M-theory and automorphic forms

Boris Pioline

LPTHE, Paris

LPM, Montpellier

Oct 23, 2002

w/ E. Kiritsis, N. Obers, 1997-99
w/ H. Nicolai, J. Plefka, A. Waldron, hep-th/0102123
w/ D. Kazhdan, A. Waldron, hep-th/0107222
w/ A. Waldron, hep-th/0209044

transparencies available from
http://www.lpthe.jussieu.fr/˜pioline/seminars.html

The many faces of string theory

String theory has taken/lost shape over a long and tumultuous history:

- late 1960s: Dual models introduced in order to describe hadronic resonances in fact correspond to a relativistic string.
- Recognized in 1974 by Scherk and Schwarz as a potentially finite theory of all fundamental interactions, where gravitons (closed strings) arise as bound states of gluons (open strings).
- The large tension limit of the few consistent superstring theories reproduces all known supergravities, except for a conspicuous 11-dimensional theory of Cremmer and Julia's.
- The stable extended solitons found in supergravity are in fact non-perturbative states of string theory, dubbed D-branes; their excitations are open strings ending on them.
- All string theories are different faces of the same object, perturbative strings states are (D-brane) solitons of a dual string theory.

? $\rightarrow$ strings $\rightarrow$ D-branes $\rightarrow$ membranes $\rightarrow$???

The theory formerly known as string theory

Worse, a theory that was believed to live in 10 dimensions actually propagates in 11 dimensions at non-zero coupling: type IIA string theory is a limit of an hypothetical M-theory when one spacelike dimension shrinks to zero radius.

Since string theory as yet is only defined by its perturbative genus expansion, the existence of M-theory is still conjectural. Several basic features are clear though:

- it admits eleven-dimensional vacua with maximal supersymmetry: $R^{1,10}, AdS_{4|7} \times S_{7|4}$ (and the KG plane wave, known for 20 years).

- It has no modulus in 11 dimensions, and a single scale, $l_M^3 = 1/T_{M2}$. It is described by Cremmer-Julia-Scherk 11D supergravity at low energies.

- It reduces to type IIA string theory with scale $1/l_s^2 = R_{11}/l_M^3$ upon compactifying on a circle of radius $R_{11} = g_s l_s$

- It has extended objects such as membranes, five-branes, (giant gravitons, Quantum Hall solitons, D(NA)-branes...).

M for M(embranes,atrix,aldacena. . .) ?

Despite these hints, no satisfactory (computable) definition of M-theory exists as yet.

- String theory lacks a prescription to include non-perturbative effects.
- Membranes might serve as fundamental degrees of freedom (κ symmetry implies 11D SUGRA eom), but non-linearities have prevented their quantization.
- M(atrix) theory purports to describe M-theory in $R^{1,10}$ in the infinite momentum frame (and regulate the membrane) but the large N limit is hardly tractable.
- $\mathcal{N} = 4$ $U(N)$ SYM in 3+1 dimensions is the same as M-theory in $AdS_5 \times S_5 \times S_1$, but the description is tied to this background.

Some of the most powerful techniques to obtain non-perturbative results remain to make use of symmetries: 32 supersymmetries, dualities...

U-duality and the Ur-theorie

- "Hidden symmetries" were discovered as early as 1979 by Cremmer and Julia as continuous non-compact symmetries $E_d(R)$ of the classical eom of 11D SUGRA (compactified on a torus T^d).

- A discrete arithmetic subgroup $E_d(Z)$ (compatible with Dirac- Zwanziger quantization condition), dubbed U-duality, was conjectured to hold as an exact quantum symmetry by Townsend and Hull in 1994. It combines the perturbative T-duality of string theory with the S-duality $g_s \to 1/g_s$ of type IIB.

- In contrast to string dualities that relate equivalent perturbative descriptions of a single physical process, U-duality is a symmetry which relates processes at different values of a same coupling constant. Combining with analytic properties gives powerful constraints.

M-automorphic forms

- Mathematically, the moduli space is given by a symmetric space $K\backslash G$, acted upon (from the right) by the U-duality group $G(Z)$. Hence physical amplitudes have to be automorphic forms of U-duality.

- Together with analyticity, this can be so strong as to fix BPS amplitudes completely: ex, the $Riemann^4$ amplitude in M-theory on torii. One can then read off (D-brane, membrane) instantons effects not otherwise calculable.

- Perturbative string theory provides numerous examples of automorphic forms, in particular correspondences between $Sp(g, Z)$ (on the worldsheet) and $SO(d, d, Z)$ (in 10D target space): just integrate on world-sheet moduli !

- Can one similarly reproduce fully non-perturbative amplitudes by quantizing the (BPS) membrane ? can the membrane teach mathematicians a correspondence between $Sl(3, Z)$ (on the world-volume) and $E_d(Z)$ (in 11D target space) ?

Outline

1. Symmetries of M-theory

   ...Non-compact ADE and arithmetic subgroups

2. Duality and non-perturbative R^4 couplings

   ...Continuous unireps, Eisenstein series

3. Quantizing the BPS membrane

   ...Non-Gaussian Theta series, correspondences

4. Quantum cosmology and conformal quantum mechanics

   ...Quantization of nilpotent coadjoint orbits

5. Outlook

1. Symmetries of M-theory

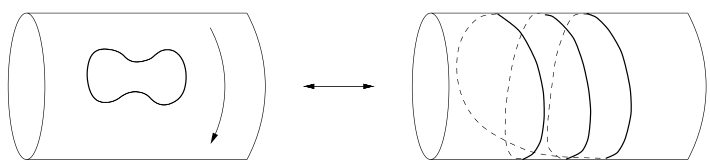
- Any diffeomorphism invariant theory compactified on a torus T^d has a discrete global symmetry, the mapping class group of the torus :

$$S_{IJ}: \quad \gamma_I \leftrightarrow \gamma_J \ , \quad R_I \leftrightarrow R_J$$
$$\Gamma_{IJ}: \quad \gamma_I \to \gamma_I + \gamma_J \ , \quad A_J^{(I)} \to A_J^{(I)} + 1$$

where $ds^2 = R_I^2 (dx^I + A_J^{(I)} dx^J)^2 + dx^J g_{JK} dx^K$

$$S_{IJ} \bowtie \Gamma_{IJ} = Sl(d, Z)$$

- Perturbative type IIA theory on a torus T^d has in addition a discrete global symmetry valid to all orders, namely (double) T-duality :



$$T: R \to l_s^2/R, \quad g_s \to g_s l_s / R$$

$$Sl(d, Z) \bowtie (T_1 T_2) = SO(d, d, Z)$$

- M-theory on a torus T^d should exhibit quantum mechanically both diffeomorphism invariance on T^d and T-duality invariance on T^{d-1} : this is U-duality.

