Non-Gaussian Theta series and the supermembrane

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Strings vs. membranes

- In the Polyakov formulation and after going to the conformal gauge, string theory is Gaussian on the worldsheet:

$$S = \frac{1}{l_s^2} \int d^2\sigma \sqrt{\gamma}\ \gamma^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j\ G_{ij} + i\ \epsilon^{\alpha\beta}\ \partial_\alpha X^i \partial_\beta X^j\ B_{ij}$$

- By contrast, membranes are interacting on their world volume, including cubic interactions:

$$S = \frac{1}{l_p^3} \int d^3\sigma \sqrt{\gamma} \left(\gamma^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j\ G_{ij} - 1 \right) + i\ \epsilon^{\alpha\beta\gamma}\ \partial_\alpha X^i \partial_\beta X^j \partial_\gamma X^k\ C_{ijk}$$

Bergshoeff Sezgin Townsend

There is neither a conformal gauge nor a genus expansion.

- Yet supermembranes (or their M(atrix) regularization) are the only candidates to date to describe the microscopic degrees of freedom of M-theory. In particular κ symmetry implies 11D SUGRA eom.

Q: Can we tame the membrane non-linearities ?

BPS strings and Gaussian theta series

- For specific "BPS saturated" amplitudes, supersymmetry guarantees the cancellation between bosonic and fermionic fluctuations, leaving the contribution of bosonic zero-modes.

- E.g, the $Riemann^4$ amplitude in type IIA/B compactified on a torus T^n reads at one-loop

$$f_{R^4}^{1-loop} = \int_{U(1)\backslash Sl(2)/Sl(2,Z)} \frac{d^2\tau}{\tau_2^2} Z_n(\tau; g, B)$$

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where Z_n is the partition function

$$Z_n = V_n \sum_{m^i, n^i \in Z} \exp\left(-\pi \frac{|m^i + n^i \tau|^2}{\tau_2} + 2\pi i m^i B_{ij} n^j \right)$$

for the constant winding configurations

$$X^i = m^i \sigma_1 + n^i \sigma_2 \ , \qquad \gamma_{\alpha\beta} = \frac{1}{\tau_2} \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}$$

- Z_n is manifestly invariant under the modular group $Sl(2, Z)$, and also (manifestly after Poisson resummation) under the T-duality group $SO(n, n, Z)$.

- In fact, Z_n is a standard symplectic theta series

$$Z_n(g, B; \tau) = \theta_{Sp}(T) := \sum_{m \in Z^{2n}} \exp\left(2\pi i \ m^I T_{IJ} m^J\right)$$

restricted to $Sl(2, Z) \times SO(d, d, Z) \subset Sp(2n, Z)$.

BPS membranes and non-Gauss. theta series

- Similarly, the insertion of four graviton vertices on the membrane with topology T^3 just saturates the fermionic zero-modes, and leaves the partition function of the constant winding modes,

$$X^i = n^i_\alpha \sigma^\alpha, \qquad \gamma = cste \in Gl(3)/SO(3)$$

- Invariance under the modular group $Sl(3, Z)$ and the target-space U-duality group

$$E_{n+1}(Z) \supset SO(n, n, Z) \bowtie Sl(n+1, Z)$$

Hull Townsend

should fix the summation measure, related to the index of $2+1$ dimensional $U(N)$ SYM for $N = \det(n^i_\alpha)$ coinciding membranes.

- One should construct a theta series invariant under a larger group containing $R^+ \times Sl(3) \times E_{n+1}$:

$$M/T^3: \qquad [Sl(3) \times Sl(2)] \times [R \times Sl(3)] \subset E_6$$
$$M/T^6: \qquad E_6 \times Sl(3) \subset E_8$$

BP Nicolai Plefka Waldron

BPS membranes and exact amplitudes

- Integrating over the worldvolume moduli, i.e. the fundamental domain of $Gl(3)/SO(3)$, one should reproduce the full non-perturbative R^4 amplitude, including toroidal membrane instantons:

$$\int_{SO(3)\backslash Sl(3)/Sl(3,Z)\times R^+} d\gamma\ Z_d(\gamma; G, C, \dots)$$

$$? = \mathrm{Eis}^{E_d(Z)}_{string;s=3/2}(G, C, \dots)$$

Green Gutperle Vanhove
Kiritsis Obers BP

- A naive attempt based on the Polyakov action for the membrane and assuming unit summation measure reproduces the correct instanton saddle points and mass spectrum, but the summation measure / degeneracy is off: (unfortunately) math has to rescue string theory, not reverse.

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- $E_{n\geq 6}$ is not contained in a symplectic group. In addition one expects a cubic action due to coupling to C_3: we need non-Gaussian theta series.

