The Quantum Attractor Mechanism

Boris Pioline

LPTHE and LPTENS, Paris

KITP, Dec 6, 2005

based on BP, hep-th/0506228 and work in progress
with M. Gunaydin, A. Neitzke and A. Waldron

slides available from
`http://www.lpthe.jussieu.fr/~pioline/seminars.html`

(Black hole degeneracies, BPS geodesic motion, unipotent representations, automorphic forms, and the adelic wave function of the universe...)

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Black hole thermodynamics and statistical mechanics

- In classical GR, black holes behave as thermodynamical systems with energy M, temperature $T = \kappa/2\pi$ and entropy $S_{BH} = A/(4G_N)$. Understanding the microscopic origin of this behavior is a challenge for quantum theories of gravity.

  Christodoulou, Bekenstein, Hawking

- For a class of extremal (BPS) 4D black holes, the Bekenstein-Hawking entropy is well reproduced by assuming that the micro-states are (effectively) weakly coupled open strings around some D-brane configuration.

  Strominger Vafa; Maldacena Strominger

- This agreement holds in the “thermodynamical” limit $A \gg G_N$ (equiv. $Q \gg 1$) where classical gravity is reliable, and is insensitive to the detailed microscopic dynamics. An important question it whether it continues to hold beyond leading order.

Some recent progress

- 1. On the macroscopic side, corrections to the Bekenstein-Hawking entropy have been analyzed, at least for a class of higher-derivative "F-term" interactions in $N = 2$ SUGRA, controlled by the topological string on CY. Hopefully, this is sufficient knowledge for BPS physics.

Wald; Cardoso Mohaupt de Wit

- 2. On the microscopic side, corrections to the Cardy formula may be studied using the M5-brane realization of $N = 2$ black holes, or quiver gauge theories. For $N = 4$, and more recently $N = 8$ black holes, an exact formula for the BH degeneracies has been conjectured on the basis of U-duality and 4D/5D lift.

Maldacena Witten Strominger;Dijkgraaf Verlinde Verlinde; Shih Strominger Yin,BP

- 3. In relating microscopic to macroscopic physics beyond leading order, one should also specify the thermodynamical ensemble / density matrix. Conjecturally, the ensemble implicit in the Bekenstein-Hawking-Wald entropy [in a given duality frame] is a mixed ensemble at fixed magnetic charge p^A, electric potential ϕ^A.

Ooguri Strominger Vafa

The OSV Conjecture

- Combining 1 and 3, Ooguri, Strominger and Vafa (OSV) have proposed a simple relation between micro-canonical degeneracies $\Omega(p^A, q_A)$ and the topological string amplitude:

$$\Omega(p^A, q_A) \sim \int d\phi^A |\Psi(p^A + i\phi^A)|^2 e^{\phi^A q_A} \qquad (*)$$

where $\Psi(X^A) = \exp\left(\frac{i\pi}{2} F(X^A)\right)$ is the topological wave function. Equivalently,

$$\sum_{q_A \in \Lambda_{el}} \Omega(p^A, q_A) e^{-\phi^A q_A} \sim \sum_{k^A \in \Lambda^*_{el}} \Psi^*(p^A + k^A + i\phi^A)\Psi(p^A - k^A + i\phi^A) \qquad (**)$$

- The $\sim$ sign in (**) allegedly denotes an equality to all orders in an expansion at large charges $(\lambda p^A, \lambda q_A)$, $\lambda \to \infty$. A non-perturbative generalization might hold upon completing the perturbative topological string amplitude and specifying a contour.

The OSV fact

- Semi-classically, the integral in (*) (or the sum in **) is dominated by a saddle point $(X, \bar{X})$ such that

$$\mathrm{Re}(X^A) = p^A, \quad Re(F_A) = q_A$$

These are the attractor equations, which determine the values of the scalar fields $(X, \bar{X})$ at the horizon in terms of the electric and magnetic charges.

- The saddle point approximation to the Laplace transform is a Legendre transform

$$\ln \Omega(p,q) \sim \mathsf{L}egendre[\mathcal{F}(p,\phi)] \,, \quad \mathcal{F} = \mathsf{Im} F(p^I + i\phi^I)$$

which agrees with CDM.

- Corrections around the saddle point lead to further corrections to the Bekenstein-Hawking entropy, beyond those already implied by the instanton or higher genus corrections to F:

$$S_{BH} = 2\pi\sqrt{I_4(Q)} + O(\log Q) + O(1/Q^2) + \cdots + O(e^{-Q})$$

Checks on the OSV conjecture

- The proposal has been tested in the case of non-compact CY: $O(-m) \oplus O(m) \to T^2$: BPS states are counted by topologically twisted SYM on N D4-brane wrapped on a 4-cycle $O(-m) \to T^2$, which is equivalent to 2D Yang Mills. At large N, this "factorizes" into $\sum_l \Psi_{top}(t + mlg_s)\Psi_{top}(\bar{t} - mlg_s)$.

  Vafa

- This was generalized for $O(-m) \oplus O(2g - 2 + m) \to \Sigma_g$, whose topological amplitude is related to q-deformed 2D Yang-Mills. The agreement with OSV for genus $g > 1$ however requires modular properties of YM_q which are less than obvious.

  Aganagic Ooguri Saulina Vafa

- Exact degeneracies are known in a class of "small black holes" dual to perturbative heterotic states. The OSV formula works beautifully in all $N = 4$ models, with some important subtleties in $N = 2$ orbifold models.

  Dabholkar Denef Moore Pioline

- Using the conjectural formulae for 1/4-BPS black hole degeneracies in $N = 4$ and $1/8$-BPS in $N = 8$, the OSV formula is again warranted, with some "volume factor corrections".

  Shih Yin

OSV conjecture and channel duality

- The OSV relation (*) may be rewritten suggestively as

$$\Omega(p,q) \sim \int d\chi \; \Psi^*_{p,q}(\chi) \; \Psi_{p,q}(\chi)$$

where the dependence on p, q is absorbed in Ψ:

$$\Psi_{p,q}(\chi) := e^{iq\chi}\Psi(\chi - p) := V_{p,q} \cdot \Psi(\chi)$$

- This is reminiscent of open/closed duality on the cylinder,

$$\mathrm{Tr} e^{-\pi t H_{open}} = \langle B | e^{-\frac{\pi}{t} H_{closed}} | B \rangle$$

In this analogy, $\Omega(p,q)$ is the trace of the open string Hamiltonian in the Hilbert space with charge (p,q), and $\Psi_{p,q}$ is the closed string boundary state. For the analogy to hold, both H_{open} and H_{closed} should vanish.

Ooguri Vafa Verlinde

Topological amplitude and quantum radial flow

- Indeed, the near-horizon geometry $AdS_2 \times S^2$ has the topology of a cylinder, and can in principle be quantized in two ways:

(global or Poincaré) time	$\leftrightarrow$	Conformal Quantum Mechanics
Radial quantization	$\leftrightarrow$	Quantum Attractor Flow

  Both Hamiltonians vanish due to the diffeomorphism invariance.

  Ooguri Vafa Verlinde;Dijkgraaf Gopakumar Ooguri Vafa; Gukov Saraikin Vafa

- In this interpretation, the topological amplitude is understood as the wave function for the radial attractor flow. In particular, it should satisfy the Wheeler-DeWit constraint $H = 0$. If one really thinks of radius as time, it is the wave function of the universe...
- Radial quantization of black holes is not a new idea: in fact much work was done on this problem in the gr-qc community. The novelty here is that one works in a SUSY context, for which the “mini-superspace” truncation to spherically symmetric geometries has some chance (perhaps) of being exact.

