Exact degeneracies of small black holes and the topological string amplitude

Boris Pioline
LPTHE and LPTENS, Paris

Ecole Polytechnique
March 22, 2005

based on `hep-th/0502157` in collaboration w/ A. Dabholkar, F. Denef, G.W.Moore

slides available from
`http://www.lpthe.jussieu.fr/~pioline/seminars.html`

Bekenstein-Hawking entropy and D-brane counting

- General relativity associates to a black hole with horizon area A a "geometric" entropy

$$S_{BH} = A/(4G_N)$$

which satisfies properties analogous to the standard laws of thermodynamics

$$1)\ dS_{BH} = T_H dM + \dots\ , \qquad 2)\ d(S_{BH} + S_{matter}) > 0$$

where $T_H = \kappa/2\pi$ is the Hawking temperature of the black hole.

- One of string theory's strongest claims to fame is to provide a microscopic description of black hole microstates, reproducing the macroscopic Bekenstein-Hawking entropy. Eg, for "4-charge" extremal black holes in 4D,

$$S_{BH} = 2\pi\sqrt{Q_1 Q_5 Q_{KK} P}\ , \quad S_{micro} = \ln\Omega \sim 2\pi\sqrt{cN/6} \sim S_{BH}$$

- This agreement relies on the "thermodynamical" limit where $A \gg G_N$, or $Q \gg 1$, and classical gravity can be trusted. Can we test this beyond leading order, and compare gravitational corrections to the Bekenstein-Hawking entropy to finite size effects on the microscopic side ?

Black hole entropy beyond leading order

- On the macroscopic side, the Bekenstein-Hawking “area law” receives corrections due to higher-derivative interactions in the low energy effective action. E.g, for 4D Einstein with polynomial interactions in $R_{\mu\nu\rho\sigma}$,

$$S_{BHW} = 2\pi \int_\Sigma \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \epsilon^{\mu\nu} \epsilon^{\rho\sigma} \sqrt{h} d\Omega \sim \frac{1}{4} A + \dots$$

Wald; Jacobson Kang Myers

where $\epsilon^{\mu\nu}$ is the binormal on the horizon Σ. In addition, the geometry itself is deformed (sometimes in a drastic way).

- On the microscopic side, the entropy is defined as the Legendre transform of the free energy, which depends on a choice of statistical ensemble. In the thermodynamical limit, the entropy is universal, but subleading corrections are not.

Entropy of $N=2$ black holes beyond leading order

- In the case of extremal black holes in type II/CY, carrying magnetic and electric charges p^A, q_A, the Bekenstein-Hawking-Wald entropy $S_{BHW}(p^A, q_A)$ taking into account an infinite set of "F-term" higher derivative corrections,

$$\int d^4\theta F(W, X^A) = \int d^4\theta \sum_{h=0}^{\infty} W^{2h} F_h(X^A) \sim \sum_{h=0}^{\infty} R_+^2 F_+^{2h-2} F_h(X^A)$$

  (and only those) has been computed to all orders in $1/Q$, by generalizing the tree-level "attractor mechanism".

  de Wit, Lopes Cardoso, Mohaupt

- Recently, it has been proposed that the microscopic statistical ensemble to be compared with this macrosopic result is a "mixed" ensemble, where magnetic charges are treated micro-canonically but electric charges are treated canonically:

$$Z_{OSV}(p^A, \phi^A) = e^{\mathcal{F}(p^A, \phi^A)} = \sum_{q_A \in \Lambda_{el}} \Omega(p^A, q_A) e^{-\phi^A q_A}$$

Ooguri Strominger Vafa

The OSV conjecture for BH degeneracies

- In particular, the free energy $\mathcal{F}(p, \phi)$ can be obtained from the entropy $S_{BHW}(p^A, q_A)$ by (inverse)Legendre transform, and takes a simple form:

$$\mathcal{F}(p^A, \phi^A) = -\pi \, \text{Im} \left[F(p^A + i\phi^A) \right]$$

  hence the OSV conjecture

$$Z_{OSV}(p^A, \phi^A) \equiv \left| \exp\left(\frac{i\pi}{2} F(p^A + i\phi^A) \right) \right|^2$$

- If correct, this provides a way to compute the microscopic degeneracies $\Omega(p^A, q_A)$ (or rather a suitable index) from the topological string amplitude $F(W, X)$, by inverse Laplace transform,

$$\Omega(p^A, q_A) \equiv \int d\phi^A \left| \exp\left(\frac{i\pi}{2} F(p^A + i\phi^A) \right) \right|^2 e^{\phi^A q_A}$$

- Conversely, one may hope to understand the non-perturbative completion of the topological string from knowledge of black hole micro-states.

