Black hole entropy and topological string theory

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CERN, May 24, 2005

based on hep-th/0502157 and hep-th/0506nnn

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Black hole thermodynamics and microscopic counting

- In general relativity, one associates to a macroscopic black hole with mass M, horizon area A and surface gravity κ an entropy $S_{BH} = A/4G_N$ and temperature $T = \kappa/2\pi$ such properties analogous to the standard laws of thermodynamics are obeyed

$$1)\ dS_{BH} = \frac{1}{T_H} dM + \omega dJ \ldots\ , \qquad 2)\ d(S_{BH} + S_{matter}) > 0$$

Christodoulou, Bekenstein, Hawking

- String theory is famously known to provide a microscopic description of black hole microstates, reproducing the Bekenstein-Hawking entropy. Eg, "4-charge" extremal black holes in 4D have a macroscopic entropy:

$$S_{BH} = 2\pi \sqrt{Q_1 Q_5 Q_{KK} P}\ ,$$

They can be represented as a D1-D5-KKM-P bound state, whose microstates are described by a 2D CFT. Their entropy can be counted by using the Ramanujan-Hardy (Cardy) formula

$$S_{micro} = \ln \Omega \sim 2\pi \sqrt{cN/6} \sim S_{BH}$$

Strominger Vafa; Maldacena Strominger; Johnson Khuri Myers

Black hole entropy beyond leading order

- This agreement relies on the "thermodynamical" limit where $A \gg G_N$, or $Q \gg 1$, and classical gravity can be trusted. Can we test this beyond leading order ?
- On the macroscopic side, both the Bekenstein-Hawking "area law" and the actual geometry receive corrections due to higher-derivative interactions in the low energy effective action. E.g, for 4D Einstein with polynomial interactions in $R_{\mu\nu\rho\sigma}$,

$$S_{BHW} = 2\pi \int_\Sigma \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \epsilon^{\mu\nu} \epsilon^{\rho\sigma} \sqrt{h} d\Omega \sim \frac{1}{4} A + \dots$$

Wald; Jacobson Kang Myers

  where $\epsilon^{\mu\nu}$ is the binormal on the horizon Σ. These corrections become important when the horizon area becomes $O(l_P)$ or $O(l_s)$.
- On the microscopic side, the entropy (defined as the Legendre transform of the free energy) receives non-extensive contributions away from the thermodynamical limit. These finite size corrections depend on a choice of statistical ensemble.

Beyond the BH entropy in $N = 2$ supergravity

- In the context of $N = 2$, $D = 4$ supergravity, Cardoso, de Wit Mohaupt (CDM) have computed the Bekenstein-Hawking-Wald entropy of extremal black holes including an infinite class higher derivative F-term corrections

$$\mathcal{L} = \mathcal{L}_{tree} + \sum_{h=0}^{\infty} F_h(X^A)({}^-C^-)^2(T^-)^{2h-2}$$

where ${}^-C^-$ and T^- are the self-dual part of the Weyl tensor/ graviphoton field strength.

- Ooguri Strominger and Vafa (OSV) observed that the CDM result takes a specially simple form if one assumes that the statistical ensemble implicit in the macroscopic computation is a “mixed” ensemble, where the magnetic charges are fixed but electric charges are allowed to fluctuate at a fixed electric potential:

$$\text{Legendre}[S_{CDM}] = \langle S_{CDM}(p^A, q_A) - \pi\phi^A q_A\rangle_{q_A} = \frac{1}{\pi}\text{Im}F(p^A + i\phi^A, 2^8)$$

where $F(X^A, W^2) := \sum_h F_h(X^A)W^{2h-2}$.

The OSV Conjecture

- In combination with the computation of CDM, this motivates a simple relation between micro-canonical degeneracies $\Omega(p^A, q_A)$ and the higher derivative amplitudes F:

$$Z(p^A, \phi^A) := \sum_{q_A \in \Lambda_{el}} \Omega(p^A, q_A) e^{-\pi \phi^A q_A} \overset{?}{=} \left| \exp\left(\frac{i\pi}{2} \mathrm{Im} F(p^A + i\phi^A, 2^8) \right) \right|^2$$

Ooguri Strominger Vafa

- In type II/CY, F_h is computable as a genus-h topological string amplitude. In Het/K3, F_1 appears at tree-level and all F_h appear already at one-loop. $F(X^A, W^2)$ could in principle contain non-perturbative corrections $O(-1/W^2)$.

