0(x^+, x^-, x) = \int dp_+ \; dp_i \; \chi(p_+, p_i) \left(\frac{f(x^+)}{f(x_0^+)} \right)^{\frac{D-2}{2}}$$

$$\times \exp \left[i(p_+ x^- + p_i y_i) - i \frac{p_i^2}{4p_+}(x^+ - x_0^+) - i \frac{m^2}{4p_+} \int_{x_0^+}^{x^+} \frac{du}{f^2(u)} \right]$$

where $\chi(p_+, p)$ is the Fourier transform of the boundary data at $x^+ = x_0^+$.

The Killing vectors commute with the Laplacian. Hence they can be commuted through the propagator, to act on the boundary data only. The dependence on x_0^+ can be eliminated by redefining

$$\phi(p_+, p_1, p_2) = \tilde{\phi}(p_+, p_1, p_2) \exp \left(-i \frac{p_1^2 + p_2^2}{4p_+} x_0^+ \right)$$

The non-trivial generators, back in position space, are:

$$\begin{aligned} \tilde{J}_L + \tilde{J}_R &= -(y_1^2 + y_2^2)\partial_- + (4m^2 - \Delta)\partial_-^{-1} \\ \tilde{P}_{iL} + \tilde{P}_{iR} &= 2y_i \partial_- \end{aligned}$$

where $\partial_-^{-1} = \int dx^-$, $\partial_-^{-1}\partial_- = \partial_- \partial_-^{-1} = 1$

The representation is singular at $p_+ = 0$, as usual in light-cone quantization.

Boundary correlators from the bulk

For simplicity, we consider a massive scalar field in the bulk,

$$S = -\frac{1}{2}\int d^D x\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\ \partial_\nu\phi + m^2\phi^2 + \frac{2\lambda}{3}\phi^3\right)$$

Expanding perturbatively $\phi = \phi_0 + \lambda\phi_1 + \ldots$,

$$(\Delta - m^2)\phi_0 = 0\ ,\ \ (\Delta - m^2)\phi_1 = \phi_0^2$$

ϕ_0 is determined from the boundary data by the boundary-to-bulk prop. ϕ_1 follows from ϕ_0 by the bulk-to-bulk propagators. We evaluate the action on the classical solution between x_0^+ and x_1^+:

$$\begin{aligned} S_0 &= \int \left[\partial_+(f^{2-D}\phi_0\partial_-\phi_0) + \partial_-(f^{2-D}\phi_0\partial_+\phi_0)\right] \\ S_1 &= \lambda\int \left[\partial_+(f^{2-D}\phi_1\partial_-\phi_0) + \partial_-(f^{2-D}\phi_1\partial_+\phi_0) + \partial_+\partial_-()\right] \end{aligned}$$

Keeping the boundary terms at $x^+ = x_0^+, x_1^+$, we find $S_0 = 0$: the two-point functions vanish ! In general, Ward identities of the $H_L \times H_R$ symmetry allow

$$F_2(p, p_+; q, q_+) = f(p_+)\delta(p_+ + q_+)\delta^{(D-2)}(p+q)$$

Three point functions

At next to leading order in λ, we can extract the three point function. A renormalization of the boundary fields is necessary,

$$\phi_0 \to \phi_0 \; f(x_0^+)^{\frac{2-D}{2}}$$

The result is a non-trivial function,

$$\begin{aligned} F_3(p,q,r) &= \lambda \frac{(2\pi)^{D-1}}{12} \int_{x_0^+ \to}^{\infty} du \; f(u)^{\frac{D-6}{2}} \\ &\times \exp\left[-\frac{im^2}{4} \left(\frac{1}{p_+} + \frac{1}{q_+} + \frac{1}{r_+} \right) \gamma(u, x_0^+) \right] \\ &\times \exp\left[-\frac{i}{4} \left(\frac{p^2}{p_+} + \frac{q^2}{q_+} + \frac{r^2}{r_+} \right) (u - x_0^+) \right] \end{aligned}$$

with

$$\gamma(x^+, x_0^+) \equiv \int_{x_0^+}^{x^+} \frac{du}{f^2(u)}$$

It can be checked that it satisfies the Ward identity of $H_L \times H_R$.