$$Sl(d, Z) \bowtie SO(d-1, d-1, Z) = E_d(Z)$$

T-duality Weyl group

- The action of the Weyl group can be represented on a weight space :

$$\mathcal{T} = g_s^{x^0} R_1^{x^1} R_2^{x^2} \dots R_d^{x^d} \to \lambda = x^0 e_0 + x^1 e_1 + \dots + x^d e_d$$

- The Weyl generators T and S_i act as orthogonal reflections $\rho_{\alpha_0}, \rho_{\alpha_i}, i = 1 \dots d-1$

$$\lambda \to \rho_\alpha(\lambda) = \lambda - 2\frac{\alpha \cdot \lambda}{\alpha \cdot \alpha}\alpha$$

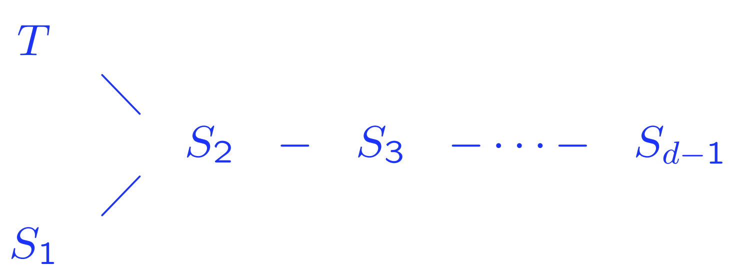
w.r.t. the signature $(-++\dots)$ metric

$$\mathrm{d}s^2 = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^i)^2 + \mathrm{d}x^0(dx^1 + \dots + \mathrm{d}x^d)$$

- The Planck length $V/g_s^2 l_s^8$ is invariant, so the action restricts to the spacelike hyperplane $\delta \cdot x = x^0 = 0$ normal to

$$\delta = e_1 + \dots + e_d - 2e_0$$

- The Coxeter group is characterized by its Cartan matrix or Dynkin diagram



This is the Dynkin diagram of $SO(d,d)$.

U-duality Weyl and Borel generators

- In M-theory variables, the T-duality $(R_i, R_j) \to (1/R_j, 1/R_i)$ reads

$$T_{ij} : R_i \to \frac{l_M^3}{R_j R_s} \; , \; \; R_j \to \frac{l_M^3}{R_s R_i} \; , \; \; R_s \to \frac{l_M^3}{R_i R_j} \; , \; \; l_M^3 \to \frac{l_M^6}{R_i R_j R_s}$$

- By a $Sl(d, Z)$ global diffeomorphism, it can be conjugated to

$$T_{IJK} : R_I \to \frac{l_M^3}{R_J R_K} \; , \quad l_M^3 \to \frac{l_M^6}{R_I R_J R_K}$$

  Hence T-duality really involves a set of three directions. A minimal set of Weyl transformations can be chosen as $T = T_{123}$ and $S_I \; : R_I \leftrightarrow R_{I+1}$.

- In addition, there are Borel generators corresponding to positive roots: Dehn twists, shifts of gauge backgrounds $C_{IJK} \to C_{IJK} + 1, \ldots$.

U-duality Weyl group

- The action of the Weyl group can again be represented on a weight space :

$$\mathcal{T} = l_M^{3x^0} R_1^{x^1} R_2^{x^2} \dots R_d^{x^d} \to \lambda = x^0 e_0 + x^1 e_1 + \dots + x^d e_d$$

- The Weyl generators T and S_i act as *orthogonal reflections*

$$\lambda \to \rho_\alpha(\lambda) = \lambda - 2\frac{\alpha \cdot \lambda}{\alpha \cdot \alpha}\alpha$$

for the signature $(- + + \dots)$ metric

$$\mathrm{d}s^2 = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^i)^2$$

- The Planck length V/l_M^9 being invariant, the action restricts to the spacelike hyperplane $\delta \cdot x = 0$ normal to

$$\delta = e_1 + \dots + e_d - 3e_0 \ ,$$

except when $d = 9$ where δ is null.
- The Coxeter group is characterized by its Cartan matrix or Dynkin diagram

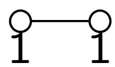
$$\begin{array}{ccccccccc} & & & & T & & & & \\ & & & & | & & & & \\ S_1 & - & S_2 & - & S_3 & - & S_4 & - \cdots - & S_{d-1} \end{array}$$

This is the Dynkin diagram of (split) E_d, or the *extended* Dynkin diagram of $\widehat{E}_8 = E_9$ for $d = 9$.

