M-theory and U-duality,

An algebraic approach

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N. Obers, BP and E. Rabinovici, NPB `hep-th/9712084`,
N. Obers and BP, Phys. Rept. `hep-th/9809039`,

transparencies available from
`http://cpht.polytechnique.fr/cpht/pioline/academic.html`

Beyond string perturbation theory

- String dualities have played an important role in uncovering the relevant degrees of freedom of string theories at short distance or strong coupling, either by relating a given string theory to another one in a distinct regime:

$$IIA(R) \leftrightarrow IIB(1/R) \ , IIA/CY \leftrightarrow IIB/\tilde{CY}$$

$$I(g) \leftrightarrow HO(1/g) \ , II/CY_n(g) \leftrightarrow Het/CY_{n-1}(1/g) \ ,$$

  or to the same string theory in another regime:

$$Het(R) \leftrightarrow Het(1/R) \ , Het/CY \leftrightarrow Het/\tilde{CY}$$

$$IIB(g) \leftrightarrow IIB(1/g)$$

  or to a limit of an unknown theory:

$$IIA(g) \leftrightarrow M/S_1(R \sim g^{2/3}) \ , \quad HO(g) \leftrightarrow M/I_1(R \sim g^{2/3})$$

$$IIB(g) \leftrightarrow F/T^2(U \sim 1/g)$$

- Due to the asymptotic nature of string perturbation theory, and lack of (tractable) definition of M-theory, this covers only a zero-measure set of the moduli space.

- Analytic continuation to finite g_s can be controlled for quantities protected from quantum corrections: BPS spectrum, BPS-saturated amplitudes.

BPS saturated amplitudes

- Based on SUSY, some BPS-saturated amplitudes can be one-loop–exact on one side, and string-string dualities lead to a non-perturbative result, by mapping *worldsheet instantons* to *space-time instantons*:

$$O(e^{-1/\alpha'}) \to \begin{cases} O(e^{-1/g^2}) & (Het/II) \\ O(e^{-1/g}) & (Het/I, IIB/IIB) \end{cases}$$

- This applies to amplitudes corresponding to chiral densities in superspace:

SUSY	Coupling	Computed in	Instantons
8	$(\partial\phi)^2$	II/CY_3	Het 5-brane
8	R^2,R^2F^{2g-2}	II/CY_3	Het 5-brane
16	R^2	II/K_3	Het 5-brane
16	$(\partial\partial\phi)^2$	Het	II D-branes
16	F^4,tR^4	Het	I D1-brane
16	?	Type I	Het D-branes ?

- From these non-perturbative results, one may hope to derive a set of rules for semi-classical calculus in string/M-theory.

Symmetries and exact amplitudes

- Non-perturbative symmetries, on the other hand, identify different points in the moduli space, and apply to *all* amplitudes:

SUSY	Theory	Symmetry
16	Het/T^6	$SO(6,22,\mathbb{Z}) \times Sl(2,\mathbb{Z})$
16	IIB/K_3	$SO(5,21,\mathbb{Z})$
32	IIB	$Sl(2,\mathbb{Z})$
32	M/T^d	$E_{d(d)}(\mathbb{Z})$

- Amplitudes are given by (non-holomorphic) modular forms of the duality group, which one may hope to select from their leading behaviour (first perturbative terms) and SUSY constraints (Partial differential equations).

- The prototypical example is the type IIB ttR^4 coupling:

$$\begin{cases} f = 2\zeta(3)e^{-3\phi/2} + \frac{2\pi^2}{3}e^{\phi/2} + \dots \\ \Delta f = \frac{3}{4}f \ , \Delta = \tau^2\partial_\tau\partial_{\bar\tau} \end{cases}$$

$$\Rightarrow f = E_{s=3/2}^{Sl(2,\mathbb{Z})} = tree + 1 - loop + D - instantons$$

Green, Gutperle; Berkovits; BP; Green,Sethi

- Non-perturbative symmetries are non-trivial only in situations with high SUSY. One may hope to learn rules of semi-classical calculus of more general validity.

Highlights of the talk

I. M-theory, BPS spectrum and hidden symmetries

$$E_{d(d)}(\mathbb{R}) \to E_{d(d)}(\mathbb{Z})$$

II. T-duality from a Dynkinese view point

$$\begin{array}{ccccccc} & & T & & & & \\ & & | & & & & \\ S_1 & - & S_2 & - & S_3 & - \cdots - & S_{d-1} \end{array}$$

III. U-duality along the same lines

$$\begin{array}{ccccccccc} & & & & T & & & & \\ & & & & | & & & & \\ S_1 & - & S_2 & - & S_3 & - & S_4 & - \cdots - & S_{d-1} \end{array}$$

IV. T- and U-invariant modular forms

$$E^{G(\mathbb{Z})}_{s;R}(g) = \sum_{m \in \Lambda_R} [m \cdot R(g) \cdot m]^{-s} \, \delta(m^2)$$

M-theory half-BPS states

$$R_s = l_s g \ , \quad \frac{R_s}{l_p^3} = \frac{1}{l_s^2} \ , \quad \frac{R_s}{l_p} = g^{2/3} \ , \quad \frac{R_s}{l_s} = g$$