Non-Gauss. Poisson resum. : a toy model

- The invariance of the standard theta series under $\tau \to -1/\tau$ relies on Poisson resummation formula,

$$\sum_{n \in Z} f(n) = \sum_{m \in Z} \tilde{f}(m) \ , \quad \tilde{f} = Fourier(f)$$

and the fact that the Gaussian is preserved under Fourier transform:

$$\int dx \, \exp(i x^2/\hbar + i p x) = \sqrt{\hbar} \exp(-i\hbar \ p^2)$$

In other words, for a Gaussian the semi-classical (saddle) approximation is exact. Perturbative QFT is arises from generalizing to ∞ x's.

- Interestingly, there exists a generalization of this to cubic characters:

$$\int dx^{0123} (1/x_0) \exp\left(i \frac{x_1 x_2 x_3}{\hbar \ x_0} + p_i x^i \right) = (\hbar/p_0) \exp\left(-i\hbar \frac{p_1 p_2 p_3}{p_0} \right)$$

Again, the saddle point approximation is exact. Such cubic forms are classified by (A)DE:

$$\begin{aligned}
D_n &: \quad I_3 = x_1(x_2 x_3 + x_4 x_5 + \dots) \\
E_6 &: \quad I_3 = \det(3 \times 3) \\
E_7 &: \quad I_3 = \mathrm{Pf}(6 \wedge 6) \\
E_8 &: \quad I_3 = 27^3|_1
\end{aligned}$$

Etingof Kazhdan Polischuk

- This observation is at the heart of the construction of theta series for simply laced groups. *Kazhdan Savin*

Theta series under the hood

The standard theta series can be deconstructed as

$$\theta(\tau) = \sum_{m \in Z} \exp(i\pi\tau m^2) = \langle \delta_Z, \rho(g_\tau) f \rangle \ , \qquad g_\tau = \begin{pmatrix} 1 & \tau_1 \\ 0 & \tau_2 \end{pmatrix} / \sqrt{\tau_2}$$

- $\rho(g)$ is a unitary representation of $g \in Sl(2)$ on functions of one variable:

  $$E_+ = i\pi\, x^2 \, , \qquad H = \frac{1}{2} (x\partial_x + \partial_x x) \ , \qquad E_- = \frac{i}{4\pi} \partial_x^2 \, ,$$

  satisfying the $Sl(2,R)$ algebra,

  $$[H, E_\pm] = \pm 2\, E_\pm \ , \qquad H = [E_+, E_-] \ ,$$

- $f(x) = e^{-x^2/2}$ is a spherical vector, i.e. a function ϕ (quasi) annihilated by the compact generator $K = E_+ + E_-$; in particular invariant under the Weyl generator $\exp(i\pi K) = Fourier$.

- δ_Z is a distribution invariant under $Sl(2,Z)$,

  $$\delta_Z(x) = \sum_{m \in Z} \delta(x - m) =^{***} \prod_{p \text{ prime}} f_p(x) \ ,$$

  where each f_p is invariant under Fourier transform over the p-adic field.

All these parts can be engineered for any simply-laced G

Min. rep. and conformal quantum mechanics

- The representation space is constructed as the Hilbert space of a conformal quantum mechanical system whose phase space is the minimal nilpotent orbit of G.

de Alfaro Fubini Furlan

- Classically, the Lagrangian is manifestly invariant under G_0,

$$\mathcal{L} = \dot{x}_0 \dot{y} + 2x_0 \sqrt{I_3(\dot{x}_i)} + \frac{d}{dt}\left(\frac{x_0 I_3(x_i)}{y}\right)$$

the Hamiltonian is invariant under $G_1 \supset G_0$ mixing positions and momenta,

$$\mathcal{H} = p^2 + y^2 + \frac{1}{y^2} I_4(x_I, p_I)$$

and the conformal transformations, $t \to (at+b)/(ct+d)$ extend the symmetry group to $G \supset G_1 \supset G_0$.

BP Waldron; Gunaydin Koepsell Nicolai

- The quantization of this system produces the minimal representation of G as differential operators acting on wave functions.

Quantization and spherical vector

- Quantization is carried out by replacing $p_i \to i d/dx_i$ and adding normal ordering terms so that the generators still close. More abstractly, it proceeds by a sequence of induced representations.

  Kazhdan Savin; Brylinsky Kostant

- The Weyl generators

$$(Sf)(y, x_0, \ldots, x_{N-1}) = \int \frac{\prod_{i=0}^{N-1} dp_i}{(2\pi y)^{N/2}} f(y, p_0, \ldots, p_{N-1})\, e^{\frac{i}{y} \sum_{i=0}^{N-1} p_i x_i}$$

$$(Af)(y, x_0, x_1, \ldots, x_{N-1}) = \exp\left(-\frac{i I_3}{x_0 y}\right) f(-x_0, y, x_1, \ldots, x_{N-1})$$

  satisfy the correct relation $(AS)^3 = (SA)^3$ thanks to the invariance of the cubic character under Fourier transform.