Berschadsky Cecotti Ooguri Vafa;Antoniadis Gava Narain Taylor

- If correct, this relation may allow us to test our microscopic understanding of black hole microstates at unprecedented accuracy. Our aim will be to test this proposal in cases where the two sides of the equation are known exactly.

Outline of the talk

1. Attractor mechanism and the OSV conjecture
2. A benchmark case: $K_3 \times T^2$
3. $N = 4$ CHL strings
4. $N = 2$ orbifolds
5. Towards an exact OSV-type formula
6. Discussion

The attractor mechanism

- Consider a general ansatz for a static, spherically symmetric BH in type IIA/CY:

$$ds^2 = -e^{2U(r)+2r} dt^2 + e^{-2U(r)} \left(dr^2 + r^2 d\Omega_2^2 \right) + ds^2_{CY}$$

  The shape of the CY is parameterized by Kähler moduli $X^A(r)$ (vectors) and complex structure moduli (hypers). The latter decouple and can be taken to be constant.

- The tree-level $N = 2$ lagrangian is controlled by the prepotential, an homogeneous holomorphic function $F_0(X^A)$ given by

$$F_0(X^A) = -\frac{1}{6} C_{ABC} \frac{X^A X^B X^C}{X^0} + \sum_{\beta \in H_2(Y,Z)} N_{0,\beta}\, e^{2\pi i \beta_A X^A / X^0}$$

  Notation: $F_A = \partial F_0 / \partial X^A$.

- The SUSY equations can be rewritten as

$$Re \left(X^A - \frac{d}{dr} X^A \right) = p^A \,, \quad Re \left(F_A - \frac{d}{dr} F_A \right) = q_A$$

The attractor mechanism (cont)

- At the horizon, the geometry becomes $AdS^2 \times S^2 \times CY$ and the Kähler moduli are fixed by the attractor equations,

$$Re(X^A) = p^A, \quad Re(F_A) = q_A$$

Ferrara Kallosh Strominger

- The Bekenstein-Hawking entropy is thus independent from the moduli at infinity, and a function of the charges only,

$$S_{BH} = |Z(p,q)|^2 = \frac{i\pi}{2}\left(q_A \bar{X}^A - p^A \bar{F}_A\right)$$

  where $Z(p,q)$ is the “graviphoton” central charge of the $N=2$ superalgebra.

- S_{BH} is an homogeneous function of degree 2, so the horizon area is generically $\gg (l_s, l_P)$ at large charges. For special choices of charges, the area vanishes and the geometry is singular.

The attractor mechanism, revisited

- The attractor equations are usefully recast as follows: set $X^A = p^A + i\phi^A$ where ϕ^A is real. The second equation becomes

$$q_A = \frac{1}{2}\left(\partial F_0/\partial X^A + \partial \bar{F}_0/\partial \bar{X}^A\right) = \frac{1}{2i}\left(\partial F_0/\partial \phi^A - \partial \bar{F}_0/\partial \bar{\phi}^A\right)$$

hence

$$q_A = \pi\, \partial \mathcal{F}/\partial \bar{\phi}^A \quad \text{where} \quad \mathcal{F}_0(p^A, \phi^A) = \frac{1}{\pi}\text{Im}F_0(p^A + i\phi^A)$$

- In addition, the BH entropy may be rewritten as

$$S_{BH} = \mathcal{F}_0(p^A, \phi^A) + \pi\, q_A \phi^A$$

- The BH entropy $S_{BH}(p^A, q_A)$ is thus recognized as the Legendre transform of the free energy $\mathcal{F}_0(p^A, \phi^A)$. To compute the latter, no need to solve the attractor equations !

Ooguri Vafa Strominger

Leading entropy of large black holes

- As an application, let us compute the tree-level entropy of a black hole with arbitrary charges (set $p^0 = 0$ for simplicity). Neglecting GW instantons, the tree-level prepotential is

$$F_0 = -\frac{1}{6} C_{ABC} \frac{X^A X^B X^C}{X^0} \Rightarrow \mathcal{F}_0(p,\phi) = -\frac{\pi}{6}\frac{C(p)}{\phi^0} + \frac{\pi}{2}\frac{C_{AB}(p)\phi^A\phi^B}{\phi^0}$$

$$C(p) = C_{ABC} p^A p^B p^C \ , \quad C_{AB}(p) = C_{ABC} p^C \ , \quad A = 1, \dots n_V - 1$$