More on the OSV conjecture

- From the point of view of Witten's interpretation of $\Psi = e^F$ as a wave function, and after rotating $\phi^A \to i\chi^A$, $\Omega(p^A, q_A)$ appears to be the Wigner function associated to the pure state $|\Psi\rangle\langle\Psi|$.
- The OSV proposal is somewhat formal: what is the precise integration measure and contour ? Should we compare to absolute microscopic degeneracies, or to some suitable index ? Should we count micro-states with arbitrary angular momentum or only $J = 0$? How about holomorphic anomalies, curves of marginal stability, etc ?
- The proposal has been "tested" in the case of non-compact CY: $O(-m) \oplus O(m) \to T^2$: BPS states are counted by topologically twisted SYM on N D4-brane wrapped on a 4-cycle $O(-m) \to T^2$, which is equivalent to 2D Yang Mills. The latter factorizes in the large N limit into

$$Z_{2dYM} = \sum_{l=-\infty}^{\infty} Z_+(t + lG_{YM}^2) Z_-(\bar{t} - lG_{YM}^2)$$

On the macroscopic side, one can show that $Z_+(t)$ is indeed the topological string amplitude.

Vafa; Aganagic Ooguri Saulina Vafa

Testing OSV: small black holes

- Our goal will be to test the OSV conjecture in cases where black holes degeneracies are exactly known. For this, restrict to K_3-fibered CY, which admit a dual description as heterotic / $K^3 \times T^2$.
- The heterotic string admits a class of perturbative BPS states, known as Dabholkar-Harvey states:

$$|osc, N\rangle \otimes \overline{|osc, 0\rangle} \times |n_i, w^i\rangle$$

  satisfying the matching condition $N - 1 = n_i w^i$. They carry purely electric charge, in the natural heterotic polarization. They are counted by simple modular forms.
- At strong coupling, these states remain stable and become black holes, carrying both electric and magnetic charges, in the natural type II polarization. They are singular at tree-level, but acquire a smooth horizon due to R^2-type corrections.

Sen 95; Dabholkar 04; Kallosh Maloney Dabholkar; Hubeny Maloney Rangamani; Bak Kim Rey

Testing OSV: small black holes

- We find that in a large set of models, the OSV conjecture predicts the correct (indexed) microscopic degeneracies to all orders in $1/Q$.
- However, it misses non-perturbative corrections, computable on the heterotic side by the Rademacher expansion.
- In some particular cases, OSV fails to reproduce even the leading entropy of some states. This is presumably due to decay of pairs of BPS states.

Outline of the talk

- Attractor formalism and the topological string amplitude
- OSV prediction for large and small black holes
- A benchmark case: $K_3 \times T^2$
- Other $N = 4$ examples
- $N = 2$ orbifolds
- Towards an exact OSV-type formula
- Discussion

The attractor mechanism

- In general, the near horizon geometry of extremal black holes is independent of the value of the moduli at spatial infinity. In particular, the Bekenstein-Hawking entropy is a function of the electric and magnetic charges q_A, p^A only.
- For spherically symmetric RN black holes in $N = 2$ SUGRA, the vector multiplet moduli are fixed at the horizon by the (tree-level) attractor equations

$$\text{Re}(X^A) = p^A \,, \qquad \text{Re}(F_A) = q_A$$

where $F_A = \partial F_0 / \partial X^A$ and $F_0(X)$ is the tree-level prepotential.
- Hypermultiplets on the other hand are not sourced by the black hole. The near-horizon geometry is independent of their value.
- The (tree-level) BH entropy for those charges is

$$S_{BH} = \frac{i\pi}{2} \left(q_A \bar{X}^A - p^A \bar{F}_A \right)$$

Ferrara Kallosh Strominger

The attractor mechanism, revisited

- This can be recast as follows: set $X^A = p^A + i\phi^A$ where ϕ^A is real. The second equation becomes

$$q_A = \frac{1}{2}\left(\partial F_0/\partial X^A + \partial\bar{F}_0/\partial\bar{X}^A\right) = \frac{1}{2i}\left(\partial F_0/\partial\phi^A - \partial\bar{F}_0/\partial\bar{\phi}^A\right)$$

hence

$$q_A = \pi\ \partial\mathcal{F}/\partial\bar{\phi}^A \quad \text{where} \quad \mathcal{F}_0(p^A,\phi^A) = \frac{1}{\pi}\mathrm{Im}F_0(p^A + i\phi^A)$$

- In addition, the BH entropy may be rewritten as

$$S_{BH} = \mathcal{F}_0(p^A,\phi^A) + \pi\ q_A\phi^A$$

- The BH entropy $S_{BH}(p^A, q_A)$ can thus be viewed as the Legendre transform of the free energy $\mathcal{F}_0(p^A,\phi^A)$! To compute the latter, no need to solve the attractor equations !