  Cavaglia de Alfaro Filippov; Kuchar; Thiemann Kastrup; Breitenlohner Hellmann

- Q: is there a physical principle that picks out Ψ_{top} from the infinite dimensional SUSY Hilbert space ?

Outline of the talk

- Our goal is to try and clarify these ideas, by considering situations with higher symmetry: $N = 8$ and $N = 4$ SUGRA, or "very special" $N = 2$ SUGRA. The complexity of CY geometry is jettisoned in favor of representation theory.
- Our approach is to reinterpret the equations governing the radial evolution of the metric and scalars as (BPS) geodesic motion on the scalar manifold $\mathcal{M}_3^*$ of the 3D SUGRA obtained by reducing 4D SUGRA along the time direction.

Breitenlohner Gibbons Maison, Gutperle Spalinski

- This geodesic motion is then quantized by replacing classical trajectories by functions on $\mathcal{M}_3^*$. BPS trajectories quantize into special (e.g. holomorphic) functions. When $\mathcal{M}_3^* = G_3/K_3^*$ is symmetric, the (BPS) Hilbert space may be understood in terms of (unusually small) irreps of G_3.

Gross Wallach; Kazhdan BP Waldron; Gunaydin Koepsell Nicolai

- Our main message is that, beyond the expected 4D U-duality symmetry, under which black hole degeneracies ought to be invariant, there is a larger "spectrum generating" symmetry G_3, the 3D U-duality group , which underlies the black hole wave function. Exact degeneracies should be expressed in terms of Fourier coefficients of automorphic forms for $G_3(\mathbb{Z})$.
- Warning: work in progress, many loose ends remain.

Plan of the talk

- Black hole entropy in $N = 8$, $N = 4$ and very special $N = 2$ SUGRA
- Attractor flow and geodesic motion
- The quantum attractor flow
- The automorphic black hole wave function
- Outlook

Black hole degeneracies in $N = 4$

- $N = 4$ theories with n_v vector multiplets have a moduli space

$$\mathcal{M}_4 = \frac{Sl(2)}{U(1)} \times \frac{SO(6, n_v)}{SO(6) \times SO(n_v)}$$

($n_v = 22$ for IIA$/K3 \times T^2$ or Het $/T^6$ model)

- Electric and magnetic charges transform like a doublet of $SO(6, n_v)$ vectors. The Bekenstein-Hawking entropy is given by

$$S_{BH} = 2\pi \sqrt{I_4}\ , \quad I_4 = \det \begin{pmatrix} \vec{p}^2 & \vec{p} \cdot \vec{q} \\ \vec{p} \cdot \vec{q} & \vec{q}^2 \end{pmatrix}$$

which is manifestly invariant under $Sl(2, \mathbb{R}) \times SO(6, n_v, \mathbb{R})$.

Counting $N = 4$ dyons

- Dijkgraav Verlinde Verlinde have made a conjecture for the 1/4-BPS black hole degeneracies in $n_v = 22$ model

$$\sum_{p^I, q_I} \Omega(p^I, q_I) e^{i(\rho \vec{p}^2 + \sigma \vec{q}^2 + (2\nu - 1)\vec{p}\cdot\vec{q})} = \frac{1}{\Phi(\omega)}, \quad \omega = \begin{pmatrix} \rho & \nu \\ \nu & \sigma \end{pmatrix} \in \frac{Sp(4)}{U(4)}$$

  where Φ is the unique weight 10 (Igusa) cusp form of $Sp(4, \mathbb{Z})$. The S-duality group $Sl(2, \mathbb{Z})$ is realized as a subgroup of the "genus 2" modular group $Sp(4, \mathbb{Z})$.

- This conjecture is supported by the recent 4D/5D lift, using the elliptic genus of $Hilb(K3)$. Variants now exist for CHL models.

  Shih Strominger Yin; Jatkar Sen

- Note however that p, q enter only via their inner products: there could exist more subtle invariants under T-duality.

Black hole degeneracies in $N = 8$

- For $N = 8$, i.e. M-theory on T^7, the scalar manifold is $\mathcal{M}_4 = \frac{E_{7(7)}}{SU(8)}$ and the electric and magnetic charges transform linearly under $E_{7(7)}$ as a 56. The BH entropy is

$$S_{BH} = 2\pi\sqrt{I_4(p,q)}$$

where I_4 is the E_7 quartic invariant:

$$Q = \begin{pmatrix} [D2]^{ij} & [F1]^i & [kkm]^i \\ -[F1]^i & 0 & [D6] \\ -[kkm]^i & -[D6] & 0 \end{pmatrix} , \quad P = \begin{pmatrix} [D4]_{ij} & [NS]_i & [kk]_i \\ -[NS]_i & 0 & [D0] \\ -[kk]_i & -[D0] & 0 \end{pmatrix} ,$$

$$\begin{aligned} I_4(P,Q) &= -\mathrm{Tr}(QPQP) + \frac{1}{4}\left(\mathrm{Tr}QP\right)^2 - 4\left[\mathrm{Pf}(P) + \mathrm{Pf}(Q)\right] \\ &= 4p^0 I_3(q_A) - 4q_0 I_3(p^A) + 4\frac{\partial I_3(q_A)}{\partial q_A}\frac{\partial I_3(p^A)}{\partial p^A} - (p^0 q_0 + p^A q_A)^2 \end{aligned}$$

Cremmer Julia;Hull Townsend;Kallosh Kol

and I_3 is the cubic invariant of the 5D U-duality group $E_{6(6)}$.

- $E_{7(7)}(\mathbb{Z})$ should be a symmetry of the exact BH degeneracies.

Counting $N = 8$ dyons

- By studying the elliptic genus of $Hilb(T^4)$, Maldecena Moore Strominger conjectured (and partially proved) that degeneracies of 5D BPS black holes in type II on T^5 were given by

$$\Omega_{5D}(N, Q_1, Q_5, \ell) = \sum_{s|(NQ_1, NQ_5, Q_1Q_5, \ell); s^2|NQ_1Q_5} s\, N(s)\, \hat{c}\left(\frac{NQ_1Q_5}{s^2}, \frac{\ell}{s}\right)$$

  where $\hat{c}(n, l)$ are the Fourier coefficients of the Jacobi form

$$-\frac{\theta_1^2(z,\tau)}{\eta^6} := \sum \hat{c}(n,l) q^n y^l \ , \quad \hat{c}(n,l) = \hat{c}(4n - l^2)$$

  and $N(s)$ is the number of divisors of $N, Q_1, Q_5, s, \frac{NQ_1}{s}, \frac{NQ_5}{s}, \frac{Q_1Q_5}{s}, \frac{NQ_1Q_5}{s^2}$

- By using the same 4D-5D lift, one may show that the exact number of micro-states is equal to

$$\Omega(p^I, q_I) = \hat{c}[I_4(p,q)]$$

  at least for black holes U-dual to a D0-D4-D6 bound state with $p^0 = 1$, and with all charges coprime. Again, there probably exist more subtle U-duality invariants than I_4.