- The Legendre transform with respect to ϕ^A leads to

$$\phi_*^A = -C^{AB}(p) q_B \phi^0 \ , \qquad \phi_*^0 = \pm\sqrt{-\hat{C}(p)/6\hat{q}_0}$$

where

$$\hat{q}_0 = q_0 + \frac{1}{2} q_A C^{AB}(p) q_B$$

- The tree-level Bekenstein-Hawking entropy is therefore the square-root of a quartic polynomial in the charges,

$$S_{BH} = 2\pi\sqrt{C(p)\hat{q}_0/6}$$

- Similarly, for 1/4-BPS states in $N = 4$ backgrounds, the BH entropy is given by the $Sl(2) \times SO(6,n)$ invariant discriminant

$$S_{BH} = 2\pi\sqrt{(\vec{P}\cdot\vec{P})(\vec{Q}\cdot\vec{Q}) - (\vec{P}\cdot\vec{Q})^2}$$

  where $\vec{Q} = (q_0, p^1, q_a)$, $\vec{P} = (p^0, q_1, p^a)$ are the electric and magnetic charges in the natural heterotic polarization (the $N = 2$ attractor mechanism applies to $n+6-4$ charges only, since 4 gauge fields belong to $N = 2$ gravitino multiplets).

- For 1/8-BPS states in $N = 8$ backgrounds,

$$S_{BH} = 2\pi\sqrt{I_4(Q)}$$

  where $I_4(Q)$ is the E_7 quartic invariant, built out of the 56=28+28 electric and magnetic charges (the $N = 2$ attractor mechanism applies to $28 - 12 = 16$ charges only).

Leading entropy of 4-dim black holes

- 1/4 BPS black holes in type IIA / $K_3 \times T^2$ can be described by a D6-D2-NS5 bound state wrapped on K_3, with momentum along S^1. After reduction along K_3, one obtains an effective black string in 6D, with central charge $c = 6Q_2Q_5Q_6$:

$$S_{micro} = 2\pi\sqrt{Q_2Q_5Q_6P} = S_{BH}$$

Maldacena Strominger

- Equivalently, the same system can be described by a bound state of D1-D5-P with Q_K KK monopole: the same entropy arises by taking into account fractional D-branes in the ALE geometry.

Johnson Khuri Myers, Constable Khuri Myers

- More generally, 4D 1/2-BPS black holes can be obtained in M-theory / CY $\times S^1$ by wrapping an M5-brane on $\gamma_4 \times S_1$. The reduced theory on γ_4 is a $(0,4)$ sigma model, and the Cardy formula predicts

$$S_{micro} = 2\pi\sqrt{(D_{ABC}p^Ap^Bp^C + c_{2A}p^A/6)q_0}$$

in the regime $q_0 \gg D(p)$. This differs from S_{BH} at order Q^0, in agreement with 1-loop R^2 corrections.

Maldacena Strominger Witten

Higher derivative interactions and the topological string

- Recall that the $(2,2)$ sigma-model on a CY threefold can be topologically twisted into the A-model topological string, which depends only on the Kähler moduli. This defines a quantum field theory of Kähler structures, known as Kähler gravity.
- The genus-h amplitude (without insertions) in the topological A-model computes the $({}^{-}C^{-})^2(T^{-})^{2h-2}$ amplitude in the physical type II superstring:

$$\int d^8\theta F_h(X) W^{2h} = F_h(X)({}^{-}C^{-})^2(T^{-})^{2h-2} + \dots$$

where $W^2 = (T^{-})^2 + \cdots + \theta^4({}^{-}C^{-})^2$

- The all-genus topological A-model thus resums an infinite number of higher-derivative F-term corrections. The precise identification is

$$F_{top} = \frac{i\pi}{2} F_{SUGRA} \, , \quad t^A = \frac{X^A}{X^0} \, , \quad \lambda = \frac{\pi}{4} \frac{W}{X^0}$$

The attractor mechanism, to all orders

- In the presence of these F-term corrections, the attractor formalism goes through upon replacing the tree-level prepotential $F_0(X)$ by the complete topological amplitude

$$F(X^A, W^2) = \sum_{h=0}^{\infty} F_h(X^A) W^2$$

  and imposing an additional attractor equation $(W/X^0)^2 = 2^8$.
- The Bekenstein-Hawking-Wald entropy is thus the Legendre transform of the free energy

$$\mathcal{F}(p^A, \phi^A) = \frac{1}{\pi} \mathrm{Im} F\left(p^A + i\phi^A; (2^4 X^0)^2\right)$$

- One may interpret $\mathcal{F}(p^A, \phi^A)$ as the free energy of a statistical ensemble of black holes with magnetic charge p^A and electric potential ϕ_A.