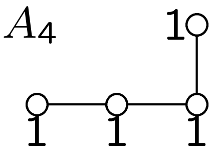
Group disintegrations

$E_2 = A_1$

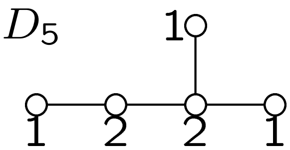
$E_3 = A_2 \oplus A_1$ ○



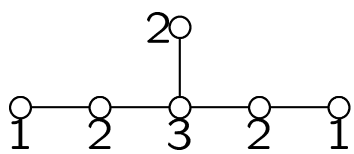
$E_4 = A_4$



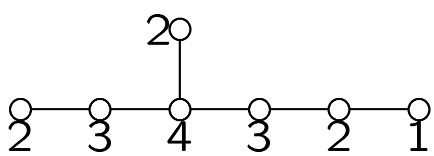
$E_5 = D_5$



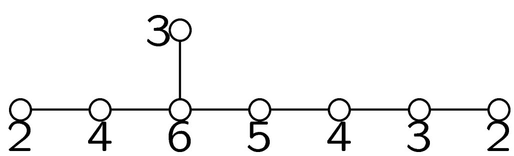
E_6



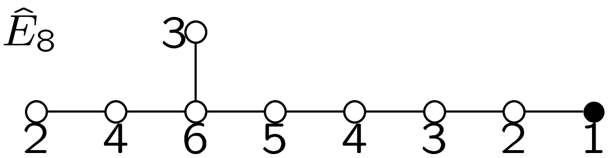
E_7



E_8

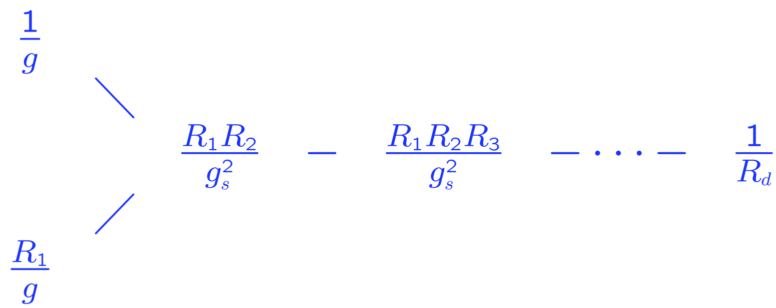


$E_9 = \widehat{E}_8$



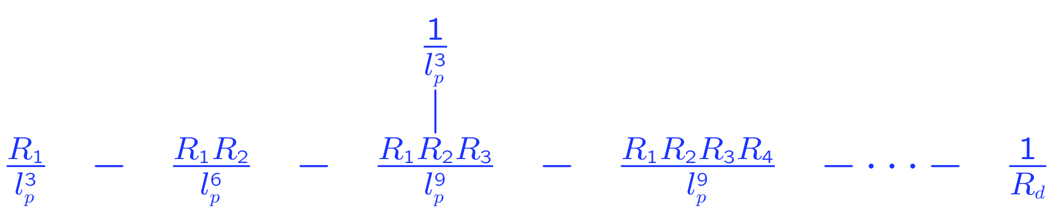
Weyl multiplets and BPS spectrum

- The representation of the finite group $\mathcal{W}(SO(d,d))$ can be obtained from the fundamental weights dual to the simple roots :



- Kaluza-Klein modes and string winding states transform as a *vector* of $SO(d,d)$, with highest weight $\mathcal{M} = 1/R_{d-1}$. D0-branes transform as a *spinor* of $SO(d,d)$ with highest weight $\mathcal{M} = 1/gl_s$, etc.

- The same holds for U-duality:



- For $d = 9$, the Weyl orbit becomes infinite. The level $k = x^1 + \cdots + x^9 + 3x^0$ is the scaling dimension of the tension. For $d = 10$, the Hell breaks loose.

Reaching boundary of moduli space

- The boundaries of the moduli space correspond to a scaling limit of the dilatonic scalars R_I, l_M parametrising the Cartan "torus". Modulo the action of the Weyl group, we can choose them to lie in a Weyl chamber such that $\alpha_+ \cdot \lambda > 0$:

$$R_1 < R_2 < \cdots < R_d \ , \qquad R_1 R_2 R_3 > l_M^3$$

- The 11D SUGRA description is valid provided all radii are larger than the Planck length :

$$11D\ SUGRA\ : \qquad l_M < R_1 < R_2 < \dots$$

- If one of the radii is smaller, then we may have a type IIA description with weak coupling $g_s^2 = (R_1/l_M)^3$, provided all radii are larger than the string length $l_s^2 = l_M^3/R_1$:

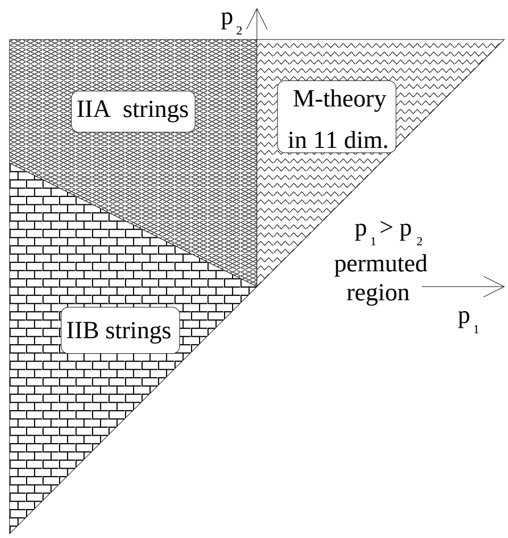
$$R_1 < l_M \ , \qquad R_1 R_2^2 > l_M^3$$

- If one of the radii is smaller than the above string length, we may instead try a type IIB description with weak coupling $g_s = R_1/R_2$, same string length $l_s^2 = l_M^3/R_1$ and 10-th radius $R_B = l_M^3/(R_1 R_2)$. The IIB radii are smaller than the string length provided

$$R_1 R_i^2 > l_M^3 \ , \qquad R_1 R_2^2 < l_M^3$$

  The first relation is automatically satisfied if $R_1 R_2 R_3 > l_M^3$, and the second implies $R_1 < l_M$.

- The 11D SUGRA, type IIA and type IIB descriptions therefore cover the complete asymptotic region of the moduli space.



- For $d > 9, 10$, the metric is now Lorentzian, and the above Weyl chamber generates the future light cone only. The space-like region admits no weakly coupled description.

Banks,Fischler,Motl

- As one approaches a space-like singularity, one can argue that a dimensional reduction down to $0 + 1$ dimensions effectively takes place : the effective dynamics is that of a free particle on the fundamental Weyl chamber of E_{10}: an hyperbolic billiard !

Damour, Henneaux, Julia, Nicolai

Moduli space and Iwasawa decomposition

- The moduli space of string/M-theory compactified on a torus can be written as a symmetric space

$$e \in K \backslash G(R)$$

where $G(R)$ is a non-compact real group and K its maximal compact subgroup:

$$(g) \in \frac{Gl(n,R)}{SO(n,R)}, \qquad (g,B) \in \frac{SO(n,n,R)}{SO(n) \times SO(n)}$$

$$(g, C, E, \dots) \in \frac{E_{n+1}}{K_{n+1}} \qquad (M/T^{n+1})$$

- The gauge symmetry can be fixed using the Iwasawa decomposition

$$e = k \cdot a \cdot n \in K \cdot A \cdot N$$

with k compact (gauged to 1), a abelian (dilatonic moduli) and N nilpotent (gauge backgrounds).

- U-duality acts by multiplication on the right by integer valued matrices, $e \to eg$. Borel shifts preserve the upper triangular gauge, but Weyl reflexions don't.

- the Mass/tension formula $\mathcal{M}^2 = m^t \cdot R(e)^t R(e) \cdot m$ is manifestly invariant.