Shih Strominger Yin; BP

Very special $N = 2$ supergravities

- For general $N = 2$ SUGRA, the moduli space is not symmetric and there is no U-duality (although we expect the monodromy group to put severe constraints on the BH degeneracies.
- There is an interesting class of $N = 2$ supergravities where the moduli space is a symmetric space. Although they still possess 8 SUSY, their extended symmetries facilitate the analysis greatly, and we shall see that some of them are related to $N = 4$ and $N = 8$ theories by analytic continuation.
- Their prepotential is purely cubic

$$F = N(X)/X^0 = C_{ABC} X^A X^B X^C / X^0$$

where $N(X)$ is the norm of a degree 3 Jordan algebra J. The moduli space is a symmetric space

$$M_4 = \frac{\text{Conf}(J)}{\text{Lorentz}^c(J) \times U(1)}$$

where $\text{Lorentz}^c(J)$ is the reduced structure group of J (in its compact form), while $\text{Conf}(J)$ is the conformal group leaving the cubic light-cone $N(X) = 0$ invariant.

Gunaydin Sierra Townsend

Very special supergravities

- Depending on the choice of the Jordan algebra J, this leads to two generic families

$$\frac{SU(n,1)}{SU(n)\times U(1)}\,, \quad \frac{SO(n,2)}{SO(n)\times SO(2)}\times\frac{Sl(2)}{U(1)}$$

  and a number of exceptional cases,

$$\frac{Sl(2)}{U(1)}\,,\frac{Sp(6)}{SU(3)\times U(1)}\,,\frac{SU(3,3)}{SU(3)\times SU(3)\times U(1)}\,,\frac{SO^*(12)}{SU(6)\times U(1)}\,,\frac{E_{7(-25)}}{E_6\times U(1)}$$

  corresponding to $N = X^0 Q_2, X^1 Q_2, (X^1)^3, \det(3\times_s 3), \det(3\times 3), \mathrm{Pf}(6\wedge 6), I_3(27)$ respectively.

- Although these may not exist as consistent string theories, they arise in the untwisted sector of type II orbifolds, or in heterotic string at tree-level.

A remark on Legendre invariance

- An important property following from the "adjoint identity" of Jordan algebras

$$X^{\sharp\sharp} = N(X) X \ , \quad X^\sharp_A := C_{ABC} X^B X^C$$

is that F is invariant under Legendre transform in all variables:

$$\langle N(X)/X^0 + X^0 Y_0 + X^A Y_A \rangle_{X^I} = -N(Y)/Y^0$$

Proof: saddle point at $Y_A = X^\sharp_A / X^0 \ , \quad Y_0 = -N(X)/(X^0)^2$, hence

$$\begin{aligned} N(X) X^A &= (X^0 Y_A)^\sharp = (X^0)^2 (Y^A)^\sharp \Rightarrow X^A = -Y^\sharp_A / Y^0 \\ N(Y) Y_A &= (-X^A Y_0)^\sharp \Rightarrow X^0 = N(Y)/(Y_0)^2 \end{aligned}$$

In fact, $(X^0)^\alpha N(X)^\beta e^{iN(X)/X^0}$ is invariant under Fourier transform, for some choice of α, β !

Etingof Kazhdan Polischchuk

BH entropy in very special SUGRA

- As an illustration of the OSV fact, let us compute the tree-level entropy of a black hole with arbitrary charges in very special SUGRA. The free energy is

$$\mathcal{F}(p,\phi) = \frac{\pi}{(p^0)^2+(\phi^0)^2}\left\{p^0\left[\phi^A p^\sharp_A - I_3(\phi)\right] + \phi^0\left[p^A\phi^\sharp_A - I_3(p)\right]\right\}$$

- In order to eliminate the quadratic term in ϕ^A, change variables to

$$x^A = \phi^A - \frac{\phi^0}{p^0}p^A\,, \quad x^0 = [(p^0)^2+(\phi^0)^2]/p^0$$

and, so as to eliminate the square root in $q_0\phi^0$, introduce an auxiliary variable t,

$$\mathcal{S} = \pi\langle -\frac{I_3(x)}{x^0} + \frac{p^\sharp_A + p^0 q_A}{p^0}x^A - \frac{t}{4}\left(\frac{x^0}{p^0}-1\right) - \frac{(2I_3(p)+p^0p^Iq_I)^2}{t\,(p^0)^2}\rangle_{\{x^I,t\}}$$

BH entropy, 4D and 5D

- Using the Legendre invariance of $N(X)/X^0$, we find

$$
\begin{aligned}
\mathcal{S} &= \pi \langle 4 \frac{I_3[p^\sharp_A + p^0 q_A]}{(p^0)^2 t} - \frac{[2I_3(p) + p^0 p^I q_I]^2}{t\,(p^0)^2} - \frac{t}{4} \rangle_t \\
&= \frac{\pi}{p^0} \sqrt{4 I_3 [p^\sharp_A + p^0 q_A] - [2 I_3(p) + p^0 p^I q_I]^2} \\
&= \pi \sqrt{4 p^0 I_3(q) - 4 q_0 I_3(p) + 4 q^A_\sharp p^\sharp_A - (p^0 q_0 + p^A q_A)^2}
\end{aligned}
$$

- By Freudenthal's triple system construction, the quartic polynomial is recognized as the quartic invariant under the 4-dimensional U-duality group.
- The intermediate equation also has an interesting interpretation: it is $1/p^0$ times the entropy of a 5D black hole with electric charge and angular momentum

$$
\begin{aligned}
Q_A &= p^0 q_A + C_{ABC} p^B p^C \\
2 J_L &= (p^0)^2 q_0 + p^0 p^A q_A + 2 I_3(p)
\end{aligned}
$$

consistent with the 4D/5D lift, generalized to include all charges.

Gaiotto Strominger Yin; BP

$N = 8$ and $N = 4$ topological amplitudes

- In particular, this holds in the very special $N = 2$ supergravity with $F = I_3(27)/X^0$, and leads to a $E_{7(-25)}$ invariant entropy formula. By analytic continuation, the same computation tells that the $E_{7(7)}$ invariant entropy of 1/8-BPS black holes in $N = 8$ can be obtained by pretending that the $N = 8$ topological amplitude is

$$\Psi_{N=8} = e^{i\frac{\pi}{2} I_3(27)/X^0}$$

  and describes all 56 electric-magnetic charges.
- Similarly, the $Sl(2) \times SO(6, n_v)$ invariant entropy of 1/4-BPS black holes in $N = 4$ with n_v multiplets can be obtained by analytic continuation from the very special $Sl(2) \times SO(2, n_v + 4)$ $N = 2$ supergravity, i.e. by pretending that the $N = 4$ topological amplitude is

$$\Psi_{N=4} = e^{i\frac{\pi}{2} X^1 X^a Q_{ab} X^b / X^0}$$

  where Q_{ab} is a signature $(5, n_v - 1)$ quadratic form.
- In either case, the 5D U-duality group is linearly realized, while the 4D group is non-linearly realized.

The attractor flow, revisited

- Stationary solutions in 4D can be parameterized in the form

$$ds_4^2 = -e^{2U}(dt+\omega)^2 + e^{-2U} ds_3^2 \,, \quad A_4^I = \zeta^I dt + A_3^I$$

where $ds_3, U, \omega, A_3^I, \zeta^I$ are independent of time. In 3D, the one-forms (A_3^I, ω) can be dualized into pseudo-scalars $(\tilde{\zeta}_I, a)$ (a is the NUT potential). The 4D Einstein-Maxwell equations reduce to 3D gravity + a non-linear sigma-model with target space $\mathcal{M}_3^*$.