Ooguri Vafa Strominger

Leading entropy of large black holes

- As an application, let us compute the tree-level entropy of a black hole with arbitrary charges, except for $p^0 = 0$: the tree-level superpotential is

$$F = -\frac{1}{6} C_{ABC} \frac{X^A X^B X^C}{X^0} \Rightarrow \mathcal{F}(p, \phi) = -\frac{\pi}{6} \frac{C(p)}{\phi^0} + \frac{\pi}{2} \frac{C_{AB}(p) \phi^A \phi^B}{\phi^0}$$

$$C(p) = C_{ABC} p^A p^B p^C \ , \quad C_{AB}(p) = C_{ABC} p^C \ , \quad A = 1, \ldots n_V - 1$$

- The Legendre transform with respect to ϕ^A leads to

$$\phi_*^A = -C^{AB}(p) q_B \phi^0 \ , \qquad \phi_*^0 = \pm \sqrt{-\hat{C}(p)/6\hat{q}_0}$$

$$\hat{q}_0 = q_0 + \frac{1}{2} q_A C^{AB}(p) q_B$$

- The tree-level Bekenstein-Hawking entropy is therefore the square-root of a quartic polynomial in the charges,

$$S_{BH} = 2\pi\sqrt{C(p)\hat{q}_0/6}$$

  in agreement from the microscopic counting at leading order.
- When $C(p) = 0$, the tree-level BH entropy vanishes, indicating a singular solution. We shall be interested in such “small black holes”, which get a non-vanishing entropy from higher order contributions.

The attractor mechanism, to all orders

- In the presence of $R_+^2 F_+^{2h-2}$ corrections, the same goes through upon replacing the tree-level prepotential $F_0(X)$ by the generating function

$$F(X^A, W^2) = \sum_{h=0}^{\infty} F_h(X^A) W^2$$

  and enforcing the additional attractor equation $W/X^0 = \pm 2^4$.
- The Bekenstein-Hawking-Wald entropy is thus the Legendre transform of the free energy

$$\mathcal{F}(p^A, \phi^A) = \frac{1}{\pi} \mathrm{Im} \mathcal{F} \left(p^A + i\phi^A; (2^4 X^0)^2 \right)$$

- One may interpret $\mathcal{F}(p^A, \phi^A)$ as the free energy of a statistical ensemble of black holes with magnetic charge p^A and electric potential ϕ_A. If so,

$$Z_{OSV}(p^A, \phi^A) := \sum_{q_A \in \Lambda_{el}} \Omega(p^A, q_A) e^{-\phi^A q_A} \equiv \left| \exp\left(\frac{i\pi}{2} F(p^A + i\phi^A) \right) \right|^2$$

The topological string amplitude

- Recall that the $(2,2)$ sigma-model on a CY threefold can be topologically twisted into the A-model topological string, which depends only on the Kähler moduli X^A. This defines a quantum field theory of Kähler structures, known as Kähler gravity.
- The topological A-model can be related to the physical type II superstring: the genus-h topological amplitude (without insertions) $F_h(X)$ is equal to the coefficient of the $R_+^2 F_+^{2h-2}$ amplitude in the low energy effective action

$$\int d^8\theta F(X;W^2) = \int d^8\theta \sum_{h=0}^{\infty} F_h(X) W^{2h} \sum_{h=0}^{\infty} F_h(X) R_+^2 F^{2h-2}$$

- The all-genus topological A-model thus resums an infinite number of higher-derivative F-term corrections. The topological coupling constant λ is proportional to the graviphoton field-strength,

$$\lambda = \frac{\pi}{4}\frac{W}{X^0}$$

Gromov-Witten vs Gopakumar-Vafa

- In general, $F(X^A, W^2)$ is an homogeneous function of degree 2 in (X^A, W):

$$F = -\frac{1}{6}C_{ABC}\frac{X^A X^B X^C}{X^0} - \frac{W^2}{64\cdot 24}\frac{c_A X^A}{X^0} - \frac{X_0^2}{(2\pi i)^3}\sum_{h=0}^{\infty}\sum_{\beta}\left(\frac{\pi W}{4X^0}\right)^{2h} N_{h,\beta}e^{2\pi i\beta_A X^A/X^0}$$

  where $C_{ABC} = \int_{CY} J_A J_B J_C$ are intersection numbers on $H_2(CY)$, $X^A/X^0 = B^A + iV^A$ are the Kähler moduli, $c_A = \int_{CY} J_A c_2(T^{1,0}(X))$ and $N_{h,\beta}$ are rational numbers known as the Gromov-Witten invariants.