M-theory				Type IIA
longitudinal KK mode	$\mathcal{M} =$	$1/R_s$	$= 1/gl_s$	D0-brane
transverse KK mode	$\mathcal{M} =$	$1/R_i$	$= 1/R_i$	mode de KK
longitudinal KK monopole	$T_6 =$	R_s^2/l_p^9	$= 1/gl_s^7$	D6-brane
transverse KK monopole	$T_5 =$	$R_s R_i^2/l_p^9$	$= R_i^2/g^2 l_s^6$	KK monopole
longitudinal membrane	$T_1 =$	R_s/l_p^3	$= 1/l_s^2$	IIA string
transverse membrane	$T_2 =$	$1/l_p^3$	$= 1/gl_s^3$	D2-brane
longitudinal 5-brane	$T_4 =$	R_s/l_p^6	$= 1/gl_s^5$	D4-brane
transverse 5-brane	$T_5 =$	$1/l_p^6$	$= 1/g^2 l_s^6$	NS5-brane
transverse' KK monopole	$T_6 =$	R_i^2/l_p^9	$= R_i^2/g^3 l_s^9$	exotic

Ur-theory and U-duality

- Any Lorentz invariant theory compactified on a torus T^d has a discrete global symmetry, the *mapping class group* of the torus :

$$S_{IJ} \ : \ \gamma_I \leftrightarrow \gamma_J \ , \ \ R_I \leftrightarrow R_J$$

$$\Gamma_{IJ} \ : \ \gamma_I \to \gamma_I + \gamma_J \ , \ \ A_J^{(I)} \to A_J^{(I)} + 1$$

where $ds^2 = R_I^2 (dx^I + A_J^{(I)} dx^J)^2 + dx^J g_{JK} dx^K$

$$S_{IJ} \bowtie \Gamma_{IJ} = Sl(d, \mathbb{Z})$$

- Perturbative type IIA theory on a torus T^d has in addition a discrete global symmetry valid to all orders, namely (double) T-duality :

$$T_{ij} \ : \ (R_i, R_j, g_s) \to \left(\frac{l_s^2}{R_j}, \frac{l_s^2}{R_i}, \frac{g_s l_s^2}{R_i R_j} \right)$$

$$\Delta_{ij} \ : \ B_{ij} \to B_{ij} + 1$$

$$Sl(d, \mathbb{Z}) \bowtie T_{ij} \bowtie \Delta_{ij} = SO(d, d, \mathbb{Z})$$

- M-theory on a torus T^d should exhibit both Lorentz invariant on T^d and T-duality invariance on T^{d-1} :

$$Sl(d, \mathbb{Z}) \bowtie SO(d-1, d-1, \mathbb{Z}) = E_{d(d)}(\mathbb{Z})$$

This defines *a* discrete subgroup of Cremmer–Julia hidden symmetry group of 11D SUGRA on T^d, which has to remain valid at the quantum level.

Hidden symmetries revealed

- 11D SUGRA reduced on T^d as a (composite) local *R-symmetry* H_d, together with a global continuous non-compact symmetry G_d. H_d is the maximal compact subgroup of G_d.

D	d	$G_d = E_{d(d)}(\mathbb{R})$	H_d
10	1	1	1
9	2	$Sl(2,\mathbb{R})$	$U(1)$
8	3	$Sl(3,\mathbb{R}) \times Sl(2,\mathbb{R})$	$SO(3) \times U(1)$
7	4	$Sl(5,\mathbb{R})$	$SO(5)$
6	5	$SO(5,5,\mathbb{R})$	$SO(5) \times SO(5)$
5	6	$E_{6(6)}$	$USp(8)$
4	7	$E_{7(7)}$	$SU(8)$
3	8	$E_{8(8)}$	$SO(16)$

- Scalars (g_{IJ},$\mathcal{C}_3$ and duals $\mathcal{E}_6$,$\mathcal{K}_{1;8}$) take values in the *coset space* G_d/H_d. The local H_d invariance can be fixed thanks to the Iwasawa decomposition :

$$G_d = K \cdot A \cdot N$$

into maximal compact $K = H_d$, Abelian $A \simeq \mathbb{C}^d$ and nilpotent N.

- Moduli can be represented as a vielbein $\mathcal{V} \in A \cdot N$, or as an H_d–invariant matrix $M = \mathcal{V}^t\mathcal{V}$.

$$A = \text{radii}\ , \quad N = \text{gauge potentials}$$

- A right action by G has to be compensated by a rotation in K. A right action by N does not need to, and generates (non-commuting) isometries of the manifold.

Gauge charges and central charges

- One-forms, two-forms, ... transform *linearly* under G_d :

d	$E_{d(d)}$	Scalars	1-f.	2-f.
1	1	1	$\mathbf{1}$	$\mathbf{1}$
2	$Sl(2,\mathbb{R})$	3-1	$\mathbf{3}$	$\mathbf{2}$
3	$Sl(3,\mathbb{R}) \times Sl(2,\mathbb{R})$	11-4	$(\mathbf{3},\mathbf{2})$	$(\mathbf{3},\mathbf{1})$
4	$Sl(5,\mathbb{R})$	24-10	$\mathbf{10}$	$\mathbf{5}$
5	$SO(5,5,\mathbb{R})$	45-20	$\mathbf{16}$	$\mathbf{10}$
6	$E_{6(6)}$	78-36	$\mathbf{27}$	$\bar{\mathbf{27}}$
7	$E_{7(7)}$	133-63	$\mathbf{56}$	$\cdot$
8	$E_{8(8)}$	248-120	$\cdot$	$\cdot$

- The abelian 1-form (resp. 2-form) gauge potentials induce gauge charges for particles (resp. strings) m (resp. n), transforming linearly under G_d.

