Fuzzy branes and Solitons

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Foam, fudge and fuzz

String theory is probably very different from usual local field theories at short distances. Evidence include the softness of high energy fixed angle scattering (in perturbation theory), T-duality, the string uncertainty principle...

What it looks like is open to imagination: quantum gravity foam ? parton-like behavior ? non-commutative geometry ? non-associative ? non-Archimedian ? nonsense ?

While the situation with full-fledged string theory seems rather hopeless, the observation that D-brane coordinates are in general **non-commuting matrices** suggests that **non-commutative geometry** may be relevant to describe *open string theory*.

A precise connection was made by Connes-Douglas-Hull-Schwarz in the case of D-branes in the presence of **background B-field** (or electromagnetic field). The validity is expected to be more general.

Whether this open string fuzziness has any relation to quantum gravity is unclear, although AdS/CFT suggests that quantum exclusion principles could be accounted by the dual D-brane description.

Non-commutativity and fuzziness

The non-commutativity of D-branes manifests itself in two different ways:

- Since the D-brane degrees of freedom correspond to boundary operators in the CFT, the **ordering** of the operators on the boundary is important and makes the vertex algebra **non-commutative**:

  ex: D-brane in B-field: $e^{ip_1 X} \cdot e^{ip_2 X} = e^{ip_1 \theta p_2} e^{i(p_1+p_2)X}$

  The worldvolume theory becomes a **non-commutative** gauge theory.

- In the presence of Chan-Paton factors, the vertex algebra is tensored with a **matrix algebra**, hence again non-commutative

  The worldvolume theory becomes a **non-Abelian** gauge theory with Higgs fields in the adjoint.

  Rq: Tree level amplitudes for 1 D-brane in B-field are isomorphic to tree amplitudes for N D-branes in no B-field !

This is an characteristic feature of open strings. For closed strings, the braiding of vertex operators on the worldsheet may play a similar role as the ordering on the boundary.

In this talk, I will be interested in **non-commuting** configurations of the D-brane matrix coordinates.

Fuzzy branes

Non-commuting configurations of the $N \times N$ matrix coordinates allow to represent interesting topologies:

Ex1: monopole configuration on a pair of D3-branes

The Higgs field Φ has two opposite eigenvalues describing the two D3 worldvolumes. The divergence at $r = 0$ corresponds to a **spike** identified as a D1-brane stretched between the 2 D3.

Callan,Maldacena;Gibbons

Ex2: Long strings

Consider a D1 wrapped on S_1 with a non-trivial holonomy for a matrix coordinate X. X can be **locally** diagonalized, but the eigenvalues are permuted along the circle. This describes multiple D-strings wrapped on the circle. In Matrix string theory, these are **multi-string states** with longitudinal momentum.

DVV, Banks, Motl

Ex3: spherical branes

take 3 coordinates of a Dp-brane to generate the $SU(2)$ algebra

$$[X^i, X^j] = \epsilon^{ijk} X^k$$

in some representation R. If R is an irrep of spin j, this describes a D$(p+2)$ brane wrapped on a **fuzzy sphere** of radius $\sim j$.

Rey, Kabat Taylor

In this talk, we will encounter examples of the 3rd type.

Charge and dipolar moment

The identification of D-branes with fuzzy coordinates and higher-dimensional D-branes requires a matching of the charges under the RR fields.

For non-trivial **gauge** configurations on D-branes, the Wezz-Zumino coupling $\int R_{p+1} + FR_{p-1} + F^2 R_{p-3}$ imply that the brane carries the charge of a **lower-dimensional** D-brane. This gives the K-theory classification of D-brane charges.

For **fuzzy** D-branes, the Non-Abelian extension of the Wezz-Zumino coupling account for a charge of **higher dimensional** D-brane:

$$S_{WZ} = \int Tr A_{\mu\nu\ldots} + [X^j, X^k] A_{\mu\nu\ldots jk} + \ldots$$

Since $Tr[,] = 0$, there is no net charge, but if A has a gradient, the coupling becomes $Tr X^i [X^j, X^k] \partial_i A_{\mu\nu\ldots jk}$ which is that of a dipole.

For fuzzy spheres, the two hemispheres have opposite charges, so the state is non-BPS in general.

One can also create a true D$(p+4)$-dimensional brane from a Dp, due to a coupling $\int Tr A_{\mu\nu\ldots ijkl} [X^i, X^j][X^k, X^l]$.

Q: Is there a **bottom-up** classification of D-brane charges analogous to K-theory ?

Classical Fuzzy branes

The coordinates of D-branes in flat space are only non-commutative when **excited** away from their ground states. At rest, the coordinates can be simultaneously diagonalized and the eigenvalues describe the individual positions of the D-branes.

For *topologically non-trivial boundary conditions* or *non-trivial backgrounds* however, non-commuting configurations may be **stable**.