- Taking into account "multi-covering" issues, the Gromov-Witten invariants $N_{h,\beta}$ can be rewritten in terms of the integer-valued Gopakumar-Vafa invariants $n_{\beta'}^{h'}$ (closely related to Donaldson-Thomas invariants)

$$\sum_{h=0}^{\infty}\sum_{\beta} N_{h,\beta}q^{\beta}\lambda^{2h-2} = \sum_{h'=0}^{\infty}\sum_{\beta'}\sum_{d=1}^{\infty} n_{\beta'}^{h'}\frac{1}{d}\left(2\sin\frac{d\lambda}{2}\right)^{2h'-2'} q^{d\beta'}$$

Pointlike instantons

- In particular, the point-like instantons with $\beta' = 0$ lead to $n_0^0 = -\chi/2$ (χ=Euler number of CY). They contribute an infinite series of higher-genus contributions to the topological amplitude:

$$F_{point} = -\frac{\chi}{2}\left[\frac{\zeta(3)}{\lambda^2} + A + \sum_{h=2}^{\infty} \lambda^{2h-2}\frac{(2h-1)B_{2h}B_{2h-2}}{(2h-2)(2h)!}\right]$$

- The $\zeta(3)$ term follows from the tree-level R^4 amplitude in 10D, the term with $h \geq 2$ is proportional to the Euler number of the moduli space of genus-h Riemann surfaces without punctures, and A is a naively divergent quantity, but, when properly regulated

$$A = \frac{1}{12}\log(2\pi/\lambda) + finite$$

- This asymptotic expansion is valid at $\lambda \ll 1$. If λ is large, an alternative representation is provided by the Mac Mahon function,

$$F_{point} = -\chi/2\sum_{n=0}^{\infty} n\log(1-q^n) \qquad q = e^{-\lambda}$$

leading to an infinite product representation for e^F.

Large Black Hole degeneracies from OSV

- Recall the OSV prescription: 1) Compute the free energy

$$\mathcal{F}(p^A, \phi^A) = \frac{1}{\pi}\text{Im}\mathcal{F}\left(p^A + i\phi^A; (2^4 X^0)^2\right)$$

  2) take the Laplace transform of $e^{\mathcal{F}}$ wrt to ϕ^A.

- The first step leads to

$$\mathcal{F}(p, \phi) = -\frac{\pi}{6}\frac{\hat{C}(p)}{\phi^0} + \frac{\pi}{2}\frac{C_{AB}(p)\phi^A\phi^B}{\phi^0} + 2\text{Re}(F_{GW})$$

  where

$$\hat{C}(p) = C(p) + c_A p^A \, , C(p) = C_{ABC}p^A p^B p^C$$

$$C_{AB}(p) = C_{ABC}p^C \, , A = 1, \ldots n_V - 1$$

- Next, drop F_{GW}, and evaluate the Laplace transform

$$\Omega_{OSV}(p^A, q_A) = \int d\phi^0 d\phi^A \exp\left(\mathcal{F}(p,\phi) + \pi\phi^A q_A\right)$$

- The ϕ^A integral is Gaussian, with saddle at $\phi^A_* = -C^{AB}(p) q_B \phi^0$:

$$\Omega_{OSV}(p^A, q_A) = \int d\phi^0 \phi_0^{(n_V-1)/2} \det[C_{AB}(p)]^{-1/2} \exp\left(-\frac{\pi}{6}\frac{\hat{C}(p)}{\phi^0} + \pi\phi^0 \hat{q}_0\right)$$

with

$$\hat{q}_0 = q_0 + \frac{1}{2} q_A C^{AB}(p) q_B$$

- The ϕ^0 integral is now of Bessel type, with saddle at $\phi^0_* = \pm\sqrt{-\hat{C}(p)/6\hat{q}_0}$. Assuming an appropriate contour, we find

$$\Omega_{OSV}(p^A, q_A) = \det[C_{AB}(p)]^{-1/2} [\hat{C}(p)]^{(n_V+1)/2} \hat{I}_{(n_V+1)/2}\left[2\pi\sqrt{\hat{C}(p)\hat{q}_0/6}\right]$$

- Using the asymptotics

$$\hat{I}_\nu(z) \sim z^{-\nu-\frac{1}{2}} e^z \left(1 + a/z + b/z^2 + \dots\right)$$

we find the corrected entropy,

$$S_{OSV} \sim 2\pi\sqrt{\hat{C}(p)\hat{q}_0/6} - \frac{n_V+2}{2}\log[\hat{C}(p)\hat{q}_0] + \dots$$

By construction, the leading term is the Bekenstein-Hawking-Wald entropy. It would be interesting to test the subleading log corrections against the microscopic description.

- In contrast to the tree-level entropy, the BHW entropy is non-vanishing for small BH, i.e. $C(p) = 0$, thanks to the one-loop correction $\hat{C}(p) = C(p) + c_A p^A$.

