Quantum Attractor Flows
(or the radial quantization of BPS black holes)

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Main references

- Motivation: Ooguri Verlinde Vafa [hep-th/0502211]
- Summary: Gunaydin, Neitzke, BP and Waldron [hep-th/0512296]
- Preview: Lecture notes, BP [hep-th/0607227]
- Neitzke, BP and Vandoren [hep-th/0701214]
- GNPW, to appear
- Early reference: Breitenlohner Gibbons Maison [hep-th/88mmnnn]

Introduction

- BPS black holes in $N = 2$ supergravity / type II string theory on a CY threefold Y enjoy simplifying properties:
  1. By the attractor phenomenon, the near-horizon solution, hence the Bekenstein-Hawking entropy, depends only on the conserved charges;
  2. Being extremal, they are not subject to Hawking radiation; Yet their entropy can be arbitrarily large;
  3. Being supersymmetric, they are expected to correspond to exact eigenstates of the quantum Hamiltonian;
  4. The string coupling can be made arbitrary small throughout the geometry;
- This has allowed a clear microscopic derivation of the macroscopic entropy, by counting open-string/membrane micro-states in the presence of D-branes/M-branes.

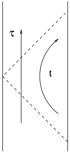
Strominger Vafa; Johnson Khuri Myers; Maldacena Strominger Witten

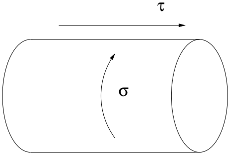
AdS_3/CFT_2 and Black String SCFT

- The modern understanding relies on AdS/CFT in the near horizon geometry $AdS_3 \times X$, where $X = S^3 \times K3$ or $S^2 \times CY_3$. The gauge theory on the boundary is a SCFT whose central charge can be computed geometrically; the density of highly excited states follows via the Ramanujan-Hardy (Cardy) formula.
- This relies on the possibility to lift the 4D black hole to a 5D black string. In general (for $[D6] \neq 0, \pm 1$), the 5D geometry is singular. Moreover, the 5-th direction can be made arbitrarily small.

$AdS_2/SCFT_1$ and Black Hole SCQM

- We expect that the entropy of 4D BPS black holes should be computed in the near-horizon geometry $AdS_2 \times X'$, in terms of superconformal quantum mechanics living on its boundary.





- Unfortunately, little is known about holography in AdS_2, partly due to the existence of two boundaries, and of a concrete $SCFT_1$.

$AdS_2/SCFT_1$ and channel duality

- A possible strategy is to try and get at the spectrum of the SQM by channel duality, as in usual open/closed string duality:

$$\mathrm{Tr} e^{-\pi t H_{open}} = \langle B | e^{-\frac{\pi}{t} H_{closed}} | B \rangle$$

  Here, H_{closed} is the Hamiltonian for string theory in AdS_2 in radial quantization. The real interest is in H_{open}.
- This is hardly doable in practice, except if one truncates to spherically symmetric SUGRA modes, and restrict to the BPS sector. It is far from clear whether this truncation is justifiable.

Topological amplitude and black hole wave function I

- Recently, OVV suggested that the OSV conjecture

$$\Omega(p^I, q_I) \sim \int d\phi^I \, |\Psi_{top}(p^I + i\phi^I)|^2 \, e^{\phi^I q_I}$$

could be interpreted in just this way (with $H_{closed} = H_{open} = 0$):

$$\Omega(p, q) = \langle \Psi^+_{p,q} | \Psi^-_{p,q} \rangle$$

where

$$\Psi^{\pm}_{p,q}(\phi) = e^{\pm \frac{1}{2} q_I \phi^I} \, \Psi_{top}(p^I \mp i\phi^I)$$

- Here $\Psi_{top}(\chi) = \langle \Psi_{top} | \chi \rangle$ is the topological amplitude in the real polarization, which guarantees that the result is invariant under changes of the electric-magnetic duality frame.

Topological amplitude and black hole wave function II

- OVV gave heuristic arguments that Ψ_{top} could be interpreted as a wave function for the radial quantization of spherically symmetric BPS geometries. If correct, this would answer a long-standing question: “What is the physical system whose “preferred” wavefunction is the topological amplitude ? ”
- One of the goals of this talk will be to perform a rigorous treatment of radial quantization, and evaluate OVV’s claim.
- Another motivation is to produce a framework for constructing an automorphic partition function, whose Fourier coefficients will count black hole micro-states.