This can follow from supersymmetry:

Ex1: The vacua of softly broken $N = 4$ SYM have non-commuting vevs for the scalars, classified by the representations of $SU(2)$

Vafa Witten

Ex2: a D-string ending on a D3-brane pops up into a fuzzy sphere in the vicinity of the D3

Diaconescu

or be ensured by energetics:

Ex3: a Dp-brane in a constant RR field strength pops up into a spherical fuzzy D$(p+2)$-brane.

Myers

Ex4: D-branes in $SU(2)_k$ WZW models correspond to fuzzy spheres S_2 embedded S_3.

Bachas Douglas Schweigert
Alexeev,Recknagel,Schomerus

All these examples have in common the fuzzy sphere...

Dielectric branes and fuzzy spheres

Consider a D3-brane along 0123, in a gauge background

$$R_{0123ij} = R\epsilon_{ijk}x^k$$

The scalar potential for the transverse coordinates reads

$$V = Tr[X^i, X^j]^2 + 2R\epsilon_{ijk}[X^i, X^j]X^k$$

and the vacua correspond to

$$[X^i, [X^i, X^j] + R\epsilon_{ijk}X^k] = 0$$

Solutions correspond either to commuting matrices, or to representations of $SU(2)$.

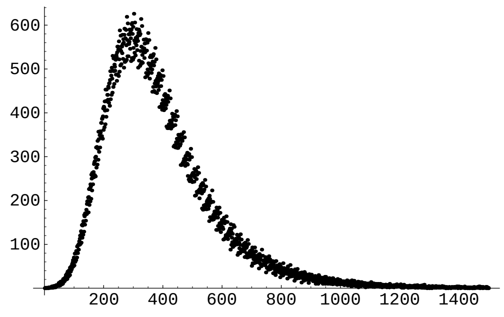
Myers

In the latter case, the brane has the topology of a **fuzzy sphere**. This is a generalization of the **polarization in dielectric media**.

The minima of the potential have energy

$$V(R) = -\text{Casimir}(R) = -R^4 \sum j(j+1)(2j+1)$$

The **ground state** corresponds to the **irreducible** representation. There are in addition many semi-stable minima, which are degenerate for $N > 7$.



The model can be **symmetrized** by introducing a mass term, corresponding to a quadrupole field strength:

$$S \to \sum Tr|[X^i, X^j] + R\epsilon_{ijk}X^k|^2$$

Fuzzy spheres and solitons

Fuzzy branes often admit **different** ground states, which may be degenerate in supersymmetric cases. In the examples we discussed above, there are stable configurations associated to **any representation of** $SU(2)$. These correspond to configurations of fuzzy spheres of different radii.

Transitions between different ground states are interesting for many reasons

Ex1: domain walls in $N = 1^*$ Super-Yang-Mills, SUSY of the true vacuum in $N = 1^*$ quantum mechanics, Scattering with transfer of longitudinal momentum in M(atrix) theory.

Ex2: monopoles in gauge theories with enhanced symmetry, RG flow in the presence of external magnetic sources, Wilson lines in higher representations

Ex3: Decay rate of the false vacuum

Ex4: Boundary RG flow in WZW and minimal models

It is the purpose of this talk is to study **solitonic** configurations of fuzzy D-branes that **interpolate** between different vacua, their moduli space and their physical interpretation.

Plan of the talk

1. Domain walls in $N = 1^*$ SYM

   Bachas Hoppe P

2. Holographic dual and fuzzy 5-branes

   Polchinski Strassler, Frey

3. Monopoles in theories with enhanced gauge symmetry

4. Radial D-strings in $AdS_5 \times S^5$ and holography

5. D-branes in $SU(2)_k$ and boundary RG flow

   Alekseev Recknagel Schomerus Yu
   Bachas Douglas Schweigert

Vacua of $N = 1^*$ SYM

The $\mathcal{N} = 4$ Yang-Mills theory in four dimensions admits a **relevant** deformation lifting all of its moduli space while preserving 4 supersymmetries:

$$W = \text{Tr} \sum \left(\frac{1}{6} \, \epsilon^{abc} \Phi^a [\Phi^b, \Phi^c] - \frac{1}{2} m \; \Phi^a \Phi^a \right) ,$$

which breaks the R-symmetry $SO(6)$ to $SO(3)$.

The supersymmetric classical vacua of the theory obey

$$\frac{\partial W}{\partial \Phi^a} = \frac{1}{2} \epsilon^{abc} [\Phi^b, \Phi^c] - m \Phi^a = 0 \; , \quad D = 0$$

with $D = \sum [\Phi^i, \Phi^i] = 0$. and are hence in one-to-one correspondence with N-dimensional **unitary representations of** $SU(2)$,

Vafa,Witten

$$\Phi^a = m \rho^a \; , \quad [\rho^a, \rho^b] = \epsilon^{abc} \rho^c \; , \quad \rho^a = -(\rho^a)^\dagger$$

The representation ρ is in general reducible,

$$\rho \; = \; \bigoplus \; n_{2j+1} \; [2j+1]$$

The unbroken gauge symmetry is then

$$G_{(\rho)} \equiv \prod U(n_j)/U(1)$$

The number of classical vacua grows as $\sim e^{\sqrt{N}}$. Quantum mechanically, the vacua with $U(n)$ non-Abelian factors split into n distinct vacua.