Small black holes and $K3$-fibered CY

- Let us now restrict to K3-fibered CY three-fold, so as to admit an heterotic dual description. The Kähler moduli split into the modulus X^1/X^0 of the base, and the moduli X^a/X^0 of the fiber ($a = 2, \dots n_V - 1$). The intersection form is

$$C_{ABC} X^A X^B X^C = X^1 C_{ab} X^a X^b$$

- Further consider a state whose only non-vanishing magnetic charge is p^1:

$$C(p) = 0\ , \quad \hat{C}(p) = 24 p^1\ , \quad C_{AB}(p) = \begin{pmatrix} 0 & 0 \\ 0 p^1 C_{ab} & \end{pmatrix}$$

- The dependence on ϕ^1 now disappears from the integrand. Since F_{top} is invariant under monodromies $\phi_1 \to \phi_1 + \phi_0$, it is natural to restrict the integration range to $[0, \phi_0]$:

$$\Omega_{OSV}(p^1, q_A) = \int d\phi^0\, \phi_0^{n_V/2} \exp\left(-\frac{4\pi p_1}{\phi^0} + \pi \phi^0 \hat{q}_0 \right) \sim \hat{I}_{(n_V+2)/2}\left[4\pi \sqrt{p^1 \hat{q}_0} \right]$$

where $\hat{q}_0 = q_0 + \frac{1}{2} C^{ab} q_a q_b / p_1$.

Comments

- Integrals have been carried out somewhat formally. Since $C_{AB}(p)$ in general has signature $(1, n_V - 2)$, the gaussian integral needs to be computed by rotating the contour for ϕ^A to the imaginary axis.
- In addition to the Bessel $\hat{I}$ function, the OSV integration measure leads to extra p-dependent factors, which, if taken literally, contradict T-duality on the heterotic side. The ratio $\Omega_{OSV}(p, q)/\Omega_{OSV}(p', q)$ seems to be free of these ambiguities.
- In the derivation, we neglected GW instanton contributions. Non-degenerate instantons contributions are exponentially suppressed in the large charge limit, and can be consistently neglected. For point-like instantons (assuming $\chi \neq 0$), $\phi_*^0 \to \infty$ so that the perturbative series diverges. One should go to the Mac-Mahon representation, which can be approximated to 1 if $q_0 \gg p_1$.

A benchmark case: $II/K3 \times T^2$ vs Het/T^6

- On the macroscopic side: thanks to $N = 4$, $F_{h>1} = 0$. F_1 can be extracted from R^2 coupling,

$$f_{R^2} \sim \log T_2 |\eta(T)|^4 \Rightarrow F_1 = \log \eta^{24}(T) \, , \quad T = X_1/X_0$$

- The gauge group is $U(1)^6 \times U(1)^{22}$, however upon decomposition into $N = 2$ multiplets 4 $U(1)$ are part of gravitino multiplets, and not covered by the attractor formalism. So $n_V = 24$.
- According to the above, the OSV prediction for small BH degeneracies is

$$\Omega_{OSV}(p^1, q_0) = \hat{I}_{13} \left[4\pi \sqrt{p^1 \hat{q}_0} \right]$$

- On the heterotic side, these small BPS BH are dual to Dabholkar Harvey states, enumerated by

$$\frac{1}{\eta^{24}} = \sum_{N=0}^{\infty} p_{24}(N) q^{N-1} \, , \quad N - 1 = p^1 q_0$$

- The leading exponential behavior is given by Cardy's formula $\log p_{24} = 2\pi \sqrt{N.24/6}$. Subleading corrections are given by the Rademacher formula...

The Rademacher expansion

Consider a vector-valued modular form $f_{\mu=1..r}(\tau)$ of weight $w < 0$,

$$f_\mu(\tau+1) = e^{2\pi i \Delta_\mu} f_\mu(\tau) \ , \quad f_\mu(-1/\tau) = (-i\tau)^w S_{\mu\nu} f_\nu(\tau)$$

with Fourier expansion $f_\mu(\tau) = q^{\Delta_\mu} \sum_{m=0}^{\infty} \Omega_\mu(m) q^m$

- Claim: the Fourier coefs can be expressed as an infinite series

$$\Omega_\nu(n) = \sum_{c=1}^{\infty} \sum_{\mu=1}^{r} \sum_{m+\Delta_\mu<0} c^{w-2} Kl(n,\nu;m,\mu;c) |m+\Delta_\mu|^{1-w}$$

$$\times \Omega_\mu(m) \hat{I}_{1-w} \left[\frac{4\pi}{c} \sqrt{|m+\Delta_\mu|(n+\Delta_\nu)} \right]$$

where $Kl(n,\nu;m,\mu;c)$ are generalized Kloosterman sums, equal to $S^{-1}_{\nu\mu}$ for $c=1$ and $\hat{I}_\nu(z)$ is a modified, modified Bessel function of the 1st kind,

$$\hat{I}_\nu(z) = 2\pi \left(\frac{z}{4\pi} \right)^{-\nu} I_\nu(z) \sim z^{-\nu-\frac{1}{2}} e^z (1 + a/z + b/z^2 + \dots)$$