- In contrast to the manifold $\mathcal{M}_3$ arising from KK reduction on along a space-like direction, $\mathcal{M}_3^*$ has an indefinite metric. It is obtained from that of $\mathcal{M}_3$ by analytic continuation $(\zeta^I, \tilde{\zeta}_I) \to i(\zeta^I, \tilde{\zeta}_I)$.

Ehlers;Geroch; Kinnersley; Breitenlohner Gibbons Maison; Hull Julia

- Importantly, $\mathcal{M}_3$ always has $2n+2$ isometries corresponding to the gauge symmetries of $A^I, \tilde{A}_I, \omega$, as well as rescalings of time t. The Killing vector fields satisfy the algebra

$$[p^I, q_J] = 2\delta^I_J \, k \,, \quad [m, p^I] = p^I, \; [m, q_I] = q_I \,, [m, k] = 2k$$

As the notation suggests, the associated conserved charges will be identified to electric and magnetic charges, NUT charge and ADM mass.

KK reduction on a time-like direction

- For $N = 8$ SUGRA,

$$\mathcal{M}_3 = E_{8(8)}/SO(16)\ , \quad \mathcal{M}_3^* = E_{8(8)}/SO^*(16)$$

- For $N = 4$ SUGRA with n_v vector multiplets,

$$\mathcal{M}_3 = \frac{SO(8, n_v + 2)}{SO(8) \times SO(n_v + 2)}\ , \quad \mathcal{M}_3^* = \frac{SO(8, n_v + 2)}{SO(6, 2) \times SO(2, n_v)}$$

- For generic $N = 2$ SUGRA, $\mathcal{M}_3$ is a quaternionic-Kahler manifold obtained from the special Kahler manifold M_4 by the "c-map". Its analytic continuation $\mathcal{M}_3^*$ is known as a "para-quaternionic-Kahler manifold".

Ferrara Sabharwal

- For very special $N = 2$ SUGRA, $\mathcal{M}_3$ is a symmetric quaternionic-Kahler manifold again obtained from Jordan algebra technology:

$$\mathcal{M}_3 = \frac{\text{QConf}(J)}{\text{Conf}^c(J) \times SU(2)}\ , \quad \mathcal{M}_3^* = \frac{\text{QConf}(J)}{\text{Conf}(J) \times Sl(2)}$$

Q	$D=5$	$D=4$	$D=3$	$D=3^*$
8		$\frac{SU(n,1)}{SU(n)\times U(1)}$	$\frac{SU(n+1,2)}{SU(n+1)\times SU(2)\times U(1)}$	$\frac{SU(n+1,2)}{SU(n,1)\times Sl(2)\times U(1)}$
8	$\mathbb{R}\times\frac{SO(n-1,1)}{SO(n-1)}$	$\frac{SO(n,2)}{SO(n)\times SO(2)}\times\frac{Sl(2)}{U(1)}$	$\frac{SO(n+2,4)}{SO(n+2)\times SO(4)}$	$\frac{SO(n+2,4)}{SO(n,2)\times SO(2,2)}$
8		$\frac{Sl(2)}{U(1)}$	$\frac{SU(2,1)}{SU(2)\times U(1)}$	$\frac{SU(2,1)}{Sl(2)\times U(1)}$
8	$\varnothing$	$\frac{Sl(2)}{U(1)}$	$\frac{G_{2(2)}}{SO(4)}$	$\frac{G_{2(2)}}{SO(2,2)}$
8	$\frac{Sl(3)}{SO(3)}$	$\frac{Sp(6)}{SU(3)\times U(1)}$	$\frac{F_{4(4)}}{USp(6)\times SU(2)}$	$\frac{F_{4(4)}}{Sp(6)\times Sl(2)}$
8	$\frac{Sl(3,C)}{SU(3)}$	$\frac{SU(3,3)}{SU(3)\times SU(3)\times U(1)}$	$\frac{E_{6(+2)}}{SU(6)\times SU(2)}$	$\frac{E_{6(+2)}}{SU(3,3)\times Sl(2)}$
24	$\frac{SU^*(6)}{USp(6)}$	$\frac{SO^*(12)}{SU(6)\times U(1)}$	$\frac{E_{7(-5)}}{SO(12)\times SU(2)}$	$\frac{E_{7(-5)}}{SO^*(12)\times Sl(2)}$
8	$\frac{E_{6(-26)}}{F_4}$	$\frac{E_{7(-25)}}{E_6\times U(1)}$	$\frac{E_{8(-24)}}{E_7\times SU(2)}$	$\frac{E_{8(-24)}}{E_{7(-25)}\times Sl(2)}$
10			$\frac{Sp(2n,4)}{Sp(2n)\times Sp(4)}$	?
12			$\frac{SU(n,4)}{SU(n)\times SU(4)}$	?
16	$\mathbb{R}\times\frac{SO(n-5,5)}{SO(n-5)\times SO(5)}$	$\frac{Sl(2)}{U(1)}\times\frac{SO(n-4,6)}{SO(n-4)\times SO(6)}$	$\frac{SO(n-2,8)}{SO(n-2)\times SO(8)}$	$\frac{SO(n-2,8)}{SO(n-4,2)\times SO(2,6)}$
18			$\frac{F_{4(-20)}}{SO(9)}$	?
20		$\frac{SU(5,1)}{SU(5)\times U(1)}$	$\frac{E_{6(-14)}}{SO(10)\times SO(2)}$	$\frac{E_{6(-14)}}{SO^*(10)\times SO(2)}$
32	$\frac{E_{6(6)}}{USp(8)}$	$\frac{E_{7(7)}}{SU(8)}$	$\frac{E_{8(8)}}{SO(16)}$	$\frac{E_{8(8)}}{SO^*(16)}$

Attractor flow and geodesic motion

- Now, restrict to spherically symmetric stationary solutions:

$$ds_3^2 = N^2(\rho)d\rho^2 + r^2(\rho)d\Omega_2^2$$

The sigma-model action becomes, up to a total derivative (g_{ij} is the metric on $\mathcal{M}_3^*$):

$$S = \int d\rho \left[\frac{N}{2} + \frac{1}{2N} \left(\dot{r}^2 - r^2 g_{ij} \dot{\phi}^i \dot{\phi}^j \right) \right]$$

- The lapse N can be set to 1, but it imposes the Hamiltonian constraint

$$H_{WDW} = (p_r)^2 - \frac{1}{r^2} g^{ij} p_i p_j - 1$$

which can be set to $N = 1$ by a gauge choice. Solutions are thus massive geodesics on the cone $\mathbb{R}^+ \times \mathcal{M}_3^*$. This separates into geodesic motion on $\mathcal{M}_3^*$, times motion along r. Keeping the variable r is crucial in defining observables such as the horizon area, $A = e^{-2U} r^2|_{U\to-\infty}$ and ADM mass $M_{ADM} = r(e^{2U} - 1)|_{U\to 0}$.