Preliminary comments

- The idea of mini-superspace radial quantization of black holes was in fact much studied by the gr-qc community, yielding as yet little insight on the nature of black hole micro-states.

  Cavaglia de Alfaro Filippov; Kuchar; Thiemann Kastrup; Breitenlohner Hellmann

- One novelty here is that one works in a SUSY context, for which the "mini-superspace" truncation to spherically symmetric geometries, and omission of D-term interactions, has some chance of being exact.
- Further interest possibly arises from the relation between black hole attractor equations and SUSY vacua in flux compactifications.

Outline

Outline

Stationary solutions and KK* reduction I

- Stationary solutions in 4D can be parameterized in the form

$$ds_4^2 = -e^{2U}(dt+\omega)^2 + e^{-2U} ds_3^2 \;, \quad A_4^I = \zeta^I dt + A_3^I$$

where $ds_3, U, \omega, A_3^I, \zeta^I$ and the 4D scalars $z^i \in \mathcal{M}_4$ are independent of time. The D=3+1 theory reduces to a field theory in three Euclidean dimensions.

- In contrast to the usual KK ansatz,

$$ds_4^2 = e^{2U}(dy+\omega)^2 + e^{-2U} ds_{2,1}^2 \;, \quad A_4^I = \zeta^I dy + A_3^I$$

where the fields are independent of y, we reduce along a time-like direction.

Stationary solutions and KK* reduction II

- For the usual KK reduction to 2+1D, the one-forms (A^I_3, ω) can be dualized into pseudo-scalars $(\tilde{\zeta}_I, \sigma)$, where σ is the twist (or NUT) potential. The 4D Einstein-Maxwell equations reduce to 3D gravity + scalars living in a Riemannian space

$$\mathcal{M}_3 = R^+|_U \times \mathcal{M}_4 \times |_{z^i} \mathbb{R}^{2n_v+3}|_{\zeta^I, \tilde{\zeta}_I, \sigma}$$

with positive-definite metric

$$ds^2 = 2(dU)^2 + g_{ij} dz^i dz^j + \frac{1}{2} e^{-4U} \left(d\sigma + \zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I \right)^2$$
$$+ - e^{-2U} \left[t_{IJ} d\zeta^I d\zeta^J + t^{IJ} \left(d\tilde{\zeta}_I + \theta_{IK} d\zeta^K \right) \left(d\tilde{\zeta}_J + \theta_{JL} d\zeta^L \right) \right]$$

Stationary solutions and KK* reduction III

- The KK* reduction is simply related to the KK reduction by letting $(\zeta^I, \tilde{\zeta}_I) \to i(\zeta^I, \tilde{\zeta}_I)$. As a result, the scalar fields live in a pseudo–Riemannian space $\mathcal{M}_3^*$, with non-positive definite signature.

Breitenlohner Gibbons Maison; Hull Julia

- $\mathcal{M}_3^*$ always has $2n_V + 4$ isometries corresponding to the shifts of $\zeta, \tilde{\zeta}_I, \sigma, U$, satisfying the graded Heisenberg algebra

$$[p^I, q_J] = 2\delta^I_J k$$
$$\left[m, p^I\right] = p^I , \quad [m, q_I] = q_I , \quad [m, k] = 2k$$

- The notation anticipates the identification of the corresponding conserved charges with the electric and magnetic charges q_I and p_I, NUT charge k and ADM mass m.

Spherically symmetric BH and geodesics I

- Now, restrict to spherically symmetric solutions, with spatial slices

$$ds_3^2 = N^2(\rho)d\rho^2 + r^2(\rho)d\Omega_2^2$$

- The sigma-model action becomes, up to a total derivative (G_{ab} is the metric on $\mathcal{M}_3^*$):

$$S = \int d\rho \left[\frac{N}{2} + \frac{1}{2N} \left(\dot{r}^2 - r^2 G_{ab} \dot{\phi}^a \dot{\phi}^b \right) \right]$$

- This is the Lagrangian for the geodesic motion of a fiducial particle with unit mass on the (hyperbolic) cone $\mathbb{R}^+ \times \mathcal{M}_3^*$. The einbein $\sqrt{N}$ enforces invariance under reparameterizations of ρ.