2. Non-perturbative BPS amplitudes

- U-duality demands that physical amplitudes be functions on the moduli space $K\backslash G$ invariant under right multiplication by $G(Z)$: automorphic forms.

- The leading terms can often be computed perturbatively in string theory ($g_s \to 0$) or 11D SUGRA ($V \to \infty$).

- For particular amplitudes, with a small number of derivatives, supersymmetry puts constraints as invariant differential operators, e.g. for 1/2 BPS amplitudes,

$$\Delta_G f = \lambda f$$

- In general, all automorphic forms are Eisenstein series or derivatives thereof (at least for $Gl(n)$):

$$\mathcal{E}^{G(Z)}_{(\rho_i)}(e) = \sum_{g\in G(Z)} \prod a_i(e\cdot g)^{\rho_i} \propto \sum_{(m,n)\neq(0,0)} \left(\frac{\tau_2}{|m+n\tau|^2}\right)^s$$

- When ρ is aligned with a weight vector, this gives a more transparent formula as a sum over BPS states,

$$\mathcal{E}^{G(Z)}_{s;R}(e) = \sum_{m\in\Lambda_R} \left[m\cdot R(e^t\cdot e)\cdot m\right]^{-s} \delta(m^2)$$

  where the δ insertion restricts to half-BPS states.

- Identifying the right Eisenstein series and computing its Fourier expansion wrt the Borel moduli N gives access to instanton effects.

R^4 couplings in IIA/T^d

- The $t_8 t_8 R^4$ couplings in type II/T^d can be computed at tree-level and one-loop:

$$f_{R_4} = 2\zeta(3)\frac{V}{g_s^2} + I_d + \dots$$

- Due to the BPS property, bosonic et fermionic oscillators cancel in the one-loop integral, leaving the partition function of zero-modes only:

$$I_n = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} V \sum \exp\left(-\frac{|m^i + n^i \tau|^2}{\tau_2} + 2\pi i m^i B_{ij} n^j \right)$$

The action is the Polyakov action

$$S = \frac{1}{l_s^2} \int d^2\sigma \sqrt{\gamma}\gamma^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j G_{ij} + i\ \epsilon^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j B_{ij}$$

for $\gamma = \frac{1}{\sqrt{\tau_2}} \begin{pmatrix} 1 & \tau_2 \\ \tau_2 & |\tau|^2 \end{pmatrix}, X^i = m^i \sigma_1 + n^i \sigma_2.$

- The integrand is manifestly invariant under world-sheet modular transformations $Sl(2,Z)_\tau$; less manifestly under $SO(n,n,Z)$ T-duality (it takes a Poisson resummation on m^i to make it manifest)

R^4 couplings in M/T^{d+1}

- For generic windings m^i, n^i and at large volume, the modular integral is dominated by saddle points that correspond to worldsheet instantons:

$$I_n \sim \sum_{m_{ij}} \mu(m^{ij}) \exp\left(-\frac{1}{l_s^2}\sqrt{(m^{ij})^2} + 2\pi i m^{ij} B_{ij}\right)$$

  where $m^{ij} = m^i n^j - m^j n^i$.

- The tree-level and one-loop contributions can be recognized as T-duality invariant Eisenstein series:

$$f_{R^4} = \frac{V_d}{g_s^2}\, \mathcal{E}^{SO(d,d,Z)}_{1;s=3/2} + \ \mathcal{E}^{SO(d,d,Z)}_{\mathbf{S};s=1} + \dots$$

- SUSY requires to have an eigenmode of the U-duality Laplacian,

$$\Delta_{E_d} f_{R^4} = \frac{3(d+1)(2-d)}{2(8-d)} f_{R^4}$$

  A solution can be found in terms of U-duality invariant Eisenstein series,

$$f_{R^4} = \frac{V_{d+1}}{l_M^9}\ \mathcal{E}^{E_d(Z)}_{\boldsymbol{string};s=3/2}$$

Green Gutperle; Kiritsis BP; Obers BP

D-instantons and membrane instantons

- Beyond the tree-level and one-loop terms, it exhibits non-perturbative $O(e^{-1/g})$ effects from instantonic D-branes wrapped on T^d:

$$f_n = f_n^{pert} + \frac{V}{g_s} \sum_{\{n\}} \frac{\mu(\{n\})}{S_{\{n\}}} K_1 \left(2\pi \frac{1}{g_s} S_{\{n\}} \right) e^{2\pi i \mathcal{R}\cdot n} + \dots$$

The summation measure has been rederived by a matrix model computation.

Green Gutperle Vanhove, Kostov

(in $D \leq 6$, there are extra contributions which scale as e^{-1/g^2}.)

BP,Kiritsis

- If one expands instead at large volume in M-theory, for $d \geq 3$, one finds membrane instantons:

$$\begin{aligned} f_{R^4} &= \frac{\pi^2 l_M^6}{3} + \pi \sum \frac{l_M^9}{\sqrt{(m^3)^2}} \\ &+ \pi l_M^6 \sum \left[\frac{l_M^6}{(n^3)^2} \right]^{1/2} \mu(n^3) e^{-\frac{2\pi}{l_M^3}\sqrt{(n^3)^2} + 2\pi i n^3 C_3} \end{aligned}$$

Q: Can one derive this result first principles in M-theory ?

3. Quantizing the BPS membrane

- We have seen that U-duality together with supersymmetry determines the R^4 amplitude exactly, as a non-trivial automorphic function. At one-loop in string theory, only zero modes contribute, and yield worldsheet instantons $T^2 \to T^n$.

- A microscopic definition of M-theory should be at least be able to reproduce this simple result. Since the large volume expansion shows that only toroidal membrane instantons appear, it is reasonable to expect that a one-loop BPS membrane computation yield the correct instantons.

- Some obvious (conservative) objections: the membrane is strongly interacting, so no clearcut separation between zero and non-zero modes (but work in a winding sector). No genus expansion for either (but fermionic zero-modes may pick up T^3 topology only). (Un)Success will decide on the correctness of our assumptions.