Geodesic motion and conserved charges

- The isometries of $\mathcal{M}_3$ imply conserved Noether charges. In particular, the electric and magnetic charges satisfy an Heisenberg algebra, whose center is the NUT charge k:

$$[p^I, q_J] = 2\delta^I_J k$$

  Furthermore, the ADM mass does NOT Poisson-commute with (p, q, k).
- If $k \neq 0$, the off-diagonal term $\omega = k\cos\theta d\phi$ in the 4D metric implies that the metric has CTC's at infinity. Bona fide 4D black holes need to have $k = 0$, which is a kind of classical limit. This meshes well with the OSV conjecture, which identifies $\Omega(p, q)$ as the Wigner function of the quantum wave function Ψ...Keeping $k \neq 0$ allows to greatly extend the symmetry.
- In addition, the motion along r is has a conformal $Sl(2)$ symmetry:

$$E_+ = H \ , \quad E_0 = r p_r \ , \quad E_- = r^2$$

De Alfaro Fubini Furlan

- BPS states need to have flat 3D slices, so we may set set $N = 1, r = \rho$ from the outset: A necessary condition for SUSY is therefore that geodesics be light-like.

Geodesic flow on special quaternionic Kahler manifolds

- Let us now reproduce the attractor flow equations of BPS black holes in $N=2$ SUGRA from geodesic flow on (the analytic continuation of) $\mathcal{M}_3 =$ c-map(M_4)

$$
\begin{aligned}
ds^2 =& 2(dU)^2 + g_{i\bar{j}}(z,\bar{z})dz^i dz^{\bar{j}} + \frac{1}{2}e^{-4U}\left(da + \zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right)^2 \\
&- e^{-2U}\left[(\mathrm{Im}\mathcal{N})_{IJ} d\zeta^I d\zeta^J + (\mathrm{Im}\mathcal{N}^{-1})^{IJ}\left(d\tilde{\zeta}_I + (\mathrm{Re}\mathcal{N})_{IK} d\zeta^K\right)\left(d\tilde{\zeta}_J + (\mathrm{Re}\mathcal{N})_{JL} d\zeta^L\right)\right]
\end{aligned}
$$

This is a quaternionic Kahler manifold, obtained from the special Kahler manifold by the "c-map".

Ferrara Sabharwal; de Wit Van Proyen Vanderseypen

- The conserved charges corresponding to the shift isometries are

$$
\begin{aligned}
q_I &= -2e^{-2U}\left[(\mathrm{Im}\mathcal{N})_{IJ} d\zeta^J + (\mathrm{Re}\mathcal{N})_{IJ}(Im\mathcal{N}^{-1})^{JL}\left(d\tilde{\zeta}_L + (\mathrm{Re}\mathcal{N})_{LM} d\zeta^M\right)\right] + 2\,k\tilde{\zeta}_I \\
p^I &= -2e^{-2U}(\mathrm{Im}\mathcal{N}^{-1})^{IL}\left(d\tilde{\zeta}_L + (\mathrm{Re}\mathcal{N})_{LM} d\zeta^M\right) - 2\,k\zeta^I \\
k &= e^{-4U}\left(da + \zeta^I d\tilde{\zeta}_I - \tilde{\zeta}^I d\zeta_I\right)
\end{aligned}
$$

Quaternionic viel-bein

- The quaternionic geometry can be exposed by defining a $SU(2) \times Sp(n_v)$ quaternionic vielbein, i.e. a $2 \times 2n_v$ pseudo-real matrix

$$V^{\alpha\Gamma} = \begin{pmatrix} u & v \\ e^A & E^A \\ -\bar{v} & \bar{u} \\ -\bar{E}^A & \bar{e}^A \end{pmatrix} = \left[\epsilon_{\alpha\beta}\rho_{\Gamma\Gamma'}V^{\beta\Gamma'}\right]^*$$

so that the three Kahler forms and metric are

$$\Omega^i = \epsilon_{\alpha\beta}\,(\sigma^i)^\beta_\gamma \rho_{\Gamma\Gamma'} V^{\alpha\Gamma} \wedge V^{\gamma\Gamma'}\,, \quad ds^2 = \epsilon_{\alpha\beta}\rho_{\Gamma\Gamma'}V^{\alpha\Gamma} \otimes V^{\beta\Gamma'}$$

In terms of the conserved charges, the one-forms entering V are

$$u = -\frac{i}{2}e^{K/2+U} X^I \left[q_I - 2k\tilde{\zeta}_I - \mathcal{N}_{IJ}(p^J + 2k\zeta^J)\right]\,, \quad v = -dU + \frac{i}{2}e^{2U}k$$

$$e^A = e_i^A\, dz^i\,, \quad E^A = -\frac{i}{2}e^U e^{Ai} g^{i\bar{j}} \bar{f}^I_{\bar{j}} \left[q_I - 2k\tilde{\zeta}_I - \mathcal{N}_{IJ}(p^J + 2k\zeta^J)\right]$$

SUSY geodesic flow and generalized attractor equations

- The BH solution preserves 1/2 SUSY iff

$$\delta\chi^{\Gamma} = V_{\mu}^{\alpha\Gamma}\sigma_{\alpha}^{\mu\beta}\epsilon_{\beta} = V^{\alpha\Gamma}\tilde{\epsilon}_{\alpha} = 0$$

Equivalently, the rectangular matrix V should have a zero eigenvector $(1,\lambda)$:

$$\begin{aligned} -dU + \frac{i}{2}e^{2U}k &= -\frac{i}{2}\lambda e^{K/2+U}X^{I}\left(q_{I} - k\tilde{\zeta}_{I} - \mathcal{N}_{IJ}(p^{J} + k\zeta^{J})\right) \\ dz^{i} &= -\frac{i}{2}\lambda e^{U}g^{i\bar{j}}\bar{f}_{\bar{j}}^{I}\left(q_{I} - k\tilde{\zeta}_{I} - \mathcal{N}_{IJ}(p^{J} + k\zeta^{J})\right) \end{aligned}$$

where λ is fixed by the requirement that dU is real.

- Using standard special geometry formulae this can be rewritten as

$$-dU + \frac{i}{2}e^{2U}k = -\frac{i}{2}\lambda e^{U}Z\ , \quad dz^{i} = -i\lambda\frac{|Z|}{Z}e^{U}g^{i\bar{j}}\partial_{\bar{j}}|Z|$$

$$Z(p,q,k) = e^{K/2}\left[(q_{I} - 2k\tilde{\zeta}_{I})X^{I} - (p^{I} + 2k\zeta^{I})F_{I}\right]$$

This generalizes the standard attractor flow equations to non zero NUT charge.

Black holes and D-instantons

- The equivalence between the BH attractor equations and geodesic motion on c-map(M_4) was first observed in the study of spherically symmetric D-instanton solutions in $N = 2$ SUGRA in 5 dimensions: p^I and q_I are M2-brane instanton charge, while k is the M5-brane instanton charge. In fact, such instantons are T-dual to stationary black holes.

  Gutperle and Spalinski; Behrndt Gaida Luest Mahapatra Mohaupt

- This suggests how to incorporate higher-derivative corrections: by mirror symmetry, the $F_h R^2 F^{2h-2}$ corrections in 4D are mapped to

$$\sum_{h=1}^{\infty} \tilde{F}_h \partial^2 S \partial^2 S (\partial C)^{2h-2}$$

  which depend on the hypers only. The reduction to 3D gives rise to higher derivative corrections to the geodesic motion.

  Antoniadis Gava Narain Taylor

- Fur the purpose of this talk, we will neglect higher-derivative F-terms.

