Irrational non-commutative gauge theories

Boris Pioline

Harvard U. and LPTHE, Paris

w/ A. Schwarz `hep-th/9908019`
w/ S. Elitzur, E. Rabinovici `hep-th/0009009`
A. Hashimoto and N. Itzakhi `hep-th/9911057`

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Non-commutativity and Non-locality

NC gauge theories seem to give *consistent non-local* deformations of ordinary gauge theories:

$$\int \frac{d^n p d^n q \ e^{ip_i \theta^{ij} q_j}}{(p^2+m^2)(q^2+m^2)} \ , \quad \theta \sim L^2$$

They capture part of the *non-locality of open strings in magnetic backgrounds*:

$$B = cste \ , \quad g_{ij} \sim \epsilon \to 0 \ , \quad \alpha' \sim \sqrt{\epsilon} \to 0$$

However, NC gauge theories are sometimes equivalent to ordinary (commutative) gauge theories:

- Morita Equivalence:

$$U(N) \text{ on } T^n_\theta \sim U(N') \text{ on } T^n_{\theta'}$$

$$\theta' = \frac{A\theta + B}{C\theta + D} \ , \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SO(N,N,Z)$$

- Seiberg-Witten map:

$$U(N) \text{ on } R^n_\theta \sim U(N) \text{ on } R^n$$

$$\widehat{\mathcal{A}}_i = A_i + A_j \theta^{jk} \partial_k A_i + \dots$$

Q: Are NC gauge theories really non-local at scale θ ?

Outline

1. Seiberg-Witten map and NC Born-Infeld

2. Morita equivalence and T-duality

3. NCSYM on an irrational torus at weak coupling

4. Phases of the Villain model

5. Strongly coupled NCSYM and multifractal gravity dual

NC gauge theories

Non-Abelian $U(N)$ gauge theories can be deformed by substituting Lie commutators with **Moyal brackets**,

$$[A, B] \to [A, B]_* = A * B - B * A$$

$$A * B(x) = A(x) e^{i\theta_{ij}\partial_i^x \partial_j^y} B(y)|_{x=y}$$

- Associativity of the *-product implies the Jacobi identity of the Moyal bracket and ensures that the gauge symmetry is intact.
- The deformation induces phase factors $e^{ip\theta q}$ in non-planar diagrams, which become (more) convergent (but the limits $\Lambda \to \infty$ and $\theta \to 0$ do not commute !)
- It is non-trivial even for $U(1)$. The one-loop beta function is negative !
- For other groups than $U(N)$, one would need to deform the Lie algebra itself.
- Works only for fields in the adjoint rep.
- Purely scalar theories can also be considered, and the scale θ allows for stable soliton solutions.
- $*$-product with spatially variable θ can also be constructed, but the derivative needs also to be deformed.

Open strings in B-field

The open string propagator on the disk in a B-field is deformed to

$$\langle X^i(\tau) X^j(\tau) \rangle = G^{ij} \ln(\tau - \tau') + i\theta^{ij} \mathrm{sgn}(\tau - \tau')$$

open string moduli are related to closed string as

$$\frac{1}{g+B} = \theta + \frac{1}{G}$$

Hence all correlators are as when $B = 0$, but for phases:

$$\langle e^{ik_1 X}(\tau_1) \ldots e^{ik_n X}(\tau_n) \rangle = \exp(i \sum_{a<b} k_a \theta k_b) \prod_{a<b} |\tau_a - \tau_b|^{k_a G k_b}$$

Hence, *the effective action of open strings in constant B-field at disk level is equal to that of open strings in no B-field but propagating in metric G and with *-products instead of ordinary ones.*

• There is only one metric, but the effective open string tension is different from the closed string tension, and anisotropic: the magnetic field pulls the ends of the string apart.