Q: *Are there SUSY domain walls (or instantons in the QM) interpolating between arbitrary vacua ?*

$N = 1^*$ *Domain walls*

Bachas, Hoppe, P.

Domain walls in $N = 1$ SUSY theories are BPS iff

$$\left(\frac{d}{dx}\Phi^a\right)^\dagger = e^{i\alpha}\frac{\partial W}{\partial \Phi^a}\ ,\ D = 0$$

where α is a constant phase, chosen =-1. The scalar fields thus follow the **gradient flow** of W. The tension of a domain wall is

$$T = \frac{2}{g^2}\ |\Delta W|$$

Choosing antihermitian matrices, we have in our case

$$\frac{d\Phi^a}{dx} = \frac{1}{2}\epsilon^{abc}[\Phi^b, \Phi^c] - m\Phi^a\ .$$

with boundary conditions

$$\begin{matrix} \Phi^a(x \to -\infty) \to m\rho^a_- \\ \Phi^a(x \to +\infty) \to m\rho^a_+ \end{matrix}\ , \quad W(\rho_-) > W(\rho_+)$$

This is a deformation of Nahm's equations. Elementary solutions are given by

$$\Phi^a(x) = \frac{m}{1 + e^{m(x-x_0)}}\ \rho^a_-\ ,$$

but for arbitrary (ρ_-, ρ_+), we need to allow for general matrices.

Variations on Nahm's equations

Under the change of variables

$$\Phi^a = -e^{-mx} X^a \qquad \text{and} \qquad s = e^{-mx}/m \ ,$$

the equation become **exactly** Nahm's equations

$$\frac{dX^a}{ds} = \frac{1}{2}\epsilon^{abc}[X^b, X^c] \ .$$

but the boundary conditions are now

$$X^a(s \to 0) \sim -\frac{\rho^a_+}{s} + \text{finite} \ ,$$

$$X^a(s \to \infty) \sim -\frac{\rho^a_-}{s} + \text{subleading} \ .$$

In contrast to the standard monopole problem, we have a **semi-infinite** interval $]0, +\infty[$, and different **reducible** poles at both ends.

Fortunately, this very problem has been considered by Kronheimer in the context of self-dual configurations on $S^4 \sim R \times S^3$ invariant under $SO(3)$.

Moduli space of $N = 1^*$ domain walls

As for general hyperkähler quotients, Nahm's equations can be split into one complex and one real equation in terms of the complex variables

$$\alpha = \frac{1}{2}(\Phi^0 - i\Phi^1) \ , \qquad \beta = -\frac{1}{2}(\Phi^2 + i\Phi^3) \ ,$$

$$\begin{aligned} \frac{d\beta}{dx} + m\beta + 2[\alpha, \beta] &= 0 \\ \frac{d}{dx}(\alpha + \alpha^\dagger) + m(\alpha + \alpha^\dagger) + 2([\alpha, \alpha^\dagger] + [\beta, \beta^\dagger]) &= 0 \end{aligned}$$

- The complex equation is invariant under **complex gauge rotations**,

  $$\alpha \to g\alpha g^{-1} - \frac{1}{2}\frac{dg}{dx}g^{-1} \ , \quad \beta \to g\beta g^{-1}$$

  It is locally solved by a pure gauge configuration. The global existence can be discussed in algebraic terms.

- The real equation is invariant under **unitary** gauge rotations only. One has to pick a representative g_0 in the complex conjugacy class of g. This can be solved by variational techniques.

Donaldson;Kronheimer

Define

$$H_\pm = \rho_\pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X_\pm = \rho_\pm \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y_\pm = \rho_\pm \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Any solution of the complex equation is gauge equivalent under G^C to

$$x \in]-\infty, 0] : \qquad \begin{cases} \alpha_1(x) & = & \frac{1}{2} H_- \\ \beta_1(x) & = & Y_- \end{cases}$$

$$x \in [0, \infty[: \qquad \begin{cases} \alpha_2(x) & = & \frac{1}{2} H_+ \\ \beta_2(x) & = & Y_+ \quad + e^{-2x}\; e^{-H_+ x} \cdot Z \cdot e^{H_+ x} \end{cases}$$

where $Z \in \mathcal{Z}(\rho_+)$, the centralizer of X_+. At $x = 0$:

$$Y_+ + Z \in \mathcal{N}(\rho_-) \cap \mathcal{S}(\rho_+)$$

where

$$\mathcal{N}(\rho) = \text{Complex conjugacy class of } X(\rho)$$

$$\mathcal{S}(\rho) = Y(\rho) + \mathcal{Z}(\rho)$$

The element $Y_+ + Z$ in the intersection $\mathcal{N}(\rho_-) \cap \mathcal{S}(\rho_+)$ **uniquely** specifies the solution of to a complex gauge transformation. of

Kronheimer's theorem

Thm: the moduli space of BPS domain walls interpolating between two $SU(2)$ reps ρ_- and ρ_+ is given by the intersection

$$\mathcal{M}(\rho_-, \rho_+) = \mathcal{N}(\rho_-) \cap \mathcal{S}(\rho_+) \ .$$

(i) There is a one-to-one correspondence between the conjugacy classes of embeddings of $SU(2)$ into $\mathcal{G}$ and the G^C-orbits of **nilpotent** elements in $\mathcal{G}^C$. For $G = SU(N)$, this is provided by the Jordan canonical form.