The universal $SU(2,1)$ sector

- It is instructive to investigate the "universal sector", which encodes the scale U, the graviphoton electric and magnetic charges, and the NUT charge k (this amounts to truncating all moduli away). The Hamiltonian is

$$H = \frac{1}{8}(p_U)^2 - \frac{1}{4}e^{2U}\left[(p_{\tilde{\zeta}} - k\zeta)^2 + (p_\zeta + k\tilde{\zeta})^2\right] + \frac{1}{2}e^{4U}k^2$$

  Gauge conditions are $U = \zeta = \tilde{\zeta} = a = 0$ at $\tau = 0$.

- The motion in the $(\tilde{\zeta}, \zeta)$ plane is that of a charged particle in a constant magnetic field. The electric, magnetic charges are the generators of translations; together with the angular momentum

$$p = p_{\tilde{\zeta}} + \zeta k \ , \quad q = p_\zeta - \tilde{\zeta} k \ , \quad J = \zeta p_{\tilde{\zeta}} - \tilde{\zeta} p_\zeta$$

  they satisfy the usual magnetic translation algebra

$$[p,q] = k \ , [J,p] = q \ , [J,q] = -p$$

- The motion in the U direction is governed effectively by

$$H = \frac{1}{8}(p_U)^2 + \frac{1}{2}e^{4U}k^2 - \frac{1}{4}e^{2U}\left[p^2 + q^2 - 4kJ\right]$$

- At spatial infinity, p_U becomes equal to the ADM mass, and J vanishes; hence the BPS mass relation

$$M^2 + k^2 = p^2 + q^2$$

- At the horizon $U \to -\infty$, $\tau \to \infty$, the last term is irrelevant and one recovers $AdS_2 \times S_2$ geometry with area

$$A = 2\pi(p^2 + q^2) = 2\pi\sqrt{(p^2 + q^2)^2}$$

- Since the space is symmetric, there is in fact a whole $su(2,1)$ matrix Q of conserved Noether charges, such that

$$H = \mathrm{Tr}(Q^2)\ , \quad \det(Q) = 0$$

The last condition can be checked explicitely, and is necessary in order for the motion not to be over-determined. Negative roots correspond to Ehlers and Harrison transformations.

SUSY geodesic motion and nilpotent co-adjoint orbits

- Since V_α^A is a 2×2 matrix, SUSY is equivalent to $H = \det(V_\alpha^A) = 0$:

$$H = \frac{1}{2}\left|p_U + ike^{2U}\right|^2 - \frac{1}{4}e^{2U}\left|p + iq\right|^2 = 0$$

- The Cayley-Hamilton theorem for 3x3 matrices implies that $Q^3 = 0$ (in the fundamental representation). In this case, SUSY is equivalent to requiring the vanishing of the Casimirs $\text{Tr}Q^2 = \det Q = 0$.
- More generally, for very special $N = 2$ SUGRA: the solution preserve 1/2 SUSY iff the conserved Noether charge Q is a nilpotent element of order 5:

$$[Ad(Q)]^5 = 0$$

  Indeed, $V_\alpha^A \epsilon^\alpha = 0$ is equivalent to requiring that Q can be conjugated into a grade 1 element in the standard 5-grading.
- In other words, the SUSY phase space is a nilpotent coadjoint orbit of G_3, in general much smaller than the generic orbit. It inherits a symplectic structure by the standard Kirillov-Kostant method.

Geodesic motion in $N = 8$

- For $N = 8$, the SUSY variation is

$$\delta\lambda_A = \epsilon_I \Gamma^I_{A\dot{A}} P^{\dot{A}}$$

where ϵ_I is a vector of the R-symmetry group in 3 dimensions $SO^*(16)$, $P^{\dot{A}}$ is a 128 spinor of $SO^*(16)$ corresponding to the tangent space to $E_{8(8)}/SO^*(16)$, and λ_A is a conjugate spinor.

- This may be interpreted as a Dirac equation in 16 dimensions, where ϵ_I is the momentum, hence ϵ_I should be light-like. In order to have an ϵ_I such that (*) vanishes, $P^{\dot{A}}$ should be a special spinor.
- For example, 1/2-SUSY trajectories correspond to pure spinors of $SO^*(16)$, of real dimension 58. This is the dimension of the minimal nilpotent orbit of $E_{8(8)}$.

Geodesic motion in $N = 4$

- For $N = 4$, the SUSY variation is

$$\delta\lambda^a_A = \epsilon_I \Gamma^I_{A\dot{A}} V^{\dot{A},a}$$

where ϵ_I is a vector R-symmetry group $SO(6,2)$, and $V^{\dot{A},a}$ ($a = 1 \dots n_v$), is a collection of n_v spinors of $SO(6,2)$ corresponding to the tangent space of $SO(8,n_v)/SO(6,2) \times SO(2,n_v-2)$.

- SUSY solutions can be obtained by requiring that $V^{\dot{A},a} = \lambda^{\dot{A}} v^a$. 1/2 SUSY trajectories correspond to pure spinors of $SO(6,2)$, hence the dimension is $n_v + 5$. This is the dimension of the minimal nilpotent orbit of $SO(8,n_v)$.
- The coincidence between the dimensions of the $K(\mathbb{C})$-orbits of elements in the tangent space p ($g = t + p$) and the dimensions of the orbits in $G(\mathbb{R})$ is a general consequence of the Kostant-Sekiguchi correspondence:

$$p \text{ : momenta } \leftrightarrow Q \text{ : Noether charges}$$

Co-adjoint orbits as phase spaces

- Recall that the Noether charges take values in the dual of the Lie algebra g^*. This is foliated into orbits of the action of G. Each orbit is a symmetric space

$$\mathcal{O}_J = \{g^{-1} J g, g \in G\} = G/Stab(J)$$

where Stab(J) is the stabilizer of J.

- Each orbit carries a natural G-invariant symplectic form, known as the Kirillov-Kostant symplectic form:

$$\omega(X, Y) = \mathrm{Tr}([X, Y] J)$$

on the tangent space around at J. This is evidently non-degenerate (its kernel is given by the commutant of J, which is orthogonal to O_J). Globally,

$$\omega = d\theta \ , \quad \theta = \mathrm{Tr}(g^{-1} dg \ J)$$

where g is a gauge-fixed element in $G/Stab$.

Nilpotent orbits as small phase spaces

- Generic orbits correspond to orbits of semi-simple (=diagonalizable) elements, whose stabilizer is $U(1)^r$, where r is the rank. Their dimension is $\dim G -$ rank$G$ (an even number).
- However, when J has a non-trivial nilpotent part (i.e. non diagonal Jordan form), the stabilizer is typically larger (and non semi-simple), hence the orbit is smaller. Nilpotent orbits are classified by homomorphisms of $Sl(2)$ into G. The smallest orbit is that of a root.
- As an example, the generic orbit of $SU(2,1)$ has dimension 6. The maximal (or regular) nilpotent orbit has the same dimension 6, but the Casimirs are forced to vanish. The minimal (or sub-regular) nilpotent orbit has dimension 4.