- The central charges in the $N = 8$ supersymmetry algebra are linear combinations of the abelian gauge charges, with moduli dependent coefficients :

$$Z = \mathcal{V} \cdot m$$

  they transform linearly under local R-symmetry H_d, and undergo moduli-dependent compensating R-symmetry rotations under global $E_{d(d)}$ rotations.

- $$\mathcal{M}^2 = Z^t Z = m^t \cdot M \cdot m \ , \ \ M = \mathcal{V}^t \mathcal{V}$$

  is invariant under global and local symmetries, and quadratic in the charges : this is the mass formula for 1/2-BPS states. 1/4-BPS states have further contributions.

Charge quanta and quantum symmetry

- Some of the charges are identified as *KK momenta* m_I on the torus T^d:

$$(m_I) \in \Gamma_{dual}$$

- Some are identified as *winding/wrapping numbers* of strings/branes on T^d:

$$(m^I), (m, m^{IJ}, m^{IJKL}, \dots) \in \Gamma_{homology}$$

- In D=4, Dirac quantization for electric and magnetic charges implies

$$m^i m'_i - m'^i m_i \in \mathbb{Z}$$

- Only discrete transformations preserving Γ, can remain as exact quantum symmetries :

$$E_{d(d)}(\mathbb{Z}) \subset E_{7(7)}(\mathbb{R}) \cap Sp(56, \mathbb{Z})$$

Hull, Townsend 95

which defines the maximal U-duality groups $E_{d\leq 7}(\mathbb{Z})$. There is also a lower bound:

$$Sl(d, \mathbb{Z}) \bowtie SO(d-1, d-1, \mathbb{Z}) \subset E_{d(d)}(\mathbb{Z})$$

- We now want to exhibit a set of generators to obtain the BPS spectrum, by applying a sequence of U-dualities on the elementary states

$$\text{particle } : \mathcal{M} = \frac{1}{R_I}, \text{ string } : \mathcal{T} = \frac{R_I}{l_p^3}$$

but let us focus on T-duality first...

T-duality, Weyl and Borel generators

- Restricting 11D SUGRA/T^{d+1} to the Neveu-Schwarz sector, we find a very similar structure :

$$\mathcal{H} = \mathbb{R}^+ \times \frac{SO(d,d,\mathbb{R})}{SO(d) \times SO(d)} \ni (g_s, \mathcal{V})$$

$$\mathcal{V} = \left(\begin{array}{cc|cc} R_1 & & & \\ & R_2 & & \\ \hline & & 1/R_1 & \\ & & & 1/R_2 \end{array}\right) \cdot \left(\begin{array}{cc|cc} 1 & A_2^1 & B_{11} & B_{12} \\ & 1 & B_{21} & B_{22} \\ \hline & & 1 & -A_2^1 \\ & & & 1 \end{array}\right) = \mathcal{A} \cdot \mathcal{N}$$

- The *Weyl* group of $SO(d,d)$ preserves the diagonal form of $\mathcal{A}$. It acts on $\mathcal{V}$ from the right and permutes R_i with R_j or $1/R_j$. A minimal subset is given by :

$$S_i : R_i \leftrightarrow R_{i+1} \ , \quad i = 1 \dots d-1$$

$$T : (g_s, R_1, R_2) \leftrightarrow \left(\frac{g_s}{R_1 R_2}, \frac{1}{R_2}, \frac{1}{R_1} \right)$$

- The *Borel* group of upper triangular matrices with 1's on the diagonal (and appropriate syms) also acts from the right, leaving R_i invariant but shifting the 0-form potentials. A basis is

$$\Gamma_{ij} : A_j^{(i)} \to A_j^{(i)} + 1$$

$$\Delta_{ij} : B_{ij} \to B_{ij} + 1$$

T-duality Weyl group

- The action of the Weyl group can be represented on a weight space :

$$\mathcal{T} = g_s^{x^0} R_1^{x^1} R_2^{x^2} \dots R_d^{x^d} \to \lambda = x^0 e_0 + x^1 e_1 + \dots + x^d e_d$$

- The Weyl generators T and S_i act as *orthogonal reflections* $\rho_{\alpha_0}, \rho_{\alpha_i}, i = 1 \dots d-1$

$$\lambda \to \rho_\alpha(\lambda) = \lambda - 2\frac{\alpha \cdot \lambda}{\alpha \cdot \alpha}\alpha$$

w.r.t. the signature $(- + + \dots)$ metric

$$\mathrm{d}s^2 = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^i)^2 + \mathrm{d}x^0(dx^1 + \dots + \mathrm{d}x^d)$$

- The Planck length $V/g_s^2 l_s^8$ is invariant, so the action restricts to the spacelike hyperplane $\delta \cdot x = x^0 = 0$ normal to

$$\delta = e_1 + \dots + e_d - 2e_0$$

- The Coxeter group is characterized by its Cartan matrix or Dynkin diagram

$$\begin{array}{ccccccc} T & & & & & & \\ & \diagdown & & & & & \\ & & S_2 & - & S_3 & - \cdots - & S_{d-1} \\ & \diagup & & & & & \\ S_1 & & & & & & \end{array}$$

This is the Dynkin diagram of $SO(d,d)$.