Spherically symmetric BH and geodesics II

- The equation of motion of N imposes the Hamiltonian constraint, or Wheeler-De Witt equation

$$H_{WDW} = (p_r)^2 - \frac{1}{r^2} G^{ab} p_a p_b - 1 \equiv 0$$

- The gauge choice $N = r^2$ allows to separate the problem into radial motion along r, and geodesic motion on $\mathcal{M}_3^*$:

$$G^{ab} p_a p_b = C^2 \,, \quad (p_r)^2 - \frac{C^2}{r^2} - 1 \equiv 0 \quad \Rightarrow \quad r = \frac{C}{\sinh C\rho} \,,$$

  Thus, the problem reduces to affinely parameterized geodesic motion on the three-dimensional moduli space $\mathcal{M}_3^*$.

Spherically symmetric BH and geodesics III

- It turns out that $C = 2T_H S_{BH}$ is the extremality parameter: extremal (in particular BPS) black holes correspond to light-like geodesics on $\mathcal{M}_3^*$. Since $r = 1/\rho$, the 3D spatial slices are flat.
- Other gauges are also possible: e.g. $N = e^U$ identifies ρ with the radial geodesic distance to the horizon.
- For the purpose of defining observables such as the horizon area, $A_H = 4\pi e^{-2U} r^2|_{U\to-\infty}$ and ADM mass $M_{ADM} = r(e^{2U} - 1)|_{U\to 0}$, it may convenient to leave the gauge unfixed.

Isometries and conserved charges

- The isometries of $\mathcal{M}_3$ imply conserved Noether charges, whose Poisson bracket reflect the Lie algebra of the isometries:

$$\begin{aligned} [p^I, q_J] &= 2\delta^I_J\, k \\ \left[m, p^I\right] = p^I, \quad [m, q_I] &= q_I\,, \quad [m,k] = 2k \end{aligned}$$

- If $k \neq 0$, the off-diagonal term in the 4D metric

$$ds_4^2 = -e^{2U}(dt + k\cos\theta d\phi)^2 + e^{-2U}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]$$

  implies the existence of closed time-like curves around ϕ direction, near $\theta = 0$.
- Bona fide 4D black holes arise in the "classical limit" $k \to 0$. Keeping $k \neq 0$ will allow us to greatly extend the symmetry.

Conserved charges and black hole potential

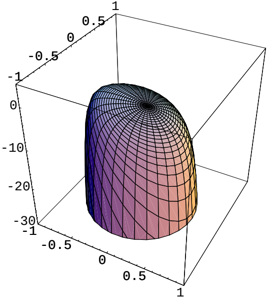
- Setting $k = 0$ for simplicity, one arrives at the Hamiltonian,

$$H = \frac{1}{2}\left[p_U^2 + p_i g^{ij} p_j - e^{2U} V_{BH}\right] \equiv C^2$$

where V_{BH} is the "black hole potential",

$$V_{BH}(z^i, p^I, q_I) = \frac{1}{2}(q_I - \mathcal{N}_{IJ}p^J)t^{IK}(q_K - \bar{\mathcal{N}}_{KL}p^L) + \frac{1}{2}p^I t_{IJ} p^J$$

- The potential $V = -e^{2U} V_{BH}$ is unbounded from below.



Quantizing geodesic motion I

- The classical phase space is the cotangent bundle $T^*(\mathcal{M}_3^*)$, specifying the initial position and velocity: non compact.
- Quantization proceeds by replacing functions on phase space by operators acting on wave functions in $L_2(\mathcal{M}_3^*)$, subject to

$$\Delta_3 \Psi(U, z^i, \zeta^I, \tilde{\zeta}_I, \sigma) = C^2 \Psi$$

where Δ_3 is the Laplace-Beltrami operator on $\mathcal{M}_3^*$.