The orbit method

- Since the action of G on $\mathcal{O}_J$ preserves the symplectic form, its action on functions on $\mathcal{O}_J$ may be expressed in terms of Poisson brackets. The moment map Q for this symplectic action takes value in the dual of the Lie algebra, in the orbit of J itself.
- The general "orbit method philosophy" indicates that (most of the) unitary representations of G may be obtained by quantizing the Hamiltonian action of G on $\mathcal{O}_J$.
- For example, the regular representation of G on $L^2(G/K)$ at fixed values of the Casimirs (assuming that G is split and K is its maximal compact subgroup) is associated to the orbit of a generic semi-simple element:

$$\dim(G/\text{Stab}) = \dim G - \text{rank}G \ , \qquad \dim(G/K) = (\dim G + \text{rank}G)/2$$

  This is the Hilbert space obtained by quantizing geodesic motion on G/K, at fixed values of the rank$G$ Casimirs !
- Similarly, nilpotent orbits are associated to "unipotent representations" of G. They will describe the Hilbert space of supersymmetric geodesic motion on G/K !

The quantum attractor mechanism

- The standard way to quantize geodesic motion of a particle on $R^+ \times \mathcal{M}_3^*$ is to replace the classical trajectories by wave functions on $R^+ \times \mathcal{M}_3^*$, satisfying the WdW equation

$$\left[-\frac{\partial^2}{\partial r^2} + \frac{\Delta}{r^2} - 1\right] \Psi(r, U, z^i, \bar{z}^{\bar{i}}, \zeta^I, \tilde{\zeta}_I, a) = 0$$

  where Δ is the Laplace-Beltrami operator on $\mathcal{M}_3^*$.
- As a matter of fact, we have to deal with the geodesic motion of a superparticle, since it comes by reduction from SUGRA in 4D. The wave function is therefore a section of the spinor bundle on $\mathcal{M}_3^*$, or equivalently a set of differential forms on $\mathcal{M}_3^*$.
- Moreover, we are really interested in the SUSY Hilbert space, satisfying the stronger constraint

$$\exists \epsilon / \ \epsilon^\alpha \frac{\partial}{\partial X_\alpha^A} \Psi = 0$$

The BPS Hilbert space

- At fixed (projective) ϵ, this implies that the function does not depend on half of the coordinates X^A. Ψ should be a holomorphic function with respect to the complex structure determined by ϵ^α.
- Better to say, Ψ should be a holomorphic function (or an element of the sheaf cohomology group $H_l(T, O(-h))$ for some l, h) on the twistor space T over the quaternionic-Kahler space $\mathcal{M}_3$. This can be viewed as a higher dimensional, quaternionic version of the Penrose - Atiyah Hitchin Singer twistor tranform.

Salamon; Baston

- More generally, it may be fruitful to consider the hyperkahler cone (HKC) over the quaternionic-Kahler manifold $\mathcal{M}_3$, by including the cone direction r and an extra conjugate variable together with the twistor fiber. The minimal representation of G, relevant for BPS states with 16 supercharges, should then consist of tri-holomorphic functions on HKC.

SUSY Hilbert space for motion on symmetric spaces

- In the case where $\mathcal{M}_3^*$ is a symmetric space G/K, the Hilbert space H may be decomposed into unitary representations $\rho_i : G \longrightarrow H_i$ of G. Furthermore their should exist a map between vectors of each representation and the unconstrained Hilbert space $L^2(G/K)$.
- CAUTION: we are dealing with unitary representations of non-compact groups, hence of infinite dimension. Their size may still be characterized by their Gelfand-Kirillov (or functional) dimension, very roughly, the number d such that $H \sim L_2(R^d)$.
- This can be achieved if the representation admits a (preferably unique) vector f_K, called "spherical vector", invariant under K. Then

$$\Psi(g) = \langle f_K, \rho(g) v \rangle$$

  is K-invariant for any choice of v. If f_K does not exist, any other finite-dim irrep of K (called K-type) will do, and yield a section of some non-trivial bundle over G/H rather than a function.
- Supersymmetric geodesic motion should correspond to unitary representations in a Hilbert space H_{BPS} of unusually small functional dimension: the unipotent representations attached to the nilpotent orbits !

Quaternionic discrete series and very special SUGRA

- Gross and Wallach have constructed unitary representations π_h of G by considering the sheaf cohomology group $H^1(T, O(-h))$ on the twistor space T over the quaternionic-Kahler space $\mathcal{M}_3 = G/K$. For $h \geq 2n_v + 1$, this representation is irreducible, lies in the "quaternionic" discrete series and has functional dimension $2n_v + 1$: this can be viewed as the "quasi-conformal" action of G on (p^I, q_I, k), from the 5-grading

$$G = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}$$

where G_{+2} is the highest root (k), G_{+1} is a symplectic space (p^I, q_I) and $G_0 = R \times M$.

- For lower values of h, the representation becomes decomposable. It admits a unitarizable submodule π'_h of smaller functional dimension:

k	$\dim$	Constraint on (p,q)
$\geq 2n_v + 1$	$2n_v + 1$	$I_4 \neq 0$
$n_v - 1$	$2n_v$	$I_4 = 0$
$(2n_v - 2)/3$	$(5n_v - 2)/3$	$\partial I_4(p, q) = 0$
$(n_v + 2)/3$	$n_v + 2$	$\partial \otimes \partial\|_M I_4(p, q) = 0$

Quaternionic discrete series and N=4,8 SUGRA

- By analytic continuation from $G = E_{8(-24)}$ to to $G = E_{8(8)}$, we expect these same representations to be relevant for 1/8, 1/8 with zero entropy, 1/4, and 1/2 BPS black holes, respectively. Since the maximal compact group changes, the spherical vector however will be different.

Ferrara Gunaydin

- In the context of very special $N = 2$ SUGRA, these representations may still be relevant for the 4-, 3-, 2- and 1-charge black holes, respectively, although all of these preserve the same SUSY. Optimistically, h may be related to the order of the helicity supertrace...
- For $G = E_{8(8)}$ (and all other simply laced groups in their split real form), the minimal representation and its spherical vector have been constructed (although with a totally different motivation). It amounts to quantizing the quasi-conformal action (p, q, k), and relies on the invariance of $\exp(I_3(X)/X^0)$ under Fourier. Remarkably,

$$\lim_{\beta\to\infty} e^{\beta H_\omega} f_H = e^{iI_3(\chi^A)/\chi^0} , \quad E_{-\omega} = p_k^2 + \frac{I_4(p,q)}{k^2}$$

reproduce the tree-level topological amplitude and the Hamiltonian of conformal quantum mechanics...

Kazhdan Pioline Waldron, Gunaydin Koepsell Nicolai, Gunaydin Pavlyk

Physical interpretation of the wave function

- As usual in diffeomorphism invariant theories (e.g. quantum cosmology), the wave function is independent of the "time" variable ρ, and some other variable should be chosen as a "clock".
- It is natural to use e^U as the "radial clock", since it goes from 0 at the horizon to ∞ at spatial infinity. One could also use the black hole area $A = e^{-2U} r^2$, although classically its range depends on the charges. We expect the wave function to be peaked towards the attractor values of the moduli and the horizon area as $U \to -\infty$.
- The natural inner product is obtained by using the Klein-Gordon inner product (also known as Wronskian, or $U(1)$ charge) at fixed values of U. E.g, the mean value of the horizon area should be roughly

$$A \sim e^{-2U} \int r^2 dr\, dz^i\, d\bar{z}^{\bar{j}}\, \Psi^* \overleftrightarrow{\partial_U}\, \Psi|_{U \to -\infty}$$

- Unfortunately, this product is famously known NOT to be positive definite. A possible way out is "third quantization", where the wave function Ψ becomes itself an operator... this may describe the possible black hole fragmentation near the horizon...