(ii) $\mathcal{S}(\rho_+)$ is the solution to a **linear** problem. $\mathcal{N}(\rho_-)$ is given by solutions to a set of **polynomial** equations. The moduli space is therefore an algebraic variety.

(iii) For $\rho_- = \rho_+$, the moduli space reduces to a point Y, which describes the vacuum (no domain wall).

ex: $N = 3,\ \rho_- = 3,\ \rho_+ = 2 \oplus 1$

$$S(\rho) \ni s = \begin{pmatrix} a & b & c \\ 1 & a & 0 \\ 0 & d & e \end{pmatrix}$$

$$s^3 = 0 \Rightarrow s = \begin{pmatrix} a & -3a^2 & c \\ 1 & a & 0 \\ 0 & d & -2a \end{pmatrix} \text{ and } cd = 8a^3$$

This is an ALE space isomorphic to C^2/Z_3.

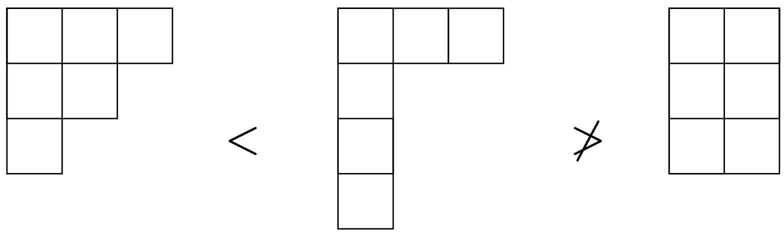
Condition for existence of domain walls

The intersection $\mathcal{N}(\rho_-) \cap \mathcal{S}(\rho_+)$ is non-empty iff

$$\mathcal{N}(\rho_+) \subset \overline{\mathcal{N}(\rho_-)} \ .$$

This defines a **partial order** between N-dimensional representations of $SU(2)$, or equivalently nilpotent orbits, or partitions of N. The trivial representation smaller than any other, and the irreducible representation is bigger than any other.

The condition is simply checked in terms of Young tableaux. To any partition of $N = n_1 + n_2 + \ldots n_k$ with decreasing n_i, associate the Young tableau with k lines:



$\mathbf{3} \oplus \mathbf{2} \oplus \mathbf{1} \qquad \mathbf{3} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1} \qquad \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{2}$

In particular, the number of irreps can only increase as W decreases.

$\rho_-\backslash\rho_+$	$\mathbf{2}$	$\mathbf{1}^2$
$\mathbf{2}$	0	C^2/Z_2
$\mathbf{1}^2$		0

$\rho_-\backslash\rho_+$	$\mathbf{3}$	$\mathbf{2}\oplus\mathbf{1}$	$\mathbf{1}^3$
$\mathbf{3}$	0	C^2/Z_3	6
$\mathbf{2}\oplus\mathbf{1}$		0	4
$\mathbf{1}^3$			0

$\rho_-\backslash\rho_+$	$\mathbf{4}$	$\mathbf{3}\oplus\mathbf{1}$	$\mathbf{2}^2$	$\mathbf{2}\oplus\mathbf{1}^2$	$\mathbf{1}^4$
$\mathbf{4}$	0	C^2/Z_4	4	6	12
$\mathbf{3}\oplus\mathbf{1}$		0	C^2/Z_2	4	10
$\mathbf{2}^2$			0	C^2/Z_2	8
$\mathbf{2}\oplus\mathbf{1}^2$				0	6
$\mathbf{1}^4$					0

$\rho_-\backslash\rho_+$	$\mathbf{5}$	$\mathbf{4}\oplus\mathbf{1}$	$\mathbf{3}\oplus\mathbf{2}$	$\mathbf{3}\oplus\mathbf{1}^2$	$\mathbf{2}^2\oplus\mathbf{1}$	$\mathbf{2}\oplus\mathbf{1}^3$	$\mathbf{1}^5$
$\mathbf{5}$	0	C^2/Z_5	4	6	8	12	20
$\mathbf{4}\oplus\mathbf{1}$		0	$xy^4=z^3$	4	6	10	18
$\mathbf{3}\oplus\mathbf{2}$			0	C^2/Z_2	4	8	16
$\mathbf{3}\oplus\mathbf{1}^2$				0	C^2/Z_2	6	14
$\mathbf{2}^2\oplus\mathbf{1}$					0	4	12
$\mathbf{2}\oplus\mathbf{1}^3$						0	8
$\mathbf{1}^5$							0

Additivity rule and elementary walls

From the tables, one observes that

$$d(\rho_-,\rho_+) = d(\rho_-,\rho_0) + d(\rho_0,\rho_+) \ ,$$

This suggests that all domain walls can be seen as **composite** of elementary domain walls, for which $W(\rho_-) - W(\rho_+)$ is minimal (but strictly positive) at fixed ρ_+.