- The electric, magnetic and NUT charges may be diagonalized as

$$\Psi(U, z^i, \zeta^I, \tilde{\zeta}_I, \sigma) = \Psi_{p,q}(U, z)\, e^{i(q_I \zeta^I + p^I \tilde{\zeta}_I)}$$

$$\left[-\partial_U^2 - \Delta_4 - e^{2U} V_{BH} - C^2\right] \Psi_{p,q}(U, z) = 0$$

Quantizing geodesic motion II

- The black hole wave function $\Psi_{p,q}(U, z)$ describes quantum fluctuations of the 4D moduli as one reaches the horizon at $U \to -\infty$. Naively, should be peaked at the attractor point.
- Restoring the variable r, one could also describe the quantum fluctuations of the horizon area $4\pi r^2 e^{-2U}$, around the classical value $4S_{BH}$.
- The natural inner product is the Klein-Gordon inner product at fixed U, famously NOT positive definite. A standard remedy in quantum cosmology is "third quantization", possibly relevant for black hole fragmentation / multi-centered solutions.

Attractor flow in $N = 2$ supergravity

- Consider $N = 2$ SUGRA coupled to n_V abelian vector multiplets [*hypers decouple at tree-level*]: the vector multiplet scalars z^i take values in a special Kähler manifold $\mathcal{M}_4$. For type IIA on $X = CY_3$, z^i parameterize the complexified Kähler structure of X.
- After reduction to 3 dimensions, the vector multiplet scalars take value in a quaternionic-Kähler space $\mathcal{M}_3$, known as the $c - map$ of the special Kähler space $\mathcal{M}_4$.
- Under T-duality along the 4th direction, this becomes the hypermultiplet space for type IIB compactified on X at tree-level.
- The manifold $\mathcal{M}_3^*$ obtained by analytic continuation is sometimes called "para-quaternionic-Kahler manifold"; it has split signature $(2n_V + 2, 2n_V + 2)$

Cortes Mayer Mohaupt Saueressig

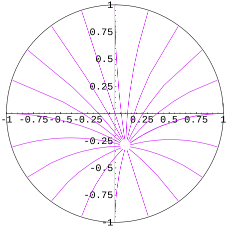
Attractor flow and semi-classical BPS wave function

- The black hole potential splits into two pieces,

$$V_{BH}(p,q;z^i,\bar{z}^i) = |Z|^2 + \partial_i|Z|\, g^{i\bar{j}}\, \partial_{\bar{j}}|Z|$$

where Z is the central charge $Z = e^{K/2}(q_I X^I - p^I F_I)$.

- Supersymmetric solutions are obtained by cancelling each term separately, leading back to the attractor flow equations,



$$\begin{aligned} r^2 \frac{dU}{dr} &= e^U |Z| \\ r^2 \frac{dz^i}{dr} &= 2 e^U g_{i\bar{j}}\, \partial_{\bar{j}}|Z| \end{aligned}$$

Attractor flow and semi-classical BPS wave function

- At this stage, one could already quantize the attractor flow equations and guess the BPS wave function:

$$\begin{cases} p_U &= -e^U |Z| \\ p_{\bar{z}^{\bar{i}}} &= -2e^U \partial_{\bar{i}} |Z| \end{cases} \Rightarrow \Psi(U, z^i, \bar{z}^{\bar{j}}, p, q) \sim \exp\left[2ie^U |Z|\right]$$

  The phase is stationary at the classical attractor points.
- Using twistor techniques, we shall be able to resolve ordering ambiguities, and compute the BPS wave function exactly.

Supersymmetric quantum mechanics

- More rigorously, the full $D = 4, N = 2$ SUGRA including fermions, reduces to $D = 1, N = 4$ supergravity:

$$S = \int d\rho \; G_{ab}\dot{\phi}^a\dot{\phi}^b + \psi^A \frac{D}{D\rho}\psi_A + (\psi^A\psi_A)(\psi^A\psi_A) + \dots$$

- The supersymmetry variations are $\delta\psi^A = V^{AA'}\epsilon_{A'}$, where $V^{AA'}$ ($A = 1, ..2n_V + 2, A' = 1, 2$) is the quaternionic vielbein afforded by the restricted holonomy $Sp(2) \times Sp(2n_V + 2)$.
- Thus, SUSY trajectories are characterized by

$$\exists\epsilon_\alpha \; / \; V^{AA'}_\mu \; \dot{\phi}^\mu \; \epsilon_{A'} = 0 \quad \Leftrightarrow \quad V^{A[A'} V^{B']B} = 0$$