Seiberg-Witten map

Instead of choosing the non-commutative representation, one may use the ordinary Born-Infeld action and replace $F \to F + B$:

$$\int \sqrt{\det(g + \alpha'(B + F))} \leftrightarrow \text{``}\int_\theta \sqrt{\det(G + \alpha' \widehat{F})}\text{''}$$

The two are related by a field redefinition,

$$\widehat{\mathcal{A}}_i = A_i + A_j \theta^{jk} \partial_k A_i + \dots$$

$$\hat{\lambda} = \lambda + A_i \theta^{ij} \partial_j \lambda + \dots$$

Hence the commutative and non-commutative frameworks are equivalent, but useful in different regimes:

- when $\alpha \to 0$ and g_{ij} fixed ("non-relativistic"):

$$S_{BI} \sim \int \sqrt{g} \int (1 + (F + B)^2 + \dots)$$

- when $\alpha \to 0$ and G_{ij} fixed ("ultra-relativistic"):

$$S_{BI} \sim \int \mathrm{Pf}(F + B)(1 + \mathrm{Tr}\ \frac{1}{F + B} + \dots$$

$$S_{NCBI} \sim \int_\theta (1 + \widehat{F}^2 + \dots)$$

The $*$-product resums an infinite number of derivative corrections.

Interpolating between NC theories

The discrete equivalence of the BI action under the Seiberg-Witten map can be generalized to a *continuous* symmetry by allowing for a magnetic background Φ on the NCBI side:

$$S[G,\theta,\Phi] = \int_\theta \sqrt{\det(G+\alpha'(\hat{F}+\Phi))}$$

Open string moduli are related to closed string thru

$$\frac{1}{g+B} = \theta + \frac{1}{G+\Phi}$$

A variation of (G,θ,Φ) at constant (g,B) leaves the NCBI action invariant (as seen on the BPS mass formula). This is a continuous *gauge* $SO(d)$ symmetry:

$$(g,B) \in \frac{SO(d,d,R)}{SO(d)\times SO(d)}\ ,(G,\Theta,\Phi) \in \frac{SO(d,d,R)}{SO(d)}$$

The physically relevant choice is when the volume of the open string metric G is maximized, ie $\Phi = 0$.

This equivalence follows from the Darboux theorem: *any symplectic form can be written as* $\theta = \sum dp_i dq_i$ *in some coordinate system.* The SW map is a worldvolume reparametrization.

Morita equivalence and T-duality

Consider $U(N)$ NCSYM on T^2_θ with flux m. There is a $Sl(2,Z)$ group of *off-shell non-local* field redefinitions such that

$$\tilde{\theta} = \frac{c+d\theta}{a+b\theta}, \qquad \tilde{\Phi} = (a+b\theta)^2\Phi - b(a+b\theta),$$
$$\tilde{g}_{ij} = (a+b\theta)^2 g_{ij}, \qquad \tilde{g}^2_{YM} = (a+b\theta)g^2_{YM},$$
$$\begin{pmatrix} \tilde{m} \\ \tilde{N} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} m \\ N \end{pmatrix},$$

A flavor of Morita equivalence: expand any field in momentum modes and "gauge modes":

$$X = \sum_{mnpq} e^{imx} e^{iny} U^p V^q x_{mnpq}$$

with U, V the clock and shift operators:

$$UV = VUe^{2\pi i/N} \ , \quad [e^{imx}, e^{iny}] = 0$$

One may exchange momentum and gauge modes, such that now

$$e^{imx} e^{iny} = e^{iny} e^{imx} e^{2\pi i/N} \ , \quad [U, V] = 0$$

We have mapped commutative $U(N)$ to NC $U(1)$.

The $Sl(2,Z)$ reflects the T-duality symmetry of the string. It is a classical symmetry of the BPS spectrum, and should remain at the quantum level for NCSYM.

Irrational NCSYM at weak coupling

Consider $N = 4$ SYM $U(N)$ gauge theory on $R^2 \times T_\theta^2$ of radius Σ:

$$[x^1, x^2] = i\Delta^2 \ , \quad \theta = \Delta^2/\Sigma^2$$

The non-commutative torus is a phase space with uncertainty relation

$$\langle \Delta x^1 \Delta x^2 \rangle \geq \Delta^2$$

A wave packet with energy E has a Compton wave-length $1/E$ along its direction of motion, and therefore a dual Compton wave-length $\Delta^2 E$ in the transverse direction.

The perturbative (ie momentum) degrees of freedom are therefore quasi-local if this is smaller than the radius of the torus,

$$\frac{1}{\Sigma} < E < \frac{\Sigma}{\Delta^2} = \frac{1}{\theta\Sigma}$$

Hashimoto Itzhaki

Outside this energy range, one may recover a quasi-local description by choosing another Morita picture.