T-duality Weyl multiplets

- The representation of the finite group $\mathcal{W}(SO(d,d))$ can be obtained from the fundamental weights dual to the simple roots :

$$
\begin{array}{ccccccc}
\frac{1}{g} & & & & & & \\
& \diagdown & & & & & \\
& & \frac{R_1 R_2}{g_s^2} & - & \frac{R_1 R_2 R_3}{g_s^2} & - \cdots - & \frac{1}{R_d} \\
& \diagup & & & & & \\
\frac{R_1}{g} & & & & & &
\end{array}
$$

- Depending on the power of the invariant string length we include, this gives the T-duality Weyl multiplets of particle masses, or string tensions.

- The Kaluza-Klein modes on the torus together with the string winding states transform as a *vector* of $SO(d,d)$, with highest weight $\mathcal{M} = 1/R_{d-1}$.

- The D0-branes transform as a *spinor* of $SO(d,d)$ with highest weight $\mathcal{M} = 1/gl_s$, just as the RR gauge fields :

$$\mathcal{R} \leftrightarrow \mathrm{d}x^i \cdot \mathcal{R} + \mathrm{d}x^i \wedge \mathcal{R} \ , \ \ \mathcal{R} = \sum_p \mathcal{R}_p$$

  which generates a Clifford algebra. The same is true for the D2-brane tension, with highest weight $\mathcal{T}_2 = 1/gl_s^3$.

- The D-strings obtained by wrapping a D2-brane on a circle transform as a spinor of opposite chirality and highest weight $\mathcal{T}_1 = R_1/gl_s^3$.

Borel generators and mass formula

- Orbits of the Weyl group make finite representations thereof. In order to get representations of the full $SO(d,d,\mathbb{Z})$ group, we need to include the action of the Borel generators Γ_{ij}, Δ_{ij}. They have infinite order, and turn the finite representations into (discrete) infinite ones.

- In the $Sl(d,\mathbb{Z})$ case: a KK state with momentum 1 along direction i can be related by boosts –*i.e.* Borel transformations – to a state with any integer momentum m_i along the circles of T^d, with mass

$$\mathcal{M} = 1/R_i \to \mathcal{M} = \sqrt{m_i g^{ij} m_j}$$

  on which Γ_{ij} acts as $m^k \to m_k + \delta_{ik} m_j$.

- Similarly, the KK-winding rep of $\mathcal{W}(SO(d,d))$ becomes a lattice (m_i, m^i) with mass

$$\mathcal{M}^2 = (m_i + B_{ij}m^j)g^{ik}(m_k + B_{kl}m^l) + m^i g_{ij} m^j$$

  The Borel transformation acts as

$$m_k \to m_k + \Delta B_{jk} m^j \ , \ \ m^k \to m^k$$

- One only reaches states with $gcd(m_i, m^i) = 1$, and satisfying the *1/2-BPS condition* $m_i m^i = 0$.

D-brane T-duality multiplets

- Similarly, the D-brane states with mass

$$1/g_s l_s \ , \ \ R_i R_j / g_s l_s^3 \ , \ \ R_i R_j R_k R_l / g_s l_s^5 \dots$$

  turn into a lattice of integer charges

$$m = \{ m \ , \ \ m^{ij} \ , \ \ m^{ijkl} \dots \}$$

  transforming as a spinor of $SO(d,d,\mathbb{Z})$:

$$T_{ij} \ \ : m \to dx^i \wedge dx^j \wedge m + (dx^i \wedge dx^j) \cdot m$$

$$\Delta_{ij} \ \ : m \to m + dx^i \wedge dx^j \wedge m$$

- The 1/2-BPS mass is given by

$$\begin{aligned}
\mathcal{M}^2 &= \tilde{m}^2 + \frac{1}{2}(\tilde{m}^{ij})^2 + \frac{1}{4!}(\tilde{m}^{ijkl})^2 + \dots \\
\tilde{m} &= m + \frac{1}{2} m^{ij} B_{ij} + \frac{1}{8} m^{ijkl} B_{ij} B_{kl} + \dots \\
\tilde{m}^{ij} &= m^{ij} + \frac{1}{2} m^{klij} B_{kl} + \dots \\
\tilde{m}^{ijkl} &= m^{ijkl} + \dots
\end{aligned}$$

  where the $\tilde{m}$ are the (partially) dressed charges $\mathcal{V} \cdot m$, subject to the conditions

$$k^{ijkl} \equiv m^{[ij} m^{kl]} + m \ m^{ijkl} = 0 \ , \dots$$

- When $n^{ijkl} \neq 0$, the BPS state is at best 1/4-BPS, and its mass as an extra contribution :

$$\mathcal{M}^2 = \tilde{m}^2 + \frac{1}{2} \left(\tilde{m}^{ij} \right)^2 + \frac{1}{4!} \left(\tilde{m}^{ijkl} \right)^2 + \sqrt{ \frac{1}{4!} \left(\tilde{k}^{ijkl} \right)^2 }$$

D-brane and Born-Infeld mass

- The D-brane charges can be understood from the Born-Infeld action, by considering a D-brane configuration wrapped on T^d, in static gauge :

$$X^i = N^i_\alpha \sigma^\alpha \ , \ \ i = 1 \dots d \ , \ \ \alpha = 1 \dots p$$

- The wrapping numbers N^i_α and world-volume gauge field $F_{\alpha\beta}$ can be changed by a world-volume diffeomorphism, which leaves invariant

$$m^{ijkl} = \epsilon^{\alpha\beta\gamma\delta} N^i_\alpha N^j_\beta N^k_\gamma N^l_\delta$$

$$m^{ij} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} N^i_\alpha N^j_\beta F_{\gamma\delta} \ \ , \ \ m = \frac{1}{8} \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}$$

  which automatically satisfies $n^{ijkl} = 0$.