The membrane one-loop amplitude (1^{st} pass)

- By analogy with the string theory case, we may construct the one-loop BPS membrane amplitude by summing over zero-mode configurations $X^i = m^i_\alpha \sigma^\alpha$ and constant γ in $Sl(3)/SO(3)$, with Polyakov action

$$S = \frac{1}{l_p^3} \int d^3\sigma \sqrt{\gamma} \left(\gamma^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j G_{ij} - 1 \right) + \\ + i\ \epsilon^{\alpha\beta\gamma} \partial_\alpha X^i \partial_\beta X^j \partial_\gamma X^k C_{ijk}$$

  Note the cubic coupling, inherent to membranes.

- Assuming uniform summation measure, the resulting partition function for $T^3 \to T^d$ maps,

$$Z = \sum_{m^i_\alpha} \exp \left(\sqrt{\gamma}\ \mathrm{Tr}[m\gamma m^t G] - 1) + iC \cdot (m \wedge m \wedge m) \right)$$

  after integration over the volume $u = \det(\gamma)$ and the fundamental domain of $SO(3)\backslash Sl(3)$ reproduces the correct membrane instanton saddle points and one-loop spectrum, but the summation measure and spectrum degeneracy are off.

  Nicolai, BP, Plefka, Waldron

- This partition function is by construction modular invariant $Sl(3)$ on the wv and $Sl(d)$ in target-space, but not under the full U-duality group $E_d(Z)$. E.g., for $d = 3$, not under modular transforms of $T = C_{123} + iV_3$.

Partition functions and correspondences

- Going back to the string partition function, the way modular $Sl(2,Z)$ and T-duality $SO(n,n,Z)$ are simultaneously realized is that Z_n is really a symplectic theta series

$$Z_n(g,B;\tau) = \theta_{Sp}(T) := \sum_{m\in Z^{2n}} \exp\left(2\pi i\ m^I T_{IJ} m^J\right)$$

  hence it is invariant under a larger group containing both as commuting subgroups:

$$Sl(2)\times SO(n,n) \subset Sp(2n) : T \to (AT+B)(CT+D)^{-1}$$

  courtesy of Gaussian Poisson resummation.

- By integrating wrt to either factor, Z_n furnishes a correspondence between automorphic forms of worldsheet $Sl(2,Z)$ and target space $SO(n,n,Z)$.

- In order to quantize the BPS membrane, one should therefore construct an automorphic correspondence between $Sl(3,Z)$ and $E_d(Z)$. For this, we need to find a group containing the two as (maximal) commuting factors, and generalize Gaussian Poisson resummation to cubic characters...

Membrane correspondences

- For compactifications on T^d, the U-duality groups can be unified with the modular $Sl(3)$ group as

$$
\begin{array}{lll}
d=3 & Sl(3)\times Sl(2)\times R^+\times Sl(3)\subset (Sl(3)^3) & \subset E_6\\
d=4 & Sl(5)\times R^+\times Sl(3)\subset Sl(6)\times Sl(3) & \subset E_7\\
d=5 & SO(5,5)\times R^+\times Sl(3)\subset SO(5,5)\times Sl(4) & \subset E_8\\
d=6 & E_6\times Sl(3) & \subset E_8
\end{array}
$$

- For $d \geq 3$ (no membrane instantons in $d < 3$) one is thus left to construct Theta series for exceptional groups: mathematicians know them in principle, but in practice...

- One should also require that after integrating over $R^+ \times Sl(3)$, one gets an eigenmode of Δ_{E_d} with the correct eigenvalue. So before integration:

$$\left(\Delta_{E_d} + \Delta_{Sl(3)} + \partial_u^2 + cste\right) Z_d = 0$$

  This will be satisfied by construction...

- For $d = 6$, no integration on a volume factor is required: the integral will be manifestly finite. In addition to membrane instantons, the R^4 coupling contains M5-brane instantons: membranes and five-branes will be unified in this framework.

Theta series under the hood

The standard theta series can be deconstructed as

$$\theta(\tau) = \sum_{m \in Z} \exp(i\pi\tau m^2) = \langle \delta_Z, \rho(e_\tau) f \rangle \ , \quad e_\tau = \begin{pmatrix} 1 & \tau_1 \\ 0 & \tau_2 \end{pmatrix} / \sqrt{\tau_2}$$

- $\rho(g)$ is a unitary representation of $g \in Sl(2)$ on functions of one variable:

  $$E_+ = i\pi\, x^2 \, , \quad D_0 = \frac{1}{2} (x\partial_x + \partial_x x) \ , \quad E_- = \frac{i}{4\pi} \partial_x^2 \, ,$$

  satisfying the $Sl(2,R)$ algebra,

  $$[D_0, E_\pm] = \pm 2\, E_\pm \ , \quad D_0 = [E_+, E_-] \ ,$$

- $f(x) = e^{-x^2/2}$ is a spherical vector, i.e. a function ϕ (quasi) annihilated by the compact generator $K = E_+ + E_-$; in particular invariant under the Weyl generator $\exp(i\pi K) = Fourier$.

- δ_Z is a distribution invariant under $Sl(2,Z)$,

  $$\delta_Z(x) = \sum_{m \in Z} \delta(x - m) =^{***} \prod_{p \text{ prime}} f_p(x) \ ,$$

  where each f_p is invariant under Fourier transform over the p-adic field.

All these parts can be engineered for any simply-laced G

Non-Gauss. Poisson resum. : a toy model

- The invariance of the standard theta series under $\tau \to -1/\tau$ relies on Poisson resummation formula,

$$\sum_{n\in Z} f(n) = \sum_{m\in Z} \tilde{f}(m) \ , \quad \tilde{f} = Fourier(f)$$

  and the fact that the Gaussian is preserved under Fourier transform:

$$\int dx \exp(ix^2/\hbar + ipx) = \sqrt{\hbar} \exp(-i\hbar\ p^2)$$

  In other words, for a Gaussian the semi-classical (saddle) approximation is exact. Perturbative QFT is arises from generalizing to ∞ x's.