The moduli of the elementary domain walls are easily understood in terms of the deformations

$$\begin{array}{ll} (a) & \text{translations} \\ (b) & \Phi^a(t) \to g\Phi^a(t)g^\dagger \ , g = cste \ , \ g\rho_+ = \rho_+ g \\ (c) & \Phi^a(t) \to \rho_+(R)(R^a{}_b\Phi^b(t))\rho_+(R)^\dagger \ , R \in SU(2) \end{array}$$

Ex: $\mathbf{3} \to \mathbf{2} \oplus \mathbf{1}$: 1 transl. + 3 rotations.

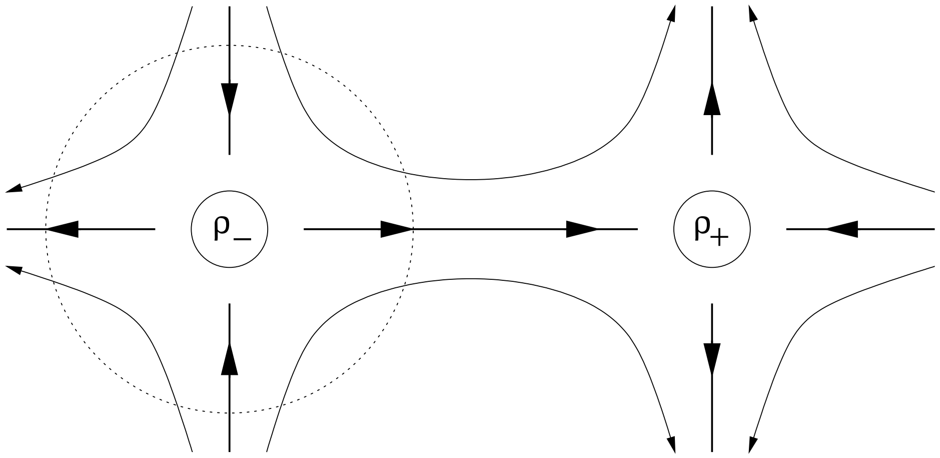
The moduli space when elementary domain walls collide is **singular**.

Morse theory and dimension of moduli space

Assuming the existence of a flow interpolating between ρ_- and ρ_+, the dimension can be computed in terms of the eigenvalues of the Hessian of W at both ends. Consider a general **gradient** flow

$$\frac{d\phi^i}{dt} = -g^{ij}\frac{\partial W}{\partial \phi^j}$$

Choose a small sphere S^{N-1} around the point ρ_- at $-\infty$:



$\phi(x \to -\infty) = \rho_-$ puts $n_>(\rho_-)$ conditions,

$\phi(x \to +\infty) = \rho_+$ puts $n_\leq(\rho_+)$ conditions.

Hence the dimension is

$$\begin{aligned} d(\rho_-, \rho_+) &= (N-1) - n_>(\rho_-) - n_\leq(\rho_+) + 1 \\ &= n_>(\rho_+) - n_>(\rho_-) \end{aligned}$$

which implies the additivity rule.

The dimension is **robust** under small deformations of the Kähler potential.

$N=1^*$ Domain walls in the gravity dual

Polchinski-Strassler, Frey

The mass deformation of $N=4$ SYM corresponds in the gravity description to switching on a **relevant** deformation on the boundary of AdS_5. It creates a singularity in the bulk which is resolved by a set of **5-branes sources** on $S_2 \subset S_5$.

In the Higgs/Coulomb phase, the 5-branes are D5-branes whose radii depend on the $SU(2)$ rep chosen for the vacuum.

Domain walls correspond to **bent D5-branes** interpolating between spheres of different radii. For charge conservation it is necessary to include 5-brane junctions.

The BPS bending equation is

$$\frac{n}{2\pi g}\partial_1\phi = \Omega\overline{W_\phi}$$

with superpotential

$$W = \frac{1}{2\pi g}\left(\frac{mn}{2}\Phi^2 + i\frac{2\sqrt{2}}{3}M\Phi^3\right) .$$

Domain walls can be constructed by gluing together elementary solutions ($\psi_v = mn/2\sqrt{2}|M|$):