- The evaluation of the Born-Infeld action

$$S_{\mathrm{BI}} = \frac{1}{l_s^{p+1}} \int \mathrm{d}^{p+1} \xi e^{-\phi} \sqrt{\hat{g} + \hat{B} + \alpha' F}$$

  on this configuration precisely gives the 1/2-BPS mass.

BP, Kiritsis

- 1/4-BPS configurations can be obtained by considering 1/2-BPS excitations on the world-volume.

Dijkgraaf, Verlinde² 96

U-duality Weyl and Borel generators

- In M-theory variables, the T-duality $(R_i, R_j) \to (1/R_j, 1/R_i)$ reads

$$T_{ij} : R_i \to \frac{l_p^3}{R_j R_s} \,, \;\; R_j \to \frac{l_p^3}{R_s R_i} \,, \;\; R_s \to \frac{l_p^3}{R_i R_j} \,, \;\; l_p^3 \to \frac{l_p^6}{R_i R_j R_s}$$

- By a $Sl(d, \mathbb{Z})$ M-theory Lorentz boost, it can be conjugated to

$$T_{IJK} : R_I \to \frac{l_p^3}{R_J R_K} \,, \quad l_p^3 \to \frac{l_p^6}{R_I R_J R_K}$$

  which involves a set of *three* directions. A minimal set of Weyl transformations can be chosen as $T = T_{123}$ and $S_I \; : R_I \leftrightarrow R_{I+1}$.

- The Borel generators include as usual the Dehn twists $\gamma_I \to \gamma_I + \gamma_J$ in $Sl(d, \mathbb{Z})$, as well as the $B_{ij} = \mathcal{C}_{sij}$ shifts in $SO(d-1, d-1, \mathbb{Z})$. These can be conjugated into

$$C_{IJK} : \mathcal{C}_{IJK} \to \mathcal{C}_{IJK} + 1$$

  and we also need, for $d \geq 6$ ($d \geq 8$),

$$E_{IJKLMN} \;\; : \mathcal{E}_{IJKLMN} \to \mathcal{E}_{IJKLMN} + 1$$

$$K_{I;JKLMNPQR} \;\; : \mathcal{K}_{I;JKLMNPQR} \to \mathcal{K}_{I;JKLMNPQR} + 1$$

U-duality Weyl group

- The action of the Weyl group can again be represented on a weight space :

  $\mathcal{T} = l_p^{3x^0} R_1^{x^1} R_2^{x^2} \dots R_d^{x^d} \to \lambda = x^0 e_0 + x^1 e_1 + \dots + x^d e_d$

- The Weyl generators T and S_i act as *orthogonal reflections*

  $$\lambda \to \rho_\alpha(\lambda) = \lambda - 2\frac{\alpha \cdot \lambda}{\alpha \cdot \alpha}\alpha$$

  for the signature $(- + + \dots)$ metric

  $$\mathrm{d}s^2 = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^i)^2$$

- The Planck length V/l_p^9 being invariant, the action restricts to the spacelike hyperplane $\delta \cdot x = 0$ normal to

  $$\delta = e_1 + \dots + e_d - 3e_0 \ ,$$

  except when $d = 9$ *where δ is null*.

- The Coxeter group is characterized by its Cartan matrix or Dynkin diagram

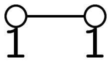
  $$\begin{array}{ccccccccc} & & & & T & & & & \\ & & & & | & & & & \\ S_1 & - & S_2 & - & S_3 & - & S_4 & - \cdots - & S_{d-1} \end{array}$$

  This is the Dynkin diagram of E_d, or the *extended* Dynkin diagram of $\hat{E}_8 = E_9$ for $d = 9$.

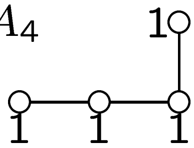
Group disintegrations

$E_2 = A_1$

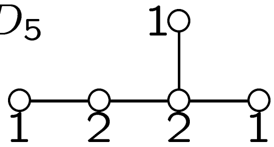
$E_3 = A_2 \oplus A_1$ 



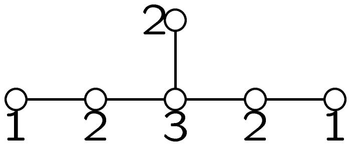
$E_4 = A_4$



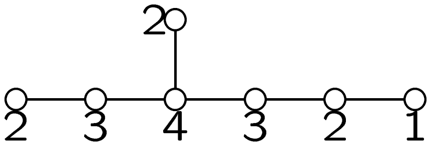
$E_5 = D_5$



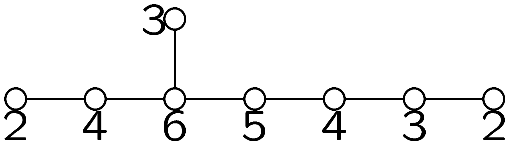
E_6



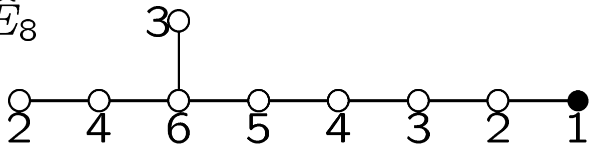
E_7



E_8



$E_9 = \widehat{E}_8$



U-duality Weyl multiplets

- The representation of the finite group $\mathcal{W}(E_{d(d)})$ ($d < 9$) can be obtained from the fundamental weights dual to the simple roots :