- Interestingly, there exists a generalization of this to cubic characters:

$$\int dx^{0123} (1/x_0) \exp\left(i \frac{x_1 x_2 x_3}{\hbar\ x_0} + p_i x^i \right) = (\hbar/p_0) \exp\left(-i\hbar \frac{p_1 p_2 p_3}{p_0} \right)$$

  Again, the saddle point approximation is exact. Such cubic forms are classified by (A)DE:

$$\begin{array}{lcl} D_n & : & I_3 = x_1(x_2 x_3 + x_4 x_5 + \dots) \\ E_6 & : & I_3 = \det(3 \times 3) \\ E_7 & : & I_3 = \mathrm{Pf}(6 \wedge 6) \\ E_8 & : & I_3 = 27^3|_1 \end{array}$$

  Etingof Kazhdan Polischuk

- This observation is at the heart of the construction of theta series for simply laced groups. *Kazhdan Savin*

Minrep and conformal quantum mechanics

- The representation space is constructed as the Hilbert space of a conformal quantum mechanical system whose phase space is G's minimal nilpotent orbit.

  de Alfaro Fubini Furlan

- Classically, the Lagrangian is manifestly invariant under G_0,

  $$\mathcal{L} = \dot{x}_0 \dot{y} + 2x_0 \sqrt{I_3(\dot{x}_i)} + \frac{d}{dt}\left(\frac{x_0 I_3(x_i)}{y}\right)$$

  the Hamiltonian is invariant under $G_1 \supset G_0$ mixing positions and momenta,

  $$\mathcal{H} = p^2 + y^2 + \frac{1}{y^2} I_4(x_I, p_I)$$

  and the conformal transformations, $t \to (at+b)/(ct+d)$ extend the symmetry group to $G \supset G_1 \supset G_0$.

  BP Waldron; Gunaydin Koepsell Nicolai

- The quantization of this system produces the minimal representation of G as differential operators acting on wave functions.

Quantization and spherical vector

- Quantization is carried out by replacing $p_i \to i d/dx_i$ and adding normal ordering terms so that the generators still close. More abstractly, it proceeds by a sequence of induced representations.

  Kazhdan Savin; Brylinsky Kostant

- The Weyl generators

$$(Sf)(y, x_0, \ldots, x_{N-1}) = \int \frac{\prod_{i=0}^{N-1} dp_i}{(2\pi y)^{N/2}} f(y, p_0, \ldots, p_{N-1})\, e^{\frac{i}{y}\sum_{i=0}^{N-1} p_i x_i}$$

$$(Af)(y, x_0, x_1, \ldots, x_{N-1}) = \exp\left(-\frac{iI_3}{x_0 y}\right) f(-x_0, y, x_1, \ldots, x_{N-1})$$

  satisfy the correct relation $(AS)^3 = (SA)^3$ thanks to the invariance of the cubic character under Fourier transform.

- The spherical vector is the ground state wave function of this quantum mechanical system, invariant under the maximal compact subgroup K of G. It can be found by solving PDEs $E_\alpha + E_{-\alpha} = 0$.

  Kazhdan BP Waldron CMP 2001

- The summation measure δ_Z is obtained by solving the same problem (with different methods) over the p-adic field Q_p.

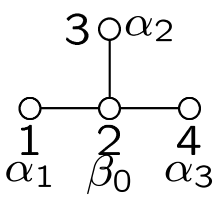
  Kazhdan Polischuk

Minimal Nilpotent Orbit

$$
\begin{array}{ccc}
Sl(n) & \supset & Sl(2)\times Sl(n-2)\times R^{+} \\
adj & = & (3,1,0)\oplus[(2,n-2,1)\oplus(2,n-2,-1)]\oplus(1,adj,0) \\
 & = & 1\oplus 2(n-2)\oplus[1\oplus adj]\oplus 2(n-2)\oplus 1 \\
SO(2n) & \supset & Sl(2)\times Sl(2)\times SO(2n-4) \\
adj & = & (3,1,1)\oplus(2,2,2n-4)\oplus(1,3,1)\oplus(1,1,adj) \\
 & = & 1\oplus(2,2n-4)\oplus[1\oplus adj]\oplus(2,2n-4)\oplus 1 \\
E_6 & \supset & Sl(2)\times Sl(6) \\
78 & = & (3,1)\oplus(2,20)\oplus(1,35) \\
 & = & 1\oplus 20\oplus[1\oplus 35]\oplus 20\oplus 1 \\
E_7 & \supset & Sl(2)\times SO(6,6) \\
133 & = & (3,1)\oplus(2,32)\oplus(1,66) \\
 & = & 1\oplus 32\oplus[1\oplus 66]\oplus 32\oplus 1 \\
E_8 & \supset & Sl(2)\times E_7 \\
248 & = & (3,1)\oplus(2,56)\oplus(1,133) \\
 & = & 1\oplus 56\oplus[1\oplus 133]\oplus 56\oplus 1
\end{array}
$$

G	dim	H_0	G_1^*	I_3
$Sl(n)$	$n-1$	$Sl(n-3)$	$[n-3]$	0
$SO(n,n)$	$2n-3$	$SO(n-3,n-3)$	$1\oplus[2n-6]$	$x_1(\sum x_{2i}x_{2i+1})$
E_6	11	$Sl(3)\times Sl(3)$	$(3,3)$	det
E_7	17	$Sl(6)$	15	Pf
E_8	29	E_6	27	$27^{\otimes_s 3}\vert_1$

Example: $D_4 = SO(4,4)$



$$\beta_i = \beta_0 + \alpha_i \ , \qquad \gamma_i = \beta_0 + \alpha_j + \alpha_k \ ,$$
$$\gamma_0 = \beta_0 + \alpha_1 + \alpha_2 + \alpha_3 \ , \qquad \omega = \beta_0 + \gamma_0$$

$$I_3 = x_1 x_2 x_3$$

$$\begin{array}{ll} E_{\beta_0} = y\partial_0 & E_{\gamma_0} = ix_0 \\ E_{\beta_1} = y\partial_1 & E_{\gamma_1} = ix_1 \\ E_{\beta_2} = y\partial_2 & E_{\gamma_2} = ix_2 \\ E_{\beta_3} = y\partial_3 & E_{\gamma_3} = ix_3 \end{array}$$
$$E_\omega = iy \, .$$

$$E_{\alpha_1} = -x_0\partial_1 - \tfrac{ix_2x_3}{y} \ , \qquad E_{-\alpha_1} = x_1\partial_0 + iy\partial_2\partial_3$$
$$E_{\alpha_2} = -x_0\partial_2 - \tfrac{ix_3x_1}{y} \ , \qquad E_{-\alpha_2} = x_2\partial_0 + iy\partial_3\partial_1$$
$$E_{\alpha_3} = -x_0\partial_3 - \tfrac{ix_1x_2}{y} \ , \qquad E_{-\alpha_3} = x_3\partial_0 + iy\partial_1\partial_2 \, .$$