Null Cauchy problem

We have so far assumed particular (vanishing) boundary conditions at $x^- \to -\infty$. Those however provide the missing "normalizable" modes (in AdS terminology).

The wave equation can however be integrated recursively for any configuration at $x^- \to \infty$, using a procedure similar as for charged particles in a null electric field:

Tomaras, Tsamis, Woodard

Defining $\phi_- = \phi f^{(2-D)/2}$ and $\phi_+ = \partial_- \phi_-$,

$$\phi_+(x^+, x^-, y) = \int_{-\infty}^{+\infty} \frac{dp_+}{2\pi} \left\{ \int_{x_0^-}^{+\infty} dv \; e^{ip_+(v-x^-)} \right.$$
$$\exp\left(\frac{i}{4p_+} \int_{x_0^+}^{x^+} \left(\frac{m^2}{f^2(u)} - \Delta_T \right) du \right) \phi_+(x_0^+, v, y)$$
$$+ \frac{ie^{-ip_+(x^- - x_0^-)}}{4p_+} \int_{x_0^+}^{x^+} du \; \left(\frac{m^2}{f^2(u)} - \Delta_T \right)$$
$$\left. \left[\exp\left(\frac{i}{4p_+} \int_{u}^{x^+} \left(\frac{m^2}{f^2(u')} - \Delta_T \right) du' \right) \right] \phi_-(u, x_0^-, y) \right\}$$

where $p_+ \to p_+ + i\epsilon$ is understood.

It would be interesting to study the mixing between these operators.

Summary 1 - strings in pp-wave

- NSR strings in NS pp-wave backgrounds are in principle amenable to covariant quantization. We have presented the spectrum with emphasis on spectral flow and long strings.

- The $p_+ = 0$ sector is accessible in this formalism. It consists of modes traveling along the wave, with same interactions as in flat space. What is its real meaning ?

- Open problems: compute one-loop amplitude (no real Euclidean continuation), scattering amplitudes (solving KZ equations is the most direct way), open string spectrum on D-branes...

- The similarity with Euclidean AdS_3 is striking. Indeed $Sl(2)$ generators can be constructed in the envelopping algebra of H_4,

  $$E_+ = \frac{i}{2K} P_1^2 \,, \quad E_- = \frac{i}{2K} P_2^2 \,, \quad H = -\frac{1}{2K}(P_1 P_2 + P_2 P_1)$$

  Can one use this observation to make the connection effective ?

Summary 2 - null holography

We have proposed an holographic correspondence between strings in the pp-wave bulk and a putative "gauge" theory living in a $D-1$ dimensional degenerate space $(x^-, y, x_\perp)$ at fixed x^+. Important differences with usual set-up are

- Evolution along the holographic direction x^+ is first order. Only one set of modes, or rather there are other modes at $x^- \to -\infty$. Can one take those into account in holographic computations ?

- Conformal invariance in the dual theory is replaced by invariance under a (non-local) Heisenberg-type group, $H_D \times H_D$. Ward identities do not restrict 2 and 3-point functions as much as in CFT case.

- Correlations functions can be computed by evaluating the classical action on a solution with specified boundary conditions. 2-point functions are zero in the supergravity limit.

- Can one formulate the dual "gauge" theory microscopically ? Infinite momentum is T-dual to critical electric field: is NCOS relevant ?

Outlook - null holography (continued)

- Our choice of holographic screen may seem rather ad hoc. To what extent is the Poincaré patch decoupled from the full geometry ?

- A concurrent proposal is to take the radial direction in transverse space as the holographic direction: is the Cauchy problem well posed ? How are symmetries represented on the boundary ?

  Leigh, Okuyama, Rozali

- Just as Minkowski is conformal to a patch of the static Einstein universe $R \times S^{D-1}$, so is the ppwave. The conformal factor is everywhere non-singular,

$$ds^2_{pp} = \left[v^2 + \left(1 + \frac{1}{4} x_i^2 \right)^2 \right] (-dt^2 + d\Omega^2_{D-1})$$

  and all the Einstein universe seems to be covered (except possibly for a conspicuous null geodesic). Is their an holographic dual at $t \to \pm\infty$?

  Maldacena