$$
\begin{array}{ccccccccc}
 & & & & \frac{1}{l_p^3} & & & & \\
 & & & & | & & & & \\
\frac{R_1}{l_p^3} & - & \frac{R_1 R_2}{l_p^6} & - & \frac{R_1 R_2 R_3}{l_p^9} & - & \frac{R_1 R_2 R_3 R_4}{l_p^9} & - \cdots - & \frac{1}{R_d}
\end{array}
$$

- The *particle multiplet* with highest weight $\mathcal{M} = 1/R_{d-1}$ transforms as the representation on the rightmost node.

- The *string multiplet* with highest weight $\mathcal{T}_1 = R_1/l_p^3$ transforms as the representation on the leftmost node.

- The *membrane multiplet* with highest weight $\mathcal{T}_2 = 1/l_p^3$ appears on the T-duality node.

- The other nodes do not correspond to other simple multiplets, but to Weyl submultiplets.

- For $d = 9$, the Weyl orbit becomes infinite. The level and degree are given by

$$k = \delta \cdot \lambda = x^1 + \cdots + x^9 + 3x^0$$

$$d = \kappa \cdot \lambda = x^9 - k/2$$

and correspond to the scaling dimension and the number of 9-th directions.

The particle multiplet

In $d = 8$: a $\mathbf{248}$ of $E_{8(8)}$.

mass $\mathcal{M}$	$Sl(8)$ irrep	charge
$\frac{1}{R_I}$	$\mathbf{8}$	m_1
$\frac{R_I R_J}{l_p^3}$	$\mathbf{28}$	m^2
$\frac{R_I R_J R_K R_L R_M}{l_p^6}$	$\mathbf{56}$	m^5
$\frac{R_I^2 R_J R_K R_L R_M R_N R_P}{l_p^9}$, 8 $\frac{V_R}{l_p^9}$	$\mathbf{1{+}63}$	$m^{1;7}$
$\frac{R_I^2 R_J^2 R_K^2 R_L R_M R_N R_P R_Q}{l_p^{12}}$	$\mathbf{56}$	$m^{3;8}$
$\frac{R_I^2 R_J^2 R_K^2 R_L^2 R_M^2 R_N^2 R_P R_Q}{l_p^{15}}$	$\mathbf{28}$	$m^{6;8}$
$\frac{R_I^3 R_J^2 R_K^2 R_L^2 R_M^2 R_N^2 R_P^2 R_Q^2}{l_p^{18}}$	$\mathbf{8}$	$m^{1;8;8}$

In lower d, drop the charges with two many antisymmetric indices :

D	d	U-duality group	irrep	$Sl(d)$ content
10	1	1	$\mathbf{1}$	$\mathbf{1}$
9	2	$Sl(2,\mathbb{Z})$	$\mathbf{3}$	$\mathbf{2} + \mathbf{1}$
8	3	$Sl(3,\mathbb{Z}) \times Sl(2,\mathbb{Z})$	$\mathbf{(3,2)}$	$\mathbf{3} + \mathbf{3}$
7	4	$Sl(5,\mathbb{Z})$	$\mathbf{10}$	$\mathbf{4} + \mathbf{6}$
6	5	$SO(5,5,\mathbb{Z})$	$\mathbf{16}$	$\mathbf{5} + \mathbf{10} + \mathbf{1}$
5	6	$E_{6(6)}(\mathbb{Z})$	$\mathbf{27}$	$\mathbf{6} + \mathbf{15} + \mathbf{6}$
4	7	$E_{7(7)}(\mathbb{Z})$	$\mathbf{56}$	$\mathbf{7} + \mathbf{21} + \mathbf{21} + \mathbf{7}$
3	8	$E_{8(8)}(\mathbb{Z})$	$\mathbf{248}$	$2(\mathbf{8} + \mathbf{28} + \mathbf{56}) + \mathbf{63} + \mathbf{1}$

The string multiplet

In $d = 8$: a $\mathbf{3875}$ of $E_{8(8)}$. In $d = 7$, a $\mathbf{133}$ of $E_{7(7)}$:

tension $\mathcal{T}$ mass	$Sl(7)$ irrep	charge
$\frac{R_I}{l_p^3}$	$\mathbf{7}$	n^1
$\frac{R_I R_J R_K R_L}{l_p^6}$	$\mathbf{35}$	n^4
$\frac{R_I^2 R_J R_K R_L R_M R_N}{l_p^9}$, $7\frac{V_R}{l_p^9}$	$\mathbf{1{+}48}$	$n^{1;6}$
$\frac{R_I^2 R_J^2 R_K^2 R_L R_M R_N R_P}{l_p^{12}}$	$\mathbf{35}$	$n^{3;7}$
$\frac{R_I^2 R_J^2 R_K^2 R_L^2 R_M^2 R_N^2 R_P}{l_p^{15}}$	$\mathbf{7}$	$n^{6;7}$

and on lower dimensional tori :