- The spherical vector is the ground state wave function of this quantum mechanical system, invariant under the maximal compact subgroup K of G. It can be found by solving PDEs $E_\alpha + E_{-\alpha} = 0$.

  Kazhdan BP Waldron CMP 2001

- The summation measure δ_Z is obtained by solving the same problem (with different methods) over the p-adic field Z_p.

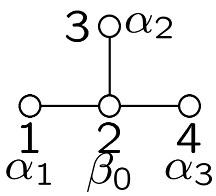
  Kazhdan Polischuk, to appear

Minimal Nilpotent Orbit

$$
\begin{array}{ccc}
Sl(n) & \supset & Sl(2) \times Sl(n-2) \times R^{+} \\
adj & = & (3,1,0) \oplus [(2,n-2,1) \oplus (2,n-2,-1)] \oplus (1,adj,0) \\
 & = & 1 \oplus 2(n-2) \oplus [1 \oplus adj] \oplus 2(n-2) \oplus 1 \\
SO(2n) & \supset & Sl(2) \times Sl(2) \times SO(2n-4) \\
adj & = & (3,1,1) \oplus (2,2,2n-4) \oplus (1,3,1) \oplus (1,1,adj) \\
 & = & 1 \oplus (2,2n-4) \oplus [1 \oplus adj] \oplus (2,2n-4) \oplus 1 \\
E_6 & \supset & Sl(2) \times Sl(6) \\
78 & = & (3,1) \oplus (2,20) \oplus (1,35) \\
 & = & 1 \oplus 20 \oplus [1 \oplus 35] \oplus 20 \oplus 1 \\
E_7 & \supset & Sl(2) \times SO(6,6) \\
133 & = & (3,1) \oplus (2,32) \oplus (1,66) \\
 & = & 1 \oplus 32 \oplus [1 \oplus 66] \oplus 32 \oplus 1 \\
E_8 & \supset & Sl(2) \times E_7 \\
248 & = & (3,1) \oplus (2,56) \oplus (1,133) \\
 & = & 1 \oplus 56 \oplus [1 \oplus 133] \oplus 56 \oplus 1
\end{array}
$$

G	dim	H_0	G_1^*	I_3
$Sl(n)$	$n-1$	$Sl(n-3)$	$[n-3]$	0
$SO(n,n)$	$2n-3$	$SO(n-3,n-3)$	$1 \oplus [2n-6]$	$x_1(\sum x_{2i}x_{2i+1})$
E_6	11	$Sl(3) \times Sl(3)$	$(3,3)$	det
E_7	17	$Sl(6)$	15	Pf
E_8	29	E_6	27	$27^{\otimes_s 3}\|_1$

Example: $D_4 = SO(4,4)$



$$\beta_i = \beta_0 + \alpha_i \ , \quad \gamma_i = \beta_0 + \alpha_j + \alpha_k \ ,$$
$$\gamma_0 = \beta_0 + \alpha_1 + \alpha_2 + \alpha_3 \ , \quad \omega = \beta_0 + \gamma_0$$

$$I_3 = x_1 x_2 x_3$$

$$\begin{array}{ll} E_{\beta_0} = y\partial_0 & E_{\gamma_0} = ix_0 \\ E_{\beta_1} = y\partial_1 & E_{\gamma_1} = ix_1 \\ E_{\beta_2} = y\partial_2 & E_{\gamma_2} = ix_2 \\ E_{\beta_3} = y\partial_3 & E_{\gamma_3} = ix_3 \end{array}$$
$$E_\omega = iy \, .$$

$$\begin{array}{ll} E_{\alpha_1} = -x_0\partial_1 - \frac{ix_2x_3}{y} \ , & E_{-\alpha_1} = x_1\partial_0 + iy\partial_2\partial_3 \\ E_{\alpha_2} = -x_0\partial_2 - \frac{ix_3x_1}{y} \ , & E_{-\alpha_2} = x_2\partial_0 + iy\partial_3\partial_1 \\ E_{\alpha_3} = -x_0\partial_3 - \frac{ix_1x_2}{y} \ , & E_{-\alpha_3} = x_3\partial_0 + iy\partial_1\partial_2 \, . \end{array}$$