$$
\begin{aligned}
E_{-\beta_0} &= -x_0\partial + \frac{i x_1 x_2 x_3}{y^2} \\
E_{-\beta_1} &= x_1\partial + \frac{x_1}{y}\,(1 + x_2\partial_2 + x_3\partial_3) - i x_0\partial_2\partial_3 \\
E_{-\gamma_0} &= 3i\partial_0 + iy\partial\partial_0 - y\partial_1\partial_2\partial_3 + i(x_0\partial_0 + x_1\partial_1 + x_2\partial_2 + x_3\partial_3)\,\partial_0 \\
E_{-\gamma_1} &= iy\partial_1\partial + i(2 + x_0\partial_0 + x_1\partial_1)\,\partial_1 - \frac{x_2 x_3}{y}\,\partial_0 \\
E_{-\omega} &= 3i\partial + iy\partial^2 + \frac{i}{y} + i x_0\partial_0\partial + \frac{x_1 x_2 x_3}{y^2}\,\partial_0 + \\
&\quad + \frac{i}{y}\,(x_1 x_2\partial_1\partial_2 + x_3 x_1\partial_3\partial_1 + x_2 x_3\partial_2\partial_3) \\
&\quad + i(x_1\partial_1 + x_2\partial_2 + x_3\partial_3)\,(\partial + \frac{1}{y}) + x_0\partial_1\partial_2\partial_3\,,
\end{aligned}
$$

$$
\begin{aligned}
H_{\beta_0} &= -y\partial + x_0\partial_0 \\
H_{\alpha_1} &= -1 - x_0\partial_0 + x_1\partial_1 - x_2\partial_2 - x_3\partial_3 \\
H_{\alpha_2} &= -1 - x_0\partial_0 - x_1\partial_1 + x_2\partial_2 - x_3\partial_3 \\
H_{\alpha_3} &= -1 - x_0\partial_0 - x_1\partial_1 - x_2\partial_2 + x_3\partial_3\,,
\end{aligned}
$$

Example: $D4 = SO(4,4)$ (continued)

- Spherical vector: solve PDE $(E_\alpha - E_{-\alpha})f = 0$:

$$f_{D_4} = \frac{4\pi}{|z|} K_0 \left(\frac{\sqrt{(|z|^2 + x_1^2)(|z|^2 + x_2^2)(|z|^2 + x_3^2)}}{|z|^2} \right) e^{-i \frac{x_0 x_1 x_2 x_3}{y|z|^2}}$$

  where $z = y + i x_0$. This is manifestly invariant under $SO(4,4)$ triality, permuting x_1, x_2, x_3.

- Rk: this minimal representation is equivalent to the one arising from the string worldsheet instantons on T^4: by Fourier transforming on x_3 and renaming variables, we can rewrite f as

$$f = \frac{e^{-2\pi\sqrt{(m^{ij})^2}}}{\sqrt{(m^{ij})^2}} , \qquad \epsilon^{ijkl} m_{ij} m_{kl} = 0$$

  This implies that the one-loop BPS amplitude of Het/T^4 is invariant under $SO(4,4)$ triality, as predicted from Heterotic-Type II duality.

  Kiritsis Obers BP

E_6 *theta series and Membrane/*T^3

- For E_6 the minimal nilpotent orbit is parameterized by 11 positions (y, x_0, M^i_α) transforming as $1 + 1 + (3, 3)$ under the linearly represented subgroup $G_0 = Sl(3) \times Sl(3)$: there are two unexpected quantum numbers (y, x_0). The cubic form is simply $I_3 = \det(M)$.

- The spherical vector invariant under the maximal compact $SU(8)$ can be obtained by integrating the PDE $(E_\alpha - E_{-\alpha})f = 0$:

$$f_{E_6} = \frac{e^{-(S_1 + i S_2)}}{|z|^2 S_1} \,, \qquad z = y + i x_0$$

$$S_1 = \frac{\sqrt{\det(M M^t + |z|^2 \mathbb{I}_3)}}{|z|^2} \,, \quad S_2 = \frac{x_0 \det(M)}{y|z|^2}$$

- The variables M can be identified with the winding numbers of the membrane $X^i = \mathcal{M}^i_\alpha \sigma^\alpha$. z looks like a complex scalar on the worldvolume, or rather an $Sl(2)$ doublet of 3-form field strengths: no dof, cosmological constant on the worldvolume.

- The representation satisfies identically

$$\Delta_{Sl(3)_1} = \Delta_{Sl(3)_2} = \Delta_{Sl(3)_3}$$

which agrees with the result expected for f_{R^4} after integrating over wv Sl(3).

4. Quantum cosmology and conformal quantum mechanics

- Conformal quantum mechanics was first introduced in 1976 by de Alfaro, Fubini and Furlan (DFF) as an attempt to understand soft breaking of conformal invariance.

- The simplest example is a particle in one dimension,

$$H = \frac{1}{2}\left(p^2 + \frac{g}{q^2}\right)$$

The Hamiltonian $H = E_+$ can be supplemented by two generators

$$E_+ = \frac{1}{2}q^2\,, \quad D_0 = \frac{1}{2}(q\,p+p\,q)\,, \quad E_- = \frac{1}{2}\left(p^2 + \frac{g}{q^2}\right)$$

that represent the conformal group $SO(2,1)$ in 0+1 dimensions,

$$\{E_+, E_-\} = 2D_0, \quad \{D_0, E_\pm\} = \pm E_\pm$$

- H is a parabolic element of $SO(2,1)$, hence has a continuous spectrum starting at 0. Adding a mass term amounts to choosing a compact generator $H = E_+ + m^2 e_-$, rendering the spectrum discrete.

Spacelike singularity and q-mechanics

- As one approaches a cosmological (spacelike) singularity, the dynamics of nearby points decouple.