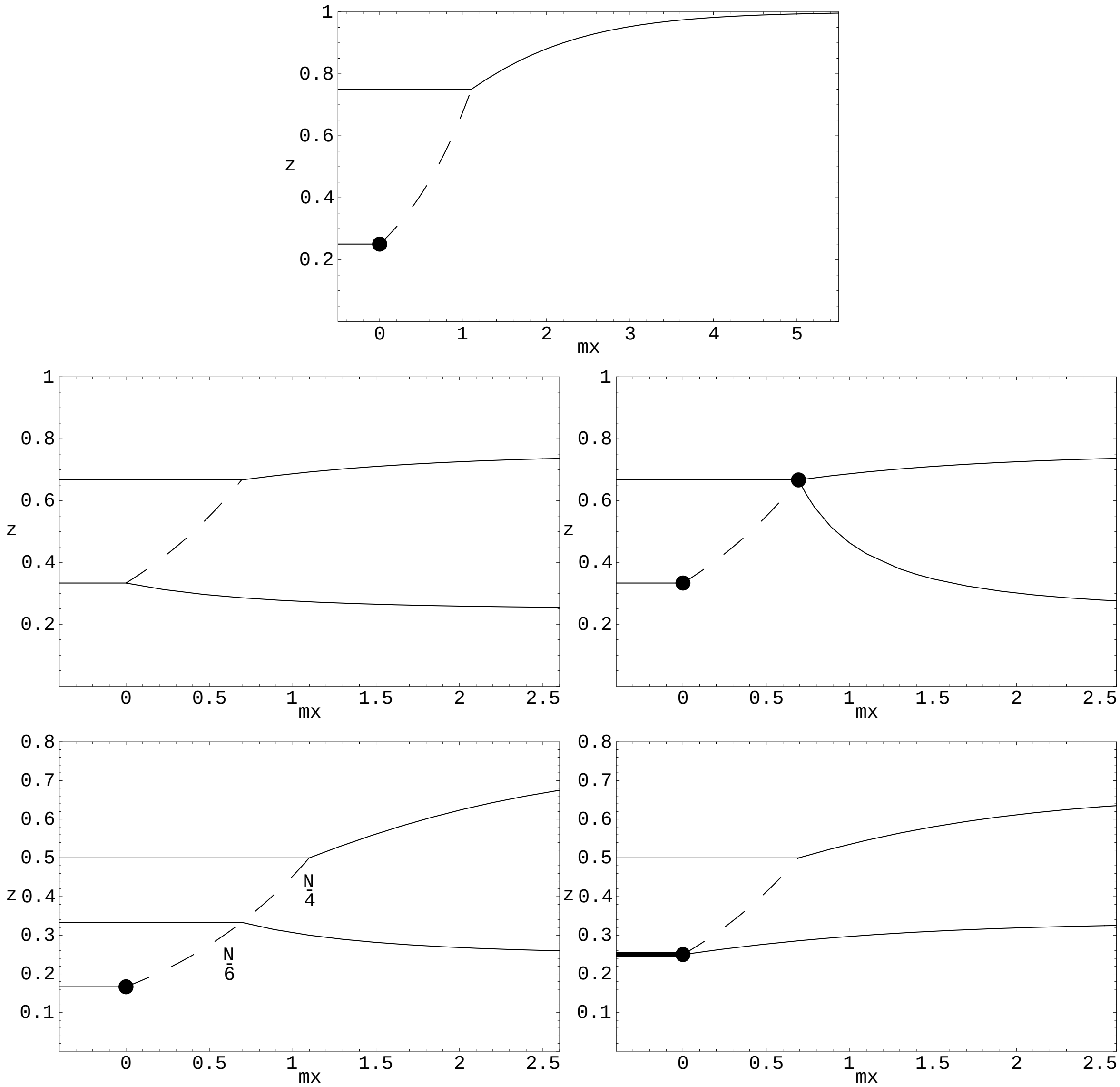
$$\begin{aligned}
\psi_<(x,x_0) &= \psi_v\frac{e^{m(x-x_0)}}{1+e^{m(x-x_0)}} \text{ for } 0<\psi<\psi_v \\
\psi_>(x,x_0) &= \psi_v\frac{e^{m(x-x_0)}}{e^{m(x-x_0)}-1} \text{ for } \psi>\psi_v\,,\ x>x_0 \\
\psi_-(x,x_0) &= \psi_v\frac{e^{m(x-x_0)}}{e^{m(x-x_0)}-1} \text{ for } \psi<0\,,\ x<x_0
\end{aligned}$$

It is also necessary to include spherical D3-branes without D5-brane charge, or **zero-5-branes**:

$$\psi_0(x, x_0) = e^{m(x-x_0)} \quad (M = 0)$$

Using charge conservation and force balance at the junction, one can construct solutions out of **spherical** D5-branes (filled), **ball-filling** D5-branes (dot) and **zero-branes** (dotted):

Frey

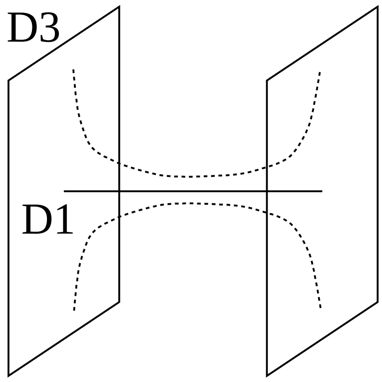


Monopoles in non-abelian phases

$N = 1^*$ domain walls satisfy the Nahm equations with unusual boundary conditions. In particular, they live on the **semi-infinite** real exis, and have **reducible** residue at 0 and ∞.

Q: what is the monopole interpretation of solutions of Nahm's equations interpolating between different $SU(2)$ reps ? In particular, solutions with a reducible residue ?