This reproduces the attractor flow equations (generalized to $k \neq 0$)

Gutperle Spalinski

Outline

Improved SUSY mechanics - HKC and twistors I

- This SUSY mechanics is rather unusual, insofar as the SUSY comes from a triplet of non-integrable almost complex structures.
- It is possible to remedy this problem by combining the Killing spinor $\epsilon_{A'} \in \mathbb{C}^2$ with the coordinates $\phi^a \in QK$, i.e. extend the QK space into its Hyperkähler cone (HKC), or Swann bundle,

$$\mathbb{R}^4 \to HKC \to QK$$

  By cancelling the $Sp(2)$ holonomy on QK against the $SU(2)$ holonomy on S^3, the three almost complex structures on QK become genuine complex structures on HKC.
- Geodesic motion on HKC is equivalent to geodesic motion on QK after gauging the $SU(2)$ and dilation symmetries. BPS property becomes just holomorphy on HKC !

The twistor space

- The relevant information is captured by the twistor space Z, a two-sphere bundle over QK with a Kähler-Einstein metric. The sphere coordinate z keeps track of the Killing spinor, $z = \epsilon_1/\epsilon_2$.
- In the presence of triholomorphic isometries, the geometry of HKC is controlled by a generalized prepotential $G(\eta^L)$,

$$\langle K(v^L, \bar{v}^L, w_L + \bar{w}_L) + x^L(w_L + \bar{w}_L)\rangle_{w+\bar{w}} = \oint \frac{d\zeta}{2\pi i\zeta} G[\eta^L(\zeta)]$$

where η^L is the "projective multiplet"

$$\eta^L = v^L/\zeta + x^L - \bar{v}^L\zeta$$

Hitchin Lindstrom Rocek; De Wit Rocek Vandoren

Twistor space for the c-map

- When HKC is the Swann bundle of the c-map of a SK manifold, the generalized prepotential is simply obtained from the prepotential F,

$$G(\eta^L, \zeta) = F(\eta^I)/\eta^\flat$$

Rocek Vafa Vandoren

- The inhomogeneous coordinates $\xi^I = v^I/v^\flat, \tilde{\xi}_I = -2iw_I$, $\alpha = 4iw_\flat - \xi^I\tilde{\xi}_I$ are complex coordinates on Z, adapted to the Heisenberg symmetries, given by the "twistor map":

$$\begin{aligned} \xi^I &= \zeta^I + i\, e^{U+\mathcal{K}(X)/2} \left(z\, \bar{X}^I + z^{-1} X^I \right) \\ \tilde{\xi}_I &= \tilde{\zeta}_I + i\, e^{U+\mathcal{K}(X)/2} \left(z\, \bar{F}_I + z^{-1}\, F_I \right) \\ \alpha &= \sigma + \zeta^I \tilde{\xi}_I - \tilde{\zeta}_I \xi^I \end{aligned}$$

- Conversely, the coordinates on the base $\mathcal{M}_3$ are $SU(2)$ invariant combinations of $\xi^I, \tilde{\xi}_I, \alpha$.

Neitkze BP Vandoren 07

BPS black holes and holomorphic curves

- Upon lifting the geodesic motion to Z, SUSY is preserved iff the momentum is holomorphic in the canonical complex structure on Z, at any point along the trajectory: 1st class constraints !
- Put differently, the SUSY phase space is the twistor space Z, equipped with its Kähler symplectic form. Its dimension is $4n_V + 6$, almost half that of the generic phase space $T^*(\mathcal{M}_3^*)$.
- BPS solutions correspond to holomorphic curves $\xi^I(\rho), \tilde{\xi}_I(\rho), \alpha(\rho)$ at constant $\bar{\xi}^I, \bar{\tilde{\xi}}_I, \bar{\alpha}$, and are algebraically determined by the conserved charges: integrable system !