Morita patches and UV behavior

For a given energy E, we have to look for integers a, b, c, d such $ad - bc = 1$ and

$$\frac{1}{|a + b\theta|\Sigma} < E < \frac{1}{|c + d\theta|\Sigma}$$

If θ is rational: one may choose $-c/d = \theta$ and the upper bound becomes ∞: beyond a certain energy, the UV behavior is that of a **local** commutative gauge theory with a magnetic background.

If θ is irrational: we have to keep changing pictures as we go to the UV.

Choose $c + d\theta \to 0$ so that $a + b\theta \to 0$ as well:

$$d < E\Sigma < \frac{1}{|c + d\theta|}$$

This is equivalent to

$$|\theta + c/d| < \frac{1}{d\ E\Sigma}\ , \quad d \leq E\Sigma$$

By an application of the Dirichlet drawer's principle, one can show that there always exists a (non-unique) solution to this problem. The solution then satisfies

$$|\theta + c/d| \leq \frac{1}{d^2}$$

which is also obeyed by the continued fraction approximations to θ...

Continued fraction decomposition

Any real number θ can be obtained as the limit of (θ_k),

$$\theta_k = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{\ddots \; c_{k-1} + \cfrac{1}{c_k}}}} \; .$$

The **convergents** $\theta_k = n_k/m_k$ are computed from the recursion relation

$$\begin{aligned} n_k &= c_k n_{k-1} + n_{k-2} \ , \\ m_k &= c_k m_{k-1} + m_{k-2} \ , \\ & \quad (n_{-1}, m_{-1}) = (1,0) \ , (n_0, m_0) = (c_0, 1) \end{aligned}$$

These equations preserve the relation

$$n_{k-1} m_k - n_k m_{k-1} = (-)^k$$

so that the integers (n_k, m_k) remain coprime. The (c_k) truncate for θ **rational**, and become periodic for a **quadratic irrational**. All convergents satisfy

$$\left| \theta - \frac{n_k}{m_k} \right| \leq \frac{1}{m_k^2} \ .$$

Together with the *intermediate convergents*, they are the **best approximations** of θ, in the sense that

$$|\theta - p/q| < |\theta - n_k/m_k| \Rightarrow q > m_k$$

A stairway to the UV

Thm: the Morita pictures given by the continued fraction approximations to θ cover the entire energy scale by disjoint intervals.

Proof: Remember that the convergents satisfy

$$n_{k-1}m_k - n_k m_{k-1} = (-)^k$$

so that

$$\theta \to (-)^{n-1}\frac{m_k\theta - n_k}{m_{k-1}\theta - n_{k-1}}$$

is indeed a $Sl(2,Z)$ Morita transformation. The new range of validity of this dual picture

$$\frac{1}{|n_{k-1} - m_{k-1}\theta|\Sigma} < E < \frac{1}{|n_k - m_k\theta|\Sigma} \ .$$

Since $|n_k - m_k\theta| \to 0$, this covers the full energy range towards the UV. (NC)QED.

Rk1: The rank and magnetic flux in the dual picture go to ∞.

Rk2: The Yang-Mills coupling in the dual picture is $|m_{k-1}\theta - n_{k-1}|g^2_{YM}$ and thus goes to zero. The 't Hooft coupling $\lambda = g^2_{YM}|N - \theta m| \to 0$.

Rk3: The size of the energy interval is constant or periodic for θ quadratic irrational, erratic otherwise.

Abelian lattice gauge theories with θ

Consider the Z_N model on a lattice,

$$Z = \sum \exp\left(-\frac{1}{2g^2} \sum_{\Box} (\Delta_\mu \phi_\nu - \Delta_\nu \phi_\mu - S_{\mu\nu})^2 + iN \sum_{\leftrightharpoons} n_\mu \phi_\mu \right)$$

ϕ_μ and n_μ are link, $S_{\mu\nu}$ plaquettes variables

• the sum over n_μ discretizes $\phi_\mu \in Z_N$

• the sum over $S_{\mu\nu}$ reproduces the Wilson action $\cos(\Delta_\mu \phi_\nu - \Delta_\nu \phi_\mu)$ in the Villain approximation.

$\Delta_\mu n_\mu = 0$: n_μ form loops of Higgs particles

$m_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho\lambda} \Delta_\nu S_{\rho\lambda}$: m_μ form loops of magnetic monopoles

Integrating out ϕ_μ gives a Lagrangian for interacting Higgs particles and monopoles:

$$Z = \sum \exp\left(-\frac{1}{2g^2} mGm - N^2 g^2 \tilde{n} G \tilde{n} + iNmKn \right)$$

where $\tilde{n}_\mu = n_\mu + \theta m_\mu$.