D	d	U-duality group	irrep	$Sl(d)$ content
10	1	1	$\mathbf{1}$	$\mathbf{1}$
9	2	$Sl(2,\mathbb{Z})$	$\mathbf{2}$	$\mathbf{2}$
8	3	$Sl(3,\mathbb{Z}) \times Sl(2,\mathbb{Z})$	$\mathbf{(3,1)}$	$\mathbf{3}$
7	4	$Sl(5,\mathbb{Z})$	$\mathbf{5}$	$\mathbf{4} + \mathbf{1}$
6	5	$SO(5,5,\mathbb{Z})$	$\mathbf{10}$	$\mathbf{5} + \mathbf{5}$
5	6	$E_{6(6)}(\mathbb{Z})$	$\mathbf{\bar{27}}$	$\mathbf{6} + \mathbf{15} + \mathbf{6}$
4	7	$E_{7(7)}(\mathbb{Z})$	$\mathbf{133}$	$\mathbf{7} + \mathbf{35} + \mathbf{49} + \mathbf{35} + \mathbf{7}$
3	8	$E_{8(8)}(\mathbb{Z})$	$\mathbf{3875}$	$\mathbf{8} + \mathbf{70} + \ldots$

A particle multiplet on T^d decomposes on T^{d-1} into a singlet $1/R_d$, a particle multiplet, a string multiplet, and extra states in $d \geq 6$.

Mass formula and BPS condition

The U-duality invariant mass formula on tori with arbitrary gauge backgrounds can again be obtained by considering the *non-commuting* $\mathcal{C}_3, \mathcal{E}_6, \mathcal{K}_{1;8}$ flows ; in $d = 7$:

$$\mathcal{M}^2 = (\tilde{m}_1)^2 + \frac{1}{2!\ l_p^6}\left(\tilde{m}^2\right)^2 + \frac{1}{5!\ l_p^{12}}\left(\tilde{m}^5\right)^2 + \frac{1}{7!\ l_p^{18}}\left(\tilde{m}^{1;7}\right)^2$$

with dressed charges given by

$$\begin{aligned}
\tilde{m}_1 &= m_1 + \frac{1}{2}\mathcal{C}_3 m^2 + \left(\mathcal{C}_3\mathcal{C}_3 + \mathcal{E}_6\right) m^5 + \left(\mathcal{C}_3\mathcal{C}_3\mathcal{C}_3 + \mathcal{C}_3\mathcal{E}_6\right) m^{1;7} \\
\tilde{m}^2 &= m^2 + \mathcal{C}_3 m^5 + \left(\mathcal{C}_3\mathcal{C}_3 + \mathcal{E}_6\right) m^{1;6} \\
\tilde{m}^5 &= m^5 + \mathcal{C}_3 m^{1;7} \\
\tilde{m}^{1;7} &= m^{1;7}
\end{aligned}$$

subject to the half-BPS conditions

$$\begin{aligned}
k^1 &= m_1 m^2 \equiv 0 \\
k^4 &= m_1 m^5 + m^2 m^2 \equiv 0 \\
k^{1;6} &= m_1 m^{1;7} + m^2 m^5 \equiv 0 \\
k^{3;7} &= m^2 m^{1;7} + m^5 m^5 \equiv 0 \\
k^{6;7} &= m^5 m^{1;7} \equiv 0
\end{aligned}$$

which transform as a *string multiplet*. When these composite charges do not vanish, the state is at most 1/4-BPS, in which case its mass formula is given by

$$\mathcal{M}^2 = \mathcal{M}_0^2(m) + \sqrt{[\mathcal{T}(k)]^2}$$

where $\mathcal{M}_0(m)$ and $\mathcal{T}(k)$ are given by the half-BPS mass and tension formulae

Green-Gutperle conjecture and its proof

- Type IIB R^4 couplings are given by the weight s=3/2 Eisenstein series of the S-duality group $Sl(2,\mathbb{Z})$:

$$E^{Sl(2,\mathbb{Z})}_{s=3/2;\mathbf{2}} = \sum_{(m,n)\neq(0,0)} \left(\frac{\tau_2}{|p+q\tau|^2}\right)^s = 2\zeta(3) \sum_{p\wedge q=1} \alpha^3_{(p,q)}$$

where the quantity $\alpha_{p,q} = \sqrt{\tau_2}\alpha'/|p+q\tau|$ is the inverse tension of a (p,q) string, and the exact result is the covariantization under $Sl(2,\mathbb{Z})$ of the perturbative result.

- The Eisenstein series are L_2 eigenmodes of the Laplacian

$$\Delta E_s = s(s-1)E_s$$

which rules out the possible contributions of the discrete cusp forms.

Green,Gutperle,Vanhove;BP;Green,Sethi

- The small coupling expansion reveals a sum of D-instanton effects:

$$\begin{aligned} f &= 2\zeta(3)e^{-3\phi/2} + \frac{2\pi^2}{3}e^{\phi/2} + 0 \\ &+ 4\pi e^{-\phi/2}\hat{\sum_m}\hat{\sum_n}\left|\frac{m}{n}\right| K_1\left(2\pi e^{-\phi}|mn|\right) e^{2\pi imna} \\ &\sim 4\pi\hat{\sum_N}\mu(N)e^{-2\pi N(e^{-\phi}\pm ia)}\left(1+O(e^{\phi})\right) \end{aligned}$$

with summation measure $\mu(N) = \sqrt{N}\sum_{n|N}\frac{1}{n^2}$.