The Penrose Transform

- At fixed values of $U, z^i, \zeta^I, \tilde{\zeta}_I, \sigma$, the complex coordinates $\xi^I, \tilde{\xi}_I, \alpha$ on Z are holomorphic functions of the twistor coordinate z: the fiber over each point is a rational curve in Z.
- Starting from a holomorphic function Φ on Z, we can produce a function Ψ on QK

$$\Psi(U, z^i, \bar{z}^{\bar{i}}, \zeta^I, \tilde{\zeta}_I, \sigma) = e^{2U} \oint \frac{dz}{2\pi i z} \, \Phi \left[\xi^I(z), \tilde{\xi}^I(z), \alpha(z) \right]$$

satisfying some generalized harmonicity condition:

$$\left(\epsilon^{A'B'} \nabla_{AA'} \nabla_{BB'} - R_{AB} \right) \Psi = 0$$

- This is a quaternionic generalization of the usual Penrose transform between holomorphic functions on CP^3 and conformally harmonic functions on S^4.

Salamon; Baston Eastwood

Outline

The BPS Hilbert space I

- In terms of geodesic motion on the QK base, the classical BPS conditions $V^{A[\alpha}V^{\beta]B}=0$ become a set of 2nd order differential operators which have to annihilate the wave function Ψ:

$$\left(\epsilon_{A'B'}\nabla^{AA'}\nabla^{BB'}-R^{AB}\right)\Psi=0$$

- In terms of the twistor space, the BPS condition $p_{\bar{L}}=0$ requires that Ψ should be a holomorphic function on Z. More precisely, taking the fermions into account, we believe it should be a section of $H^1(Z,\mathcal{O}(-2))$.
- The equivalence between the two approaches is a consequence of the Penrose transform discussed previously.

- Ignore fermionic subtleties, and go back to the simple-minded twistor transform

$$\Psi(U, z^i, \bar{z}^I, \zeta^I, \tilde{\zeta}_I, a) = e^{2U} \oint \frac{dz}{2\pi i z} \, \Phi\left[\xi^I(z), \tilde{\xi}^I(z), \alpha(z)\right]$$

- Consider a black hole with $k = 0$: p^I and q_I can be diagonalized simultaneously, and completely determine (up to normalization) the wave function as a coherent state on Z:

$$\begin{aligned}
\Phi &= \exp\left[i(p^I \tilde{\xi}_I - q_I \xi^I)\right] \\
&= \exp\left[i(p^I \tilde{\zeta}_I - q_I \zeta^I) + i e^{U + K(X)/2} (z \bar{W}_{p,q}(\bar{X}) + z^{-1} W_{p,q}(X))\right]
\end{aligned}$$

The BPS Black Hole Wave-Function II

- The integral over z is of Bessel type, leading to

$$\Psi = e^{2U} J_0 \left(2\, e^{U}\, |Z_{p,q}| \right)\; e^{i(p^I \tilde{\zeta}_I - q_I \zeta^I)}$$

  in qualitative agreement with our naive attempt at quantizing
- This is peaked around the classical attractor points, with slowly damped, increasingly faster oscillations away from them. Contrary perhaps to expectations, the wave flattens out towards the horizon ! This is because of the large fine-tuning needed to produce a BPS solution.

Relation to the topological amplitude ?

- Before integrating along the fiber, we found that $\Psi_{p,q} \sim \exp[i e^{U+K/2}(z\bar{W} + z^{-1} W)]$, in "rough" agreement with OVV's answer $\Psi_{p,q} \sim \exp(W)$.
- It is unlikely that Ψ_{top} can be identified as a black hole wave function: it naturally depends on $n_V + 1$ variables, while Ψ_{BH} depends on $2n_V + 3$ variables.
- Instead, the "super-BPS" Hilbert space of tri-holomorphic functions on HKC is the natural habitat of a one-parameter generalization of the topological string amplitude...

Gunaydin Neiztke BP

Outlook

- Higher derivative corrections remain to be incorporated: higher derivative scalar interactions on QK space.
- Multi-centered configurations can be described by certain harmonic maps from $\mathbb{R}^3$ to QK: does that correspond to "second quantization", i.e. including vertices ?
- For $N \geq 4$, this suggests that the 3D U-duality group controls the BH spectrum: can one obtain the exact degeneracies as Fourier coefs of some "BPS automorphic forms" ? Improve on DVV.
- The equivalence between BH attractor flow and geodesic flow on QK is a reflection of mirror symmetry. Can this be used to compute instanton corrections on hypermultiplet moduli space ?