The spectrum exhibits dyons with charge (m, n) and mass

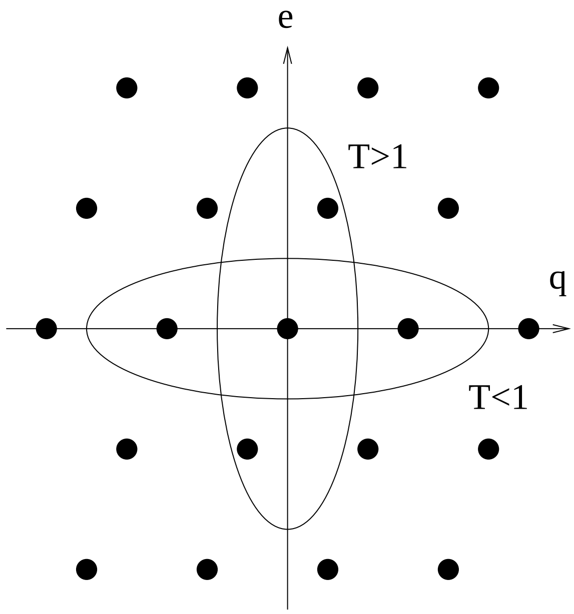
$$\mathcal{M}^2 = \frac{1}{g^2} m^2 + g^2 (n + \theta m)^2$$

Q: spectrum at finite temperature ?

Phases of the Villain model

Condensation occurs if

$$(n - \theta m)^2 T + \frac{m^2}{T} < C/N \ , \quad T = Ng^2/2\pi \ .$$



$$|m(n - \theta m)| < 1/M \ , \quad M = 2N/C \ ,$$

or equivalently

$$\left| \theta - \frac{n}{m} \right| < \frac{1}{Mm^2} \ .$$

If this inequality holds, the condensation will occur in a temperature interval

$$\frac{Mm^2}{2} < T < \frac{2}{M(n - m\theta)^2}$$

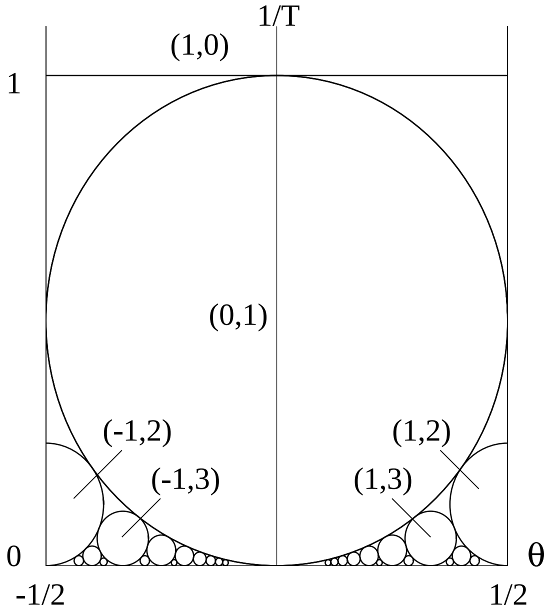
where we assumed $|\theta - n/m| \ll 1$ for simplicity.

Phases and duality

The free-energy of a (m, n) state in the Villain model is invariant under electric-magnetic duality, acting by $Sl(2, Z)$ fractional linear transformations on $\tau = \theta + i/T$

$$\tau \to \frac{c + d\tau}{a + b\tau}, \quad \begin{pmatrix} m \\ n \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}, \quad ad - bc = 1 .$$

All phase transitions are dual images of the condensation of the monopole $(1, 0)$. For $M < \sqrt{3}$, the disks cover the full upper plane. For $M = 2$, the disks are tangent.



For θ rational: there is a finite number of phases.

For θ irrational: there is an infinite number of phases if $M < M(\theta)$, or finite if $M > M(\theta)$. In that case, the high temperature phase is Coulomb !

Diophantine approximations

Q: how close can a rational come to an irrational ?

1. the best achievable power:

$$\eta(\theta) := \sup \left\{ M / \exists \infty (n,m) \text{ such that } |\theta - \frac{n}{m}| \leq \frac{1}{m^\eta} \right\}$$

Roth's theorem: for an algebraic number, $\eta = 2$.