In order to answer this question, recall how Nahm equations arise from the D-brane construction of monopoles:



The coordinates of the stretched D1-branes satisfy **Nahm's equations** on the interval $[0,1]$. The boundary conditions

$$X^i(t \to 0) \sim \frac{J^i}{t} \ , \qquad X^i(t \to 1) \sim \frac{J^i}{1-t} \ ,$$

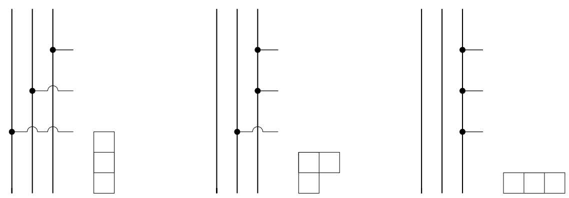
where J^i is the n-**dimensional** $SU(2)$ **irrep** correspond to the fact that the n D-strings pop up into a **fuzzy D3-brane** at $t = 0$. This follows from the coupling

$$\int_{D1} A_{0ijk} \mathrm{Tr}\ \partial_t X^i [X^j, X^k]$$

Constable;Myers;Tafjord

Magnetic sources in non-Abelian phases

A: if $X^a(s \to 0) \sim \rho^a/s$ where ρ contains r blocks, the N D-strings expand into r coinciding D3-branes. Upon separating the D3-branes, the D-strings attach according to the partition ρ:



$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$ $\quad$ $\mathbf{2} \oplus \mathbf{1}$ $\quad$ $\mathbf{3}$

The endpoint of the D-strings are magnetically charged under the Cartan subgroup. Under the non-Abelian group, they transform in the representation given by the Young tableau associated to the partition ρ.

On $[0, +\infty[$, they behave as **heavy external sources in higher representations** of the gauge group.

On $[0, 1]$: they would be finite mass dynamical quarks. It would be interesting to study the Nahm dual of these configurations.

A': The boundary condition at $+\infty$ arises more naturally in the **near horizon** geometry of a stack of D3-branes.

D-strings in $AdS_5 \times S^5$

Consider static D-strings probes stretched radially in the near-horizon geometry of $N \to \infty$ D3-branes,

$$ds^2 = L^2 \, \frac{dy^2 + dx_\mu dx^\mu}{y^2} \ , \quad \text{and} \quad C_{y0ab} = L^2 \, \frac{\epsilon_{abc} x^c}{y^5} \ .$$

The DBI+WZ action yields the energy functional

$$E = T_D L^2 \int_0^\infty \frac{dy}{2y^6} \, \text{Tr} \, \left(y^2 \frac{\partial}{\partial y} X^a - \frac{1}{2} \epsilon^{abc} [X^b, X^c] \right)^2 +$$

$$+ boundary + \ldots$$

The supersymmetric configurations of D-strings satisfy

$$y^2 \frac{dX^a}{dy} = \frac{1}{2} \epsilon^{abc} [X^b, X^c] \ .$$

The only boundary conditions that respect **scale invariance** are

$$X^a(y \to 0) = -y \rho_-^a + \text{subleading}$$

$$X^a(y \to \infty) \sim -y \rho_+^a + \text{finite}$$

After changing variables to $s = 1/y$, this is the same pb as before. In particular, the superpotential W always **decreases** away from the the boundary of AdS.

The vacuum solutions ($\rho_- = \rho_+$) correspond to **heavy magnetic sources** in the dual $\mathcal{N} = 4$ SYM theory at large N.

Domain walls and holography

We now consider unperturbed $\mathcal{N} = 4$ $U(N)$ SYM at strong 't Hooft coupling, and interpret our radial D-string from the point of view of holography.

The end of the D-string introduces a magnetic source in the gauge theory, in a representation of $U(N)$ given by the Young diagram associated to the partition of N.

Fundamental strings ending on the stack of D3 describe Wilson lines in the fundamental representation of $U(N)$. Our D-strings therefore describe **'t Hooft lines** in higher representation of the gauge group.

Domain walls change the representation of the monopole from ρ_- to ρ_+ from the UV to the IR. They should describe the RG flow following from the breaking of conformal invariance by the external source.

This is very similar to the Kondo effect, where the spin of the impurity gets **screened** by the spin of the conducting electrons. In fact the Kondo effect with k channels is precisely described by $SU(2)_k$ CFT !

The RG flow tends to increase the number of fuzzy D5-branes, hence make the representation **more antisymmetric**. Can we interpret the superpotential as an entropy ?

D-branes in $SU(2)_k$

WZW models are exactly soluble CFT's for which the spectrum of **boundary states** is known. In the large k limit, they admit a geometric description as a symmetric space with NS field-strength turned on. Can we understand the D-branes in geometrical terms ?

The gluing condition $J = -\bar{J}$ with $J = -(\partial g)g^{-1}$ shows that the brane is restricted to a conjugacy class, ie an S_2 in S_3, and that it carries an induced B-field

$$B = k(\psi - \frac{1}{2}\sin 2\psi)\sin\theta d\theta d\phi$$

Q: How can it be stable ?