- All $c > 1$ contributions are exponentially suppressed wrt to $c = 1$, yet they are exponentially large in an absolute sense.
- The Cardy-Hardy-Ramanujan formula emerges by keeping the leading term $c = 1, m = 0$, using $\Delta = c/24$:

$$\log \Omega_\nu(n) \sim 4\pi \sqrt{|\delta_\mu|(n + \Delta_\nu)} = 2\pi \sqrt{\frac{c(n + \Delta_\nu)}{6}}$$

- In addition to this leading term, there are log corrections, as well as an infinite series of power-suppressed terms.
- The Rademacher expansion depends only on the polar part $\sum_{m+\Delta_\mu<0} \Omega_\mu(m) q^{m+\Delta\mu}$ (and modular data). Indeed, one proof is to represent $f_\mu(\tau)$ (or rather its Farey transform $q\partial_q^{1-w} f$) as the Poincaré series (i.e. sum over $Sl(2, Z)$ images) of its polar part.

Back to the bench

- In particular, for the inverse of the Dedekind function, $w = -12$, $\Delta = -1$,$\Omega(0) = 1$ hence
$$p_{24}(N) = \hat{I}_{13}\left[4\pi\sqrt{p^1\hat{q}_0}\right] + 2^{-14}\hat{I}_{13}\left[2\pi\sqrt{p^1\hat{q}_0}\right] + \dots$$

- Comparing to the OSV prediction, we find agreement to ALL orders in $1/(p^1 q_0)$!
- However, OSV fails to reproduce subleading corrections which grow like $e^{2\pi\sqrt{p^1 q_0}}$.
- Note that for this to work, we had to drop non-holomorphic contributions from f_{R^2}, and consider the degeneracies of states with arbitrary angular momentum j.

$N = 4$ CHL strings

- More general $N = 4$ models with $0 \leq k \leq 22$ vector multiplets of $N = 4$ can be constructed, either as orbifolds of type II/ $K3 \times T^2$ by an Enriques involution, or as freely acting asymmetric orbifolds of Het/T^6.
- Consider the simplest case:

$$\Gamma_{6,22} = E_8(-1) \oplus E_8(-1) \oplus II^{1,1} \oplus II^{5,5}$$

orbifolded by $g|P_1, P_2, P_3, P_4\rangle = e^{2\pi i \delta \cdot P_3}|P_2, P_1, P_3, P_4\rangle$ This projects out the $U(1)$ associated to $P_1 - P_2$, leaving only the physical electric charges $Q = (P_1 + P_2, P_3, P_4)$.

- DH states arise in the untwisted sector by taking the ground state on the right, an arbitrary, orbifold invariant excitation of the 24 oscillators on the left, and level-matched internal momentum:

$$Z_{untw} = \frac{1}{2}\left(\frac{Z_{6,6}\left[{}^0_0\right]\theta^2_{E_8[1]}(\tau)}{\eta^{24}(\tau)} + \frac{Z_{6,6}\left[{}^0_1\right]\theta_{E_8[1]}(2\tau)}{\eta^8(\tau)\eta^8(2\tau)} \right)$$

- From this we need to extract the number of states with given $Q = (P_1 + P_2, P_3, P_4)$. For this, change basis from (P_1, P_2) to

$$P_1 + P_2 = 2\Sigma + \wp \,, \quad P_1 - P_2 = 2\Delta - \wp$$

where S, Δ take values in the E_8 root lattice, and $\mathcal{P}$ is an element of the finite group $Z = \Lambda_r(E_8)/2\Lambda_r(E_8)$.

- In order to sum over the "unphysical charges" Δ, introduce E_8 level-2 theta functions with characteristics:

$$\Theta_{E_8[2],\wp}(\tau) := \sum_{\Delta \in E_8(1)} e^{2\pi i \tau (\Delta - \frac{1}{2}\wp)^2}$$

and use

$$\theta^2_{E_8[1]}(\tau) = \sum_{\mathcal{P} \in E_8/2E_8} \theta_{E_8[2],\mathcal{P}}(\tau)\theta_{E_8[2],\mathcal{P}}(\tau) \,, \quad \theta_{E_8[1]}(2\tau) = \theta_{E_8[2],0}(\tau)$$

hence

$$Z_u = \frac{\theta^2_{E_8[2],\mathcal{P}}(\tau)}{\eta^{24}(\tau)} \pm \frac{1}{\eta^8(\tau)\eta^8(2\tau)} := q^{\Delta_\pm} \sum_{N=0}^{\infty} d^u_\pm(N) q^N$$

CHL strings, cont.