2. the best achievable coefficient:

$$M(\theta) := \sup \left\{ M / \exists \infty (n,m) \text{ such that } |\theta - \frac{n}{m}| \leq \frac{1}{Mm^2} \right\}$$

Rk1. $M(\theta)$ is invariant under fractional linear transformations $\theta \to (a\theta + b)/(c\theta + d)$.

Rk2. $M(\theta) > 1$ from the continued fraction construction, and in fact $M(\theta) > 2$, and even better $M(\theta) \geq \sqrt{5}$

In fact, for $\theta = \theta_0 = (1+\sqrt{5})/2$, the constant $\sqrt{5}$ cannot be improved. But if $\theta \not\equiv \theta_0$, $M(\theta) \geq 2\sqrt{2}$. Etc.

$M(\theta)$	$\sqrt{5}$	$2\sqrt{2}$	$\frac{\sqrt{221}}{5}$	$\frac{\sqrt{1517}}{13}$	3
$\theta \sim$	$\frac{1+\sqrt{5}}{2}$	$1+\sqrt{2}$	$\frac{-11+\sqrt{221}}{10}$	$\frac{-29+\sqrt{1517}}{26}$	

Irrational NCSYM at strong coupling

We consider NCSYM/$R^2 \times T^2_\theta$ at large N and large 't Hooft coupling λ. This is described by the near horizon geometry of a stack of N D3-branes with background B-field:

$$\begin{aligned} ds^2 &= l_s^2 \left\{ \frac{U^2}{\sqrt{\lambda}}(-dt^2 + dx_1^2) + \frac{\sqrt{\lambda} U^2}{\lambda + U^4 \Delta^4}(dx_2^2 + dx_3^2) \right. \\ & \left. + \frac{\sqrt{\lambda}}{U^2} dU^2 + \sqrt{\lambda} d\Omega_5^2 \right\}, \\ e^\phi &= \frac{\lambda}{4\pi N} \sqrt{\frac{\lambda}{\lambda + \Delta^4 U^4}}, \qquad B_{23} = -\frac{l_s^2 \Delta^2 U^4}{\lambda + \Delta^4 U^4}, \end{aligned}$$

with periodicities $x_2 \sim x_2 + 2\pi\Sigma$ and $x_3 \sim x_3 + 2\pi\Sigma$.

The supergravity can be trusted when the volume of the torus is bigger than the string scale, hence

$$v = \frac{u^2}{1 + \theta^2 u^4} > 1$$

where $u = U\Sigma/\lambda^{1/4}$.

Assuming a **finite radius of convergence** M for the supergravity description, or allowing a **confidence level** M to the SUGRA description, the SUGRA will be valid if

$$v = \frac{u^2}{1+\theta^2 u^4} > \frac{M}{2}$$

If this is not satisfied, one needs to include the string winding modes. v attains its maximum value $1/(2|\theta|)$ at $u = 1/\sqrt{|\theta|}$, and is bigger than $M/2$ for

$$\frac{\lambda^{1/4}}{\Sigma}\sqrt{\frac{M}{2}} < U < \frac{\lambda^{1/4}\Sigma}{\Delta^2}\sqrt{\frac{2}{M}} \qquad (\theta \gg 1)$$

Using the AdS/CFT dictionary and choosing $M = 2$, this translates into an energy range

$$\frac{1}{\lambda^{1/4}\Sigma} < E < \frac{\Sigma}{\lambda^{1/4}\Delta^2}$$

Outside this range, one may carry out a **T-duality** transformation to find a new description where the winding modes are no longer relevant.

NCSYM and the Villain model

T-duality acts as

$$\rho \to \frac{a\rho + b}{c\rho + d} \ , \quad \rho = \frac{\Sigma^2}{l_s^2} B_{23} + iv \ , \quad v e^{-2\phi} = \text{const.}$$

the volume of the dual torus in string units and the dilaton become

$$v = \frac{u^2}{d^2 + (d\theta - c)^2 u^4} \ , \quad e^{\phi} = \frac{\lambda}{4\pi N} \frac{1}{\sqrt{d^2 + (d\theta - c)^2 u^4}} \ .$$