$$\begin{aligned} E_{-\beta_0} &= -x_0\partial + \frac{ix_1x_2x_3}{y^2} \\ E_{-\beta_1} &= x_1\partial + \frac{x_1}{y}\left(1 + x_2\partial_2 + x_3\partial_3\right) - ix_0\partial_2\partial_3 \\ E_{-\gamma_0} &= 3i\partial_0 + iy\partial\partial_0 - y\partial_1\partial_2\partial_3 + i(x_0\partial_0 + x_1\partial_1 + x_2\partial_2 + x_3\partial_3)\,\partial_0 \\ E_{-\gamma_1} &= iy\partial_1\partial + i(2 + x_0\partial_0 + x_1\partial_1)\,\partial_1 - \frac{x_2x_3}{y}\,\partial_0 \\ E_{-\omega} &= 3i\partial + iy\partial^2 + \frac{i}{y} + ix_0\partial_0\partial + \frac{x_1x_2x_3}{y^2}\,\partial_0 + \\ &\quad + \frac{i}{y}\left(x_1x_2\partial_1\partial_2 + x_3x_1\partial_3\partial_1 + x_2x_3\partial_2\partial_3\right) \\ &\quad + \ i(x_1\partial_1 + x_2\partial_2 + x_3\partial_3)\left(\partial + \frac{1}{y}\right) + x_0\partial_1\partial_2\partial_3 \, , \end{aligned}$$

Example: $D4 = SO(4,4)$ (continued)

$$\begin{aligned} H_{\beta_0} &= -y\partial + x_0\partial_0 \\ H_{\alpha_1} &= -1 - x_0\partial_0 + x_1\partial_1 - x_2\partial_2 - x_3\partial_3 \\ H_{\alpha_2} &= -1 - x_0\partial_0 - x_1\partial_1 + x_2\partial_2 - x_3\partial_3 \\ H_{\alpha_3} &= -1 - x_0\partial_0 - x_1\partial_1 - x_2\partial_2 + x_3\partial_3 \,, \end{aligned}$$

- Spherical vector: solve PDE $(E_\alpha - E_{-\alpha})f = 0$:

  $$f_{D_4} = \frac{4\pi}{|z|} K_0 \left(\frac{\sqrt{(|z|^2 + x_1^2)(|z|^2 + x_2^2)(|z|^2 + x_3^2)}}{|z|^2} \right) e^{-i \frac{x_0 x_1 x_2 x_3}{y|z|^2}}$$

  where $z = y + i x_0$. This is manifestly invariant under $SO(4,4)$ triality, permuting x_1, x_2, x_3.

- Rk: this minimal representation is equivalent to the one arising from the string worldsheet instantons on T^4: by Fourier transforming on x_3 and renaming variables, we can rewrite f as

  $$f = \frac{e^{-2\pi\sqrt{(m^{ij})^2}}}{\sqrt{(m^{ij})^2}} \,, \qquad \epsilon^{ijkl} m_{ij} m_{kl} = 0$$

  This implies that the one-loop BPS amplitude of Het/T^4 is invariant under $SO(4,4)$ triality, as predicted from Heterotic-Type II duality.

Kiritsis Obers BP

E_6 theta series and Membrane/T^3

- For E_6 the minimal nilpotent orbit is parameterized by 11 positions (y, x_0, M^i_α) transforming as $1 + 1 + (3, 3)$ under the linearly represented subgroup $G_0 = Sl(3) \times Sl(3)$: there are two unexpected quantum numbers (y, x_0). The cubic form is simply $I_3 = \det(M)$.

- The spherical vector invariant under the maximal compact $SU(8)$ can be obtained by integrating the PDE $(E_\alpha - E_{-\alpha})f = 0$:

$$f_{E_6} = \frac{e^{-(S_1 + iS_2)}}{|z|^2 S_1} \,, \qquad z = y + ix_0$$

$$S_1 = \frac{\sqrt{\det(MM^t + |z|^2 \mathbb{I}_3)}}{|z|^2} \,, \quad S_2 = \frac{x_0 \det(M)}{y|z|^2}$$

- The variables M can be identified with the winding numbers of the membrane $X^i = \mathcal{M}^i_\alpha \sigma^\alpha$. z looks like a complex scalar on the worldvolume, or rather an $Sl(2)$ doublet of 3-form field strengths: no dof, cosmological constant on the worldvolume.

- The representation satisfies identically

$$\Delta_{Sl(3)_1} = \Delta_{Sl(3)_2} = \Delta_{Sl(3)_3}$$

which agrees with the result expected for f_{R^4} after integrating over wv Sl(3).

Summary - prospects

- We have obtained explicit formulae for theta series for D_n and for exceptional groups $E_{6,7,8}$. Can one understand degenerate contributions ? Can one find a simple combinatoric formula for this summation measure ?

- This construction is based on the quantization of the phase space of an exotic conformal quantum mechanical model. Any deeper meaning or application to motion on black hole moduli space ?

- It relies on the invariance of the cubic character $\exp(iI_3(x_i)/x_0)$ under Fourier transform: a class of non-Gaussian yet free cubic models. Can models be found with ∞ degrees of freedom ? the topological open membrane ?

- Applied to the membrane, it predicts new quantum numbers (y, x_0) besides the expected windings n^i_α. What is their interpretation ? Can S be generalized to include fluctuations while preserving duality ?