- In the twisted sector, the situation is simpler:

$$Z_t = \frac{1}{2}\left(\frac{1}{\eta^{12}\theta_4^4} \pm \frac{1}{\eta^{12}\theta_3^4}\right) := q^{\Delta_\pm} \sum_{N=0}^{\infty} d_\pm^t(N) q^N$$

- Using the Rademacher formula, we find

$$\begin{aligned} \dim \mathcal{H}_{BPS}(Q) &= 2^{-5} \hat{I}_9 \left(4\pi\sqrt{Q^2/2}\right) \\ &+ \hat{I}_9 \left(4\pi\sqrt{Q^2/4}\right) \begin{cases} 15 \cdot 2^{-10} + 2^{-6} e^{2\pi i P\cdot\delta}, & \wp \in \mathcal{O}_1 \\ 2^{-10}, & \wp \in \mathcal{O}_{248} \\ -2^{-10}, & \wp \in \mathcal{O}_{3875} \\ 2^{-10} e^{i\pi Q^2}, & Q \in \Lambda_1 \end{cases} + \ldots \end{aligned}$$

Hence we have agreement to all orders with OSV in all sectors. Subleading terms however are not captured by OSV, and depend crucially on the sector.

- In more generally $N = 4$ orbifolds, it is clear that degeneracies in the untwisted, unprojected sector will contain a term

$$\Theta/\eta^{24}$$

where Θ is the partition function of the lattice of charges which are projected out. Hence, the relation

$$1 - w = (n_V + 2)/2$$

will always hold. The argument of the Bessel function, however, requires more care.

Absolute degeneracies vs helicity supertraces

- We obtained agreement to all orders between the OSV prediction (at strong gravitational coupling) and the absolute degeneracy of DH states (at weak coupling). In general however, we expect that only a suitable index can be trusted in comparing weak and strong coupling results.
- The natural indexes to invoke are helicity supertraces:

$$\Omega_n = \mathrm{Tr}(-1)^F J_3^n$$

where F is the target space fermion number, and J_3 one generator of the little group of a massive particle in D=3+1. For low n, and large supersymmetry, this index receives only contributions from short multiplets, while long (non BPS) multiplets cancel out.
- For $N = 4$ SUSY, the natural index for $1/2$ (resp. $1/4$) BPS states is Ω_4 (resp. Ω_6). In heterotic orbifold constructions, Ω_4 is in fact equal to the absolute degeneracy of 1/2-BPS states, "explaining" agreement.
- For $N = 2$ SUSY, the natural index is $\Omega_2 \sim N_V - N_H$. As we shall see, in heterotic orbifolds this can be much smaller than the absolute degeneracy !

A few words on $N = 2$ models

- A number of type II/CY - Het/$K3 \times T^2$ dual pairs are known, where OSV can be tested. On the macroscopic side, the topological string amplitude receives contributions from arbitrary genus. Nevertheless, the entropy of small BH to all orders in $1/p^1 q_0$, at small p^1/q_0 depends only on the universal tree-level and one-loop terms.
- For heterotic asymmetric orbifolds with $N = 2$ supersymmetry, the DH states can be counted as before. In contrast to $N = 4$, untwisted DH states typically come in vector/hyper pairs, and are liable to decay.
- The unprojected, untwisted sector has $N = 4$ supersymmetry, hence $\Omega_2 = 0$. The absolute degeneracies typically go as $\hat{I}_\nu(4\pi\sqrt{Q^2/2})$, but the index of the Bessel function depends on the point in hypermultiplet moduli space.

N=2 models, cont.

- The projected, untwisted sector gives degeneracies of order

$$\Omega_2 \sim \hat{I}_{(n_V+2)/2}(4\pi\sqrt{|\Delta(g)|Q^2/2}) \ll \Omega_{OSV} \ , \quad \Delta(g) = -1 + \frac{1}{2}\sum_j \theta_j(g)(1-\theta_j(g))$$

is the oscillator ground state energy of the sector twisted by g. This follows from Rademacher expansion due to mixing with twisted sectors under modular transformations.

- By contrast, the twisted sectors gives degeneracies of order

$$\Omega_{abs} \sim \Omega_2 \sim \hat{I}_{(n_V+2)/2}(4\pi\sqrt{Q^2/2}) \sim \Omega_{OSV}$$

due to mixing with the untwisted sector under modular transformations.

- In a class of models such as Het/K3 with standard embedding, untwisted and twisted states cannot be distinguished, hence OSV gives the correct result to all orders.
- In other models such as FHSV, untwisted and twisted states can be distinguished by the modding of their charges, and OSV appears to fail in reproducing either Ω_{abs} or Ω_2, unless some coarse-graining is made.

Could the OSV formula be exact ?