Alekseev Schomerus

Assume that the brane also carries a $U(1)$ field strength with **quantized** flux,

$$F = -\frac{n}{2}\sin\theta d\theta d\phi$$

The minimization of the Born-Infeld action then gives minima at

$$\psi_n = \frac{n\pi}{k}$$

corresponding to D0-branes at the pole ($n = 0, k$) and a discrete spectrum of spherical branes.

Remarkably, the brane tension and spectrum at the minima fits the CFT computation, up to $k \to k + 2$:

$$M_n = 4\pi k\alpha' T_{D2}\sin\left(\frac{n\pi}{k}\right)$$

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Worldvolume theory on D-branes on $SU(2)_k$

By studying the OPE of boundary operators in the WZW model, one can show that the vertex algebra is a quantum fuzzy sphere,

$$Y_i^I * Y_j^J = \sum_{K,k} \left[\begin{array}{ccc} I & J & K \\ i & j & k \end{array} \right] \left\{ \begin{array}{ccc} I & J & K \\ \alpha & \alpha & \alpha \end{array} \right\}_q Y_k^K$$

with 6J coefficients of $U_q(SU(2))$ with $q = e^{\frac{2\pi i}{k+2}}$. In the large k limit, this reduces to the algebra of $(2\alpha+1) \times (2\alpha+1)$ matrices.

Alekseev Recknagel Schomerus Yu

The boundary fields in the $|\alpha>$ boundary states correspond to three matrix valued coordinates in this algebra. Their effective action can be computed at large k and yields

$$S = Tr\left(-\frac{1}{4}[X^i, X^j]^2 + \frac{1}{3}\epsilon^{ijk} X^i [X^j, X^k]\right)$$

We recognize Myers' action for branes in a constant field strength. Vacua correspond to

$$[X^i, [X^i, X^j] + \epsilon_{ijk} X^k] = 0$$

so that they either commuting matrices, describing D0-branes on S^3, or D2-branes on fuzzy spheres $S_2 \subset S^3$, of spin $j = \alpha$.

The value of the action on a representation is again given by the Casimir.

Mass and Boundary entropy

The tension of the fuzzy branes is given by the boundary entropy,

$$g = < 1|\alpha > = \left(\frac{2}{k+2}\right)^{1/4} \frac{\sin\frac{(2j+1)\pi}{k+2}}{\sin\frac{\pi}{k+2}}$$

At large k, this reduces to

$$g = \frac{2^{1/4}\sqrt{\pi}}{(k+2)^{3/4}}\left((2j+1) - \frac{(2j+1)^3\pi^2}{6(k+2)^2} + \dots\right)$$

For a superposition of fuzzy branes with N units of D0-brane charge $\sum(2j+1) = N$ and $\sum j(j+1)(2j+1) = C(R)$:

$$g(R) = g = \frac{2^{1/4}\sqrt{\pi}}{(k+2)^{3/4}}\left(N - \frac{(4C(R)+N)\pi^2}{6(k+2)^2} + \dots\right)$$

Up to R-independent terms, the mass is therefore proportional to the Casimir of the representation, as expected.

Under RG flow, the boundary state $|\alpha >$ will decay to a state with **a lower g-factor**. The irreducible representation is therefore the endpoint of the flow.

Q: *Which of the boundary states can be related by the flow ?*

RG flow and Solitons

Q: *Can we relate our solitons interpolating between $SU(2)$ representations to boundary RG flows ?*

The RG equation should have fixed points for solutions of eom, so

$$\mu \frac{d}{d\mu} X^i \propto [X^j, [X^j, X^i] + \epsilon_{ijk} X^k] + X^k$$

In contrast to the $N = 1^*$ case, the equation of motion is not SUSY. In a SUSY embedded $SU(2)_k$ model, such as the near-horizon geometry of the NS5, there should be a mass term generated, so that the action for the coordinates X^i comes from the $N = 1^*$ superpotential.

Still, our solitons follow the BPS gradient flow, instead of the non-BPS one. Could it be that by restricting to SUSY perturbations, the RG flow becomes

$$\mu \frac{d}{d\mu} X^i = \epsilon_{ijk} [X^j, X^k] + X^i \text{ ?}$$

Conjecture: the boundary RG flow in the WZW model can only relate fuzzy spheres such that $R_1 < R_2$ for the partial order of $SU(2)$ reps.

Concluding remarks

Fuzzy geometries are generic in backgrounds with **electric or magnetic fluxes** such as AdS backgrounds or WZW models: the tension of the brane is balanced by the flux and **stable states** are found.

"**Topological fuzzy transitions**" can occur between different vacua, either in Minkovski space (domain walls, stretched strings), or Euclidean space (instantons), or in RG space. The entropy should increase, but this is not the only condition: there is a **partial order** on fuzzy spheres.

Does this D-brane fuzziness have anything to do with the **non-locality** expected from quantum gravity ? From the IR/UV correspondance, it should rather be related to **infrared** effects in gravity. Indeed the D-brane fuzziness has been argued to explain the string exclusion principle.

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