Topological amplitude and spherical vector

- Recall the OSV proposal for BH degeneracies

$$\Omega(p,q) = \langle \Psi_{p,q} | \Psi_{p,q} \rangle \ , \quad \Psi_{p,q}(\chi) = V_{p,q} \Psi_{top} = e^{iq\chi} \Psi_{top}(\chi - p)$$

interpreted as the overlap between two wave functions associated to each boundary of AdS_2. What is so special about Ψ_{top} ? Do we really need to restrict to $k = 0$?

- On the other hand, we have shown that the proper Hilbert space for the quantum attractor flow is a sub-module $H_{BPS} \subset H \sim L_2(\mathcal{M}_3)$, corresponding to the quantization of BPS geodesic motion on $\mathcal{M}_3$. If $\mathcal{M}_3 = G/K$ is a symmetric space, there is a distinguished "spherical" vector f_K which allows for the map $H_{BPS} \to H$

$$f \longrightarrow \Psi(g) = \langle f, \rho(g) f_K \rangle$$

- We have found circumstancial evidence, at least at tree-level, that (the $k \to 0$ limit of) the spherical vector f_K is in fact the topological string amplitude ! This suggests that there should exist a 1-parameter extension of the standard topological string amplitude...

The automorphic attractor wave function

- This still leaves an infinite dimensional Hilbert space of BPS wave functions f. A natural physical principle is to select a vector invariant under the 3D U-duality group $G(Z)$:

$$\theta_G(g) = \langle f_{G(\mathbb{Z})}, \rho(g) f_K \rangle$$

  is now a function on $G(\mathbb{Z})\backslash G_3(\mathbb{R})/K$, i.e. an automorphic form. This is in fact the general construction of theta series for any group G !

- E.g, the Jacobi theta series

$$\theta(\tau) = \sum_{m \in Z} e^{i\pi m^2 \tau}$$

  fits into this frame: τ is an element of $Sl(2)/U(1)$, ρ is the metaplectic representation

$$E_+ = x^2 \;, \quad E_0 = x\partial_x + \partial_x x \;, \quad E_- = \partial_x^2 \;,$$

  f_K is the ground state of the harmonic oscillator, and $f_{G(\mathbb{Z})}$ is the "Dirac comb" distribution $\sum_{m\in\mathbb{Z}} \delta(x-m)$.

BP Waldron Les Houches lecture

Automorphic forms and adeles

- By the "Strong Approximation Theorem", $f_{G(\mathbb{Z})}$ is in fact the product over all primes p of the spherical vector over the p-adic field $\mathbb{Q}_p$. For the Jacobi theta series,

$$\sum_{m\in\mathbb{Z}} \delta(x-m) = \prod_{p\in\mathbb{Z}} \gamma_p(x)\ , \quad \gamma_p(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbb{Z}_p \\ 0 & \text{if} \quad x \notin \mathbb{Z}_p \end{cases}$$

  Indeed, $\gamma_p(x)$ is invariant under p-adic Fourier transform !

- In the language of adeles and ideles,

$$G(\mathbb{Z})\backslash G(\mathbb{R})/K(\mathbb{R}) = G(\mathbb{Q})\backslash G(\mathbb{A})/K(\mathbb{A})$$

  where $G(\mathbb{Q})$ is diagonally embedded in $G(\mathbb{A})$ and $K(\mathbb{A}) = \prod_p G(\mathbb{Z}_p) \times K(\mathbb{R})$, and the theta series is written adelically as

$$\theta_G(g) = \langle f_{G(\mathbb{Q})}, \rho(g) f_{K(\mathbb{A})} \rangle$$

- The p-adic spherical vector is in fact known for the minimal representation of any simply-laced, split group G.

Kazhdan Polischchuk

Black hole degeneracies and Fourier coefficients

- In the general theory of automorphic forms, Fourier coefficients are associated to choices of parabolic subgroups $P = LN$ of G, and are indexed by characters ξ of P:

$$\hat{\theta}(\xi) = \int_{N(\mathbb{R})/N(\mathbb{Z})} \xi(g)\, \theta_G(g)\, dg$$

- Choosing the maximal (Heisenberg) parabolic subgroup, $N \sim (\zeta^I, \tilde{\zeta}_I, a)$ has two kinds of characters,

$$\xi_{p,q} = e^{i(q_I \zeta^I + p^I \tilde{\zeta}_I)} \quad \text{or} \quad \xi_{p,k} = e^{i(p^I \tilde{\zeta}_I + ka))}$$

In the first case,

$$\hat{\theta}(p,q) = \int d\zeta^I\, d\tilde{\zeta}_I\, da\; e^{i(q_I \zeta^I + p^I \tilde{\zeta}_I)}$$

$$\sum_{(\chi^I, y) \in \mathbb{Q}} \left[e^{i(\tilde{\zeta}_I \chi^I + ay)} f^*_{G(\mathbb{Z})}(\chi^I - \zeta^I, y) \right] \left[e^{i(\tilde{\zeta}_I \chi^I + ay)} f_{K(\mathbb{R})}(\chi^I + \zeta^I, y) \right]$$

Black hole degeneracies and Fourier coefficients (cont)

- The integral of a sets $y = 0$ and the integral over $\tilde{\zeta}_I$ sets $\chi^I = p^I$, hence

$$\hat{\theta}(p,q) = \int d\zeta^I \, e^{iq_I \zeta^I} \, f^*_{G(\mathbb{Z})}(p^I - \zeta^I, 0) \, f_{K(\mathbb{R})}(p^I + \zeta^I, 0)$$

  which is tantalizingly close to the OSV for $\Omega(p,q)$!

- Said otherwise, the automorphic attractor wave function is obtained by choosing the real spherical vector at infinity, and the adelic spherical vector at the horizon. The Fourier coefficients are by construction invariant under $G_4(\mathbb{Z})$.
- It remains to show that $\log \Omega_{p,q} \sim 2\pi \sqrt{I_4(p,q)}$, and that the Fourier coefficients are integer.

Channel duality and Nahm equations

- We have seen that the black hole radial evolution is equivalent to geodesic motion on (the HKC over) a quaternionic Kahler manifold. For very special SUGRA, this is a symmetric space $G/M \times SU(2)$.
- Hyperkahler cones crop up in a completely different context, namely as moduli spaces of the Nahm equations on the semi-infinite line, or equivalently Dirac monopoles, or D1 strings attached to a D3 brane.

Kronheimer; Bachas Hoppe Pioline

- In the monopole context, the geodesic motion on moduli space describe low energy scattering, in particular time evolution. The Nahm equation on the other hand describes the radial evolution away from the D3-brane.
- Channel duality suggests that we should identify the time evolution for black holes with the radial evolution for monopoles. Hence one could think of the Nahm equations as a baby model for the conformal quantum mechanics describing the black hole !
- This is less crazy than it sounds: Recent work suggests that the CQM describing D0-D4 bound states on the quintic is a quiver quantum mechanics, not unlike Nahm's equations !

Gaiotto Guica Huang Simons Strominger Yin

Open problems

- Higher derivative corrections
- Rotating and multi-centered black holes in 4D
- Black holes and black rings in 5D
- Automorphic wave functions, and relations to other counting formulae
- Genuine N=2 theories and Kontsevitch's “very wild guess conjecture”
- Time-dependence and midi-superspace models