- Go back to the benchmark case: exact degeneracies can be extracted from

$$1/\eta^{24} = \sum_{N=0}^{\infty} p_{24}(N) q^{N-1} := 1/\Delta(q)$$

by a contour integral:

$$p_{24}(N) = \frac{1}{2\pi i} \oint q^{-N} dq/\Delta(q) = \int dt\, t^{-14} \frac{\exp\left(\frac{\pi(N-1)}{t}\right)}{\Delta\left(e^{-4\pi t}\right)}$$

- By contrast, the OSV formula can be rewritten as

$$\Omega_{OSV}(p^1, q_0) \sim \int d\tau_1\, d\tau_2\, \tau_2^{-14} \frac{\exp\left(\frac{\pi(N-1)}{\tau_2}\right)}{|\Delta\left(e^{-2\pi\tau_2+2\pi i\tau_1}\right)|^2}$$

- The two agree asymptotically when $\Delta(q) \sim q$, but the OSV formula does not appear to make sense non-perturbatively !

Reverse engineering

- Rather than extracting BH degeneracies from the topological amplitude, one may try to construct the BH partition function from our partial knowledge of exact degeneracies.
- In type II/$K3 \times T^2$, the lattices of electric charges are

$$\begin{aligned} \Lambda^{IIA}_{elec} &= D0(q_0) \oplus D2/T2(q_1) \oplus D2/\gamma_2(q_a) \oplus \dots \\ \Lambda^{IIA}_{mag} &= D6/K3 \times T^2(p^0) \oplus D4/K3(p^1) \oplus D4/T_2 \times \gamma_2(p^a) \oplus \dots \end{aligned}$$

  Exact degeneracies are known for purely electric heterotic states , i.e. for vanishing $D2/T2, D4/T^2 \times \gamma^2, D6/K3 \times T^2$.

- Setting $p^0 = p^a = 0$, the BH partition function includes terms with $q^1 = 0$:

$$Z'_{BH} = \sum_{q^0, q^a \in II^{3,19}} p_{24} \left(1 + p^1 q_0 + \frac{1}{2} q_a C^{ab} q_b \right) e^{-\pi (q_0 \phi^0 + q_a \phi^a)}$$

- Inserting the unity

$$1 = \sum_N \delta\left[N - 1 - \frac{1}{2} q_a C^{ab} q_b\right] = \sum_N \sum_{k^0=0}^{p^1-1} \frac{1}{p^1} e^{2\pi i k^0 (N-1-\frac{1}{2} q_a C^{ab} q_b)/p^1}$$

inside the sum, the sum over N reconstructs the Dedekind function

$$Z'_{BH} = \frac{1}{p^1} \sum_{k^0=0}^{p^1-1} \frac{e^{-2\pi i \tau q_a C^{ab} q_b - \pi \phi^a q_a}}{\Delta(\tau)} , \quad \tau = \frac{i\phi^0 + 2k^0}{2p^1}$$

Doing a modular transformation on τ and a Poisson resummation on q_a gives

$$Z'_{BH} = \sum_{k_0=0}^{p^1-1} \sum_{k^a \in II^{19,3}} Z_0(\phi^A) , \quad Z_0(\phi^A) = \frac{\exp\left[-\frac{\pi}{2} \frac{p^1 C_{ab} \phi^a \phi^b}{\phi^0}\right]}{(p^1)^2 \, \Delta\left(\frac{2ip_1}{\phi^0}\right)}$$

- While Z_0 looks close to the topological string amplitude, it is in fact different: no $|\Delta|^2$, and the argument has no ϕ^1 dependence !

- The sum over translations $\phi^A \to \phi^A + ik^A$ guarantees that the BH partition function has the expected periodicity due to the charge quantization. Yet much of the information in the topological string amplitude is lost in the process of averaging !

Discussion

- The OSV partition function for BPS black holes has passed several non-trivial tests, leading to agreement with microscopic degeneracies to all orders in $1/Q^2$.
- For this to hold, a number of ambiguities had to be lifted: integration contour, holomorphic anomalies, identification of Ω_{OSV} with helicity supertraces, count states with arbitrary J.
- OSV is very successful in $N = 4$ models, less so in some $N = 2$ models. When $\chi \neq 0$, the saddle point lies at strong coupling of the pointlike instanton series, requiring a non-perturbative completion of the topological amplitude in this sector.
- At the non-perturbative level, a relation like "$Z_{BH} = |e^F|^2$" cannot hold, if only because the rhs does not satisfy the required periodicities.
- From the point of view of the wave function interpretation of the topological string amplitude, this means that the BH partition function is not a "pure state", but rather a non-trivial density matrix.
- Some sort of connection between Z_{BH} and the topological amplitude seems to hold, but the precise relation remains to be found.
- In a somewhat orthogonal approach, Sen was able to reproduce the BH entropy to all orders using a different ensemble, with a chemical potential μ for Q^2 rather than Q, and keeping holomorphic corrections...