The new SUGRA picture is valid if the volume v is bigger than $M/2$, ie

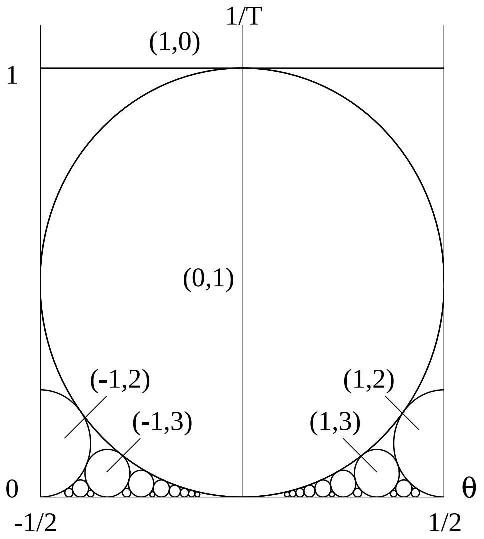
$$\frac{d^2}{u^2} + (d\theta - c)^2 u^2 < \frac{2}{M}$$

This condition is isomorphic to the one appearing in the Villain model, under the identification

$$\tau = \theta + i/T \ \text{(Villain)} \quad \leftrightarrow \quad \tau = \theta + i/u^2 \ \text{(AdS)}$$

Irrationality and the boundary of AdS

The structure is best seen in the τ plane:

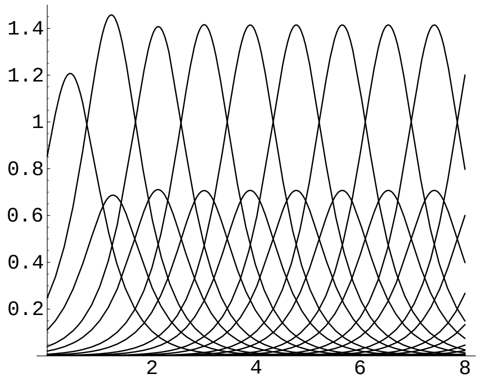


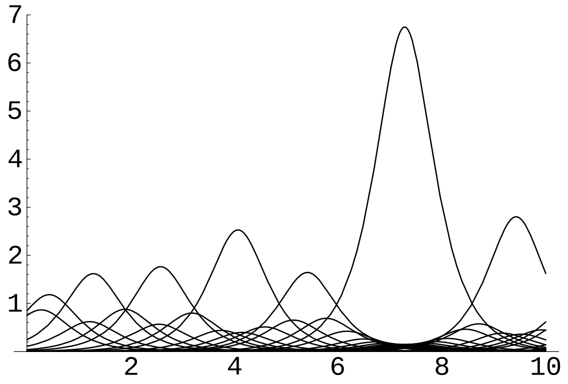
The interior of AdS ($u \to 0$, i.e. the IR of the gauge theory) is universally described by the $(c,d) = (1,0)$ picture, where the volume of the torus diverges at $u \to 0$. The system is then describable by a purely D1-brane system (*i.e.* $N = 0$), and undergoes a **Gregory-Laflamme transition**

At the boundary of AdS ($u \to \infty$, ie the UV of the gauge theory), the behavior depends crucially on the irrationality of θ. If $M(\theta) > M$, there is an infinite number of patches on the radial direction in which SUGRA is valid. If $M(\theta) < M$, beyond some radius the geometry is **stringy** all the way to the boundary of AdS.

Multifractality at the boundary of AdS

The volume of the torus along the radial direction is ambiguously defined: one should take the duality picture in which it is biggest:





If θ is quadratic irrational, the volume oscillates periodically with $\log u$: the geometry is **fractal**. Otherwise, it is **multifractal**:

Summary and Comments

Morita equivalence and Seiberg-Witten transformation relate NC theories with different deformation parameter. What is the physical scale of non-locality ?

The appropriate picture is when the higher derivative terms are suppressed (SW), and when the Compton wavelengths are bigger than the box (Morita).

Irrational NC gauge theories exhibit in the UV a *self-similar* behavior (for θ quadratic irrational) or quasi self-similar (for non quadratic irrational). Pictures are related by smooth cross-over instead of phase transitions. Thermodynamics are continuous.

Localized probes are needed to investigate the short distance structure of NCYM: Wilson line operators exhibit a universal behavior in the UV.

The relevance of irrational numbers may seem purely academic. Hierachical states in the Fractional Quantum Hall Effect follow a similar pattern and may profit of this techniques.