Moore,Nekrasov,Shatashvili;Vanhove,Kostov

Generalized Eisenstein series

- Using the mass formulae obtained before, we can construct modular functions for the T and U-duality groups:

$$E_{s;R}^{G(\mathbb{Z})}(g) = \sum_{m \in \Lambda_R} [m \cdot R(g) \cdot m]^{-s} \, \delta(m^2)$$

Obers,BP, in progress

- The δ insertion imposes the half-BPS conditions, which are required both on physical and mathematical grounds:

$$\Delta E_{s;R}^{G(\mathbb{Z})} = s\langle \lambda, s\lambda - \rho \rangle E_{s;R}^{G(\mathbb{Z})}$$

where λ is the highest weight and ρ the Weyl vector.

- The basis of invariant functions, aside from cusp forms, is expected to correspond to the extremal nodes of the Dynkin diagram:

$$\begin{array}{ccccccc} & & \bar{S} & & & & \\ & & | & & & & \\ S & - & \circ & - & \circ & - \cdots - & V \end{array}$$

up to possible identifications.

One-loop amplitude and modular integral

- For type II R^4 as well as other half-BPS couplings, the one-loop contribution is given by the modular integral of a lattice partition function:

$$I_n = \int_{\mathcal{F}} \frac{d\tau}{\tau_2^2} \Gamma_{n,n}(G,B)$$

- This integral is an eigenmode of two differential operators:

$$\Delta_{SO(n,n)} I_n = \frac{n(2-n)}{4} I_n \ , \qquad \Delta_E I_n = 0$$

Kiritsis, Kounnas, Obers

just like the weight $s=1$ spinor and $s=n/2$ vector Eisenstein series:

$$I_n = E^{SO(n,n)}_{s=1;spin} + E^{SO(n,n)}_{s=1;\overline{spin}} = E^{SO(n,n)}_{s=n/2;vect}$$

- This agrees with explicit evaluation of the integral for low n:

$$I_1 = \frac{2\pi^2}{3}\left(R + \frac{1}{R}\right) \ ,$$

$$I_2 = -2\pi \ln U_2 |\eta(U)|^4 - 2\pi \ln T_2 |\eta(T)|^4$$

Dixon, Kaplunovsky, Louis

- The large volume expansion and decompactification limits are also consistent. They also indicate an identity $E_{s=1;spin} = E_{s=1;\overline{spin}}$ for $n \geq 3$.

R^4 couplings in low dimension

- The U-duality invariant R^4 coupling has to reproduce the tree level contribution as well as the one-loop term:

$$f_n = 2\zeta(3)V + E^{SO(n,n)}_{s=1;spin} + \dots$$

- From the branching rules of $E_{n+1(n+1)}$:

$$\mathcal{M}^2_{string} = \mathcal{M}^2_{singlet} + \frac{1}{g^2}\mathcal{M}^2_{spin} + \frac{1}{g^2}\mathcal{M}^2_{\overline{spin}},$$

we see that the Eisenstein series in the string rep has the correct small coupling expansion:

$$f_n = \frac{V}{l_p^9} E^{E_d(\mathbb{Z})}_{s=3/2;string}$$

- Beyond the tree-level and one-loop terms, it exhibits non-perturbative $O(e^{-1/g})$ effects from instantonic D-branes wrapped on T^d:

$$f_n = f_n^{pert} + Ve^{-\phi} \sum_{m,\{n\}} \left| \frac{m}{S_{\{n\}}} \right| K_1\left(2\pi m e^{-\phi} S_{\{n\}}\right) e^{2\pi i m \mathcal{R}\cdot n} + \dots$$

- In $D \leq 6$, there are also contributions from the $\overline{spin}$ rep, which scale as e^{-1/g^2}.

BP,Kiritsis

Summary - conclusions

- T-duality $SO(d-1, d-1, \mathbb{Z})$ together with 11D Lorentz invariance $Sl(d, \mathbb{Z})$ imply U-duality of M-theory on a torus T^d. BPS states fit into multiplets of $E_d(\mathbb{Z})$. Stable non-BPS states do not seem to exist in this theory, but can be constructed in less supersymmetric backgrounds, such as M/K_3.

- In $D = 3$ (M-theory on T^8), new states are required which have unconventional string coupling, Planck length or radius dependence. They correspond to non asymptotically flat backgrounds.

- In $D = 2$, U-duality becomes a *discrete affine Lie symmetry*, and requires infinite multiplets of BPS states. M-theory Finite in all (but one) directions has dramatic symmetry enhancement (Kac-Moody E_{11}, hyperbolic E_{10} resp.).

- Half-BPS saturated couplings can be represented as generalized Eisenstein series for the appropriate U-duality group, and exhibit infinite sums of instantonic contributions. Quarter-BPS couplings may in principle be obtained by relaxing the constraints, and may satisfy higher-order PDEs.

- Eventually, a proper understanding of semi-classical calculus in string theory may provide an alternative to more fundamental definitions of M-theory such as M(atrix) theory.