  Belinski Khalatnikov Lifschitz

$$g_{\mu\nu}(t,x) \sim t^{\alpha} g_{\mu\nu}(t,x_0) + o(t^{\alpha})$$

  yielding a one-dimensional quantum mechanical system at each point on a spacelike slice. This validates a minisuperspace ansatz,

$$ds^2 = -\frac{\eta(t)}{V(t)}\,dt^2 + V^{2/n}(t)\,\widehat{g}_{ij}(t)\,dx^i dx^j\,,$$

- the Einstein-Hilbert action reduces to

$$\int dt\ d^n x\ \sqrt{-g}\,(R-2\Lambda) =$$

$$\int\ dt\left\{\frac{1}{2\eta}\left[-\frac{2(n-1)}{n}\dot{V}^2 + V^2\,\dot{U}^M G_{MN}\dot{U}^N\right] - 2\Lambda\eta\right\}$$

  where V denotes the volume of the spatial metric and U^M coordinatize the symmetric space $S = Sl(n)/SO(n)$

- One recognizes the Lagrangian for a free particle propagating on the Lorentzian cone

$$d\sigma^2 = -\frac{2(n-1)}{n}\,dV^2 + V^2 dU^M G_{MN} dU^N\,.$$

BP Waldron

Spacelike singularity and conformal q-mechanics

- The effect of spatial gradients is to add potential terms, behaving as reflection walls towards the singularity. These can be mimicked by modding out by an arithmetic subgroup, $Sl(n,Z)$.

- Since the moduli space admits an homothetic Killing vector $V\partial_V$, the free particle should exhibit conformal invariance.

- Choose $\eta = V$, $\rho = \sqrt{8(n-1)V/n}$ and p its canonical conjugate. The eom for η is the Wheeler-DeWitt equation,

$$H = \frac{1}{2}p^2 + \frac{4(n-1)}{n\rho^2}\Delta - \frac{n\Lambda}{4(n-1)}\rho^2 \equiv 0\,.$$

  We recognize the DFF Hamiltonian, with $g = 8(n-1)\Delta/n$. The cosmological constant induces a mass term $m^2 = -n\Lambda/(4(n-1))$.

- For $\Lambda < 0$, the wave function of the universe is therefore the spherical vector of $SO(2,1)$!

- The $SO(2,1)$ algebra fixes ordering for the quantized model.

DFF vs WDW

Despite formal identity between the two problems, there are some important differences:

- The WDW equation picks out zero-energy states only. So boundedness from below of H is no longer a requirement. Indeed, the sign of g depends on boundary conditions on S (square integrable wave functions have $g < 0$), and the sign of m^2 depends on Λ (discrete spectrum for $\Lambda < 0$)

- Usual quantum mechanics analysis requires wave functions to be square integrable. Here ρ should be thought as a time variable, square integrability along ρ should not be imposed. Instead perhaps, use a Klein-Gordon type norm on spacelike slices (and "third" quantize the system in order to get rid of negative norm states)

Those are problems in any quantum cosmology investigation, so we proceed anyway.

Quantum cosmo. and nilpotent orbits

Our quantization of nilpotent coadjoint orbits produces a wealth of conformal quantum mechanical systems - could they be related to those appearing in q-cosmology ? Yes, consider D_4 again:

$$\begin{array}{rcl} D_4 & \supset & Sl(2) \times Sl(2) \times Sl(2) \\ adj & = & [(1,1,1) + (3,1,1) + \text{perm}]_0 \oplus (2,2,2)_1 \oplus 1_2 \end{array}$$

The coordinates and momenta transform as a $(2,2,2)$, and satisfy the Heisenberg algebra

$$[q^{aA\alpha}, q^{bB\beta}] = \epsilon^{ab}\epsilon^{AB}\epsilon^{\alpha\beta}$$

The actions of each $Sl(2)$ factor in H are represented by the angular momentum-like operators

$$\Sigma^\mu = \sigma^\mu_{\alpha\beta} \;\; \epsilon_{ab} \; \epsilon_{AB} \; q^{aA\alpha} \;\; q^{bB\beta} \;, \quad [\Sigma^\mu, \Sigma^\nu] = \epsilon^{\mu\nu\rho}\Sigma^\rho$$

The quadratic Casimirs of all three $Sl(2)$'s are identical and equal to the unique quartic invariant I_4 of the $(2,2,2)$ representation.

Choose $Q^{A\alpha} = q^{1A\alpha}$ as positions, $q^{2A\alpha}$ as momenta. The bispinor $Q^{A\alpha}$ is a vector Q^I of $SO(2,2)$, parameterized by three polar angles $\Omega \in H_3 = SO(2,2)/SO(2,1) = SO(2,1)$ and its length squared $\kappa^2 = Q^I \eta_{IJ} Q^J$, where $\eta_{IJ} = (++--)$. The quadratic Casimir is then the angular momentum squared on the pseudo-sphere H_3, *i.e.* the Laplacian on $Sl(2)$.

This is the conformal mechanical model coming from dimensional reduction of $2+1$ dimensional gravity near a spacelike singularity, except for a decoupled degree of freedom κ.

Summary - prospects

- Non-perturbative dualities such as U-duality of M-theory bring automorphic forms into the realm of physics, allowing to obtain exact amplitudes including instanton effects.

- Deriving these amplitudes from a microscopic definition of M-theory is an important challenge. A naive one-loop membrane computation produces the right saddle points and spectrum, but misses the non-trivial summation measure.

- The correct one-loop membrane amplitude should provide a correspondence between $Sl(3, Z)$ and $E_d(Z)$ automorphic forms. Since naive physics failed, maths has to come to rescue.

- We have obtained explicit formulae for the spherical vector of the minimal rep for D_n and exceptional groups $E_{6,7,8}$. In order to write a Theta series completely, one needs to understand degenerate contributions. Can one find a simple combinatoric formula for the summation measure ?

- The construction relied on the invariance of the cubic character $\exp(iI_3(x_i)/x_0)$ under Fourier transform: a class of non-Gaussian yet free cubic models. Can models be found with ∞ degrees of freedom ? the topological open membrane ?

Summary - prospects 2

- Applied to the membrane, it predicts new quantum numbers (y, x_0) besides the expected windings n^i_α. What is their interpretation ? Can their action be generalized to include fluctuations while preserving duality ? Finally, do we reproduce the right R^4 amplitude after integrating out $R \times Sl(3)$?

- The quantization of coadjoint nilpotent orbits provides the Hilbert space for the minimal representation. This yields a new class of conformal quantum mechanical model. Can they be useful for black holes or other systems ?

- We have found that quantum cosmology near a spacelike singularity exhibits conformal invariance. Not so surprising, since we are expanding around a solution with power-like behavior. Can one identify other models than D_4 ?

- The chaotic mixing behavior is very very reminiscent of fully developped turbulency in fluid mechanics. Does this conformal "inertial range" extend to the singularity, or should pair creation provide a dissipation cut-off for this cascade ?