The OSV conjecture for BH degeneracies

- It is thus natural to conjecture that the relevant microscopic statistical ensemble is a “mixed” ensemble, where magnetic charges are treated micro-canonically but electric charges are treated canonically:

$$Z(p^A, \phi^A) := \sum_{q_A \in \Lambda_{el}} \Omega(p^A, q_A) e^{-\phi^A q_A} \stackrel{?}{=} e^{\mathcal{F}(p^A, \phi^A)} = \left| \exp\left(\frac{i\pi}{2} F(p^A + i\phi^A) \right) \right|^2$$

Ooguri Strominger Vafa

- If correct, this provides a way to compute the microscopic degeneracies $\Omega(p^A, q_A)$ (or rather a suitable index) from the topological string amplitude $F(W, X)$, by inverse Laplace transform,

$$\Omega(p^A, q_A) \equiv \int d\phi^A \left| \exp\left(\frac{i\pi}{2} F(p^A + i\phi^A) \right) \right|^2 e^{\phi^A q_A}$$

- Conversely, one may hope to understand the non-perturbative completion of the topological string from the knowledge of black hole micro-states.

More on the OSV conjecture

- There are several versions of the OSV conjecture: the weaker form is supposed to relate the topological string amplitude with some suitable index, and hold only asymptotically to all orders in inverse charges.
- The conjecture encounters some immediate problems: the sum does not converge (i.e. the ensemble is thermodynamically unstable), the topological amplitude does not satisfy the required periodicity $\phi \to \phi + 2iZ$,
- The OSV proposal is somewhat formal: what is the precise integration measure and contour ? What precise degeneracies should we count ? How about holomorphic anomalies, curves of marginal stability ? Should we count micro-states with arbitrary angular momentum or only $J = 0$? etc

Elaborations of the OSV conjecture

- The proposal has been tested in the case of non-compact CY: $O(-m) \oplus O(m) \to T^2$: BPS states are counted by topologically twisted SYM on N D4-brane wrapped on a 4-cycle $O(-m) \to T^2$, which is equivalent to 2D Yang Mills. At large N, this "factorizes" into $\sum_l \Psi_{top}(t + mlg_s)\Psi_{top}(\bar{t} - mlg_s)$.

Vafa

- This was generalized for $O(-m) \oplus O(2g - 2 + m) \to \Sigma_g$, whose topological amplitude is related to q-deformed 2D Yang-Mills. The agreement with OSV for genus $g > 1$ however requires modular properties which are less than obvious.

Aganagic Ooguri Saulina Vafa

- The OSV conjecture can be further motivated by interpreting the BH partition function as the inner product of two wave functions of the Universe in a minisuperspace formulation.

Ooguri Verlinde Vafa

- Exponentially suppressed corrections originating from the mixing of two Fermi seas, are understood in terms of multi-centered black holes / baby universe configurations.

Dijkgraaf Gopakumar Ooguri Vafa

Large Black Hole degeneracies from OSV

- The A-model topological string amplitude $F(X^A, W^2)$ is an homogeneous function of degree 2 in (X^A, W):

$$F = -\frac{1}{6} C_{ABC} \frac{X^A X^B X^C}{X^0} - \frac{W^2}{64 \cdot 24} \frac{c_A X^A}{X^0} - \frac{X_0^2}{(2\pi i)^3} \sum_{h=0}^{\infty} \sum_{\beta} \left(\frac{\pi W}{4 X^0} \right)^{2h} N_{h,\beta} e^{2\pi i \beta_A X^A / X^0}$$

where $A = 1, \ldots n_V - 1$ runs over a base of 2-cycles of Y, $C_{ABC} = \int_Y J_A J_B J_C$ are triple intersection numbers, $X^A/X^0 = B^A + iV^A$ are the Kähler moduli, $c_A = \int_Y J_A c_2(T^{1,0}(X))$ and $N_{h,\beta}$ are rational numbers known as the Gromov-Witten invariants.

- OST instruct us to compute the topological free energy

$$\mathcal{F}(p, \phi) = -\frac{\pi}{6} \frac{\hat{C}(p)}{\phi^0} + \frac{\pi}{2} \frac{C_{AB}(p) \phi^A \phi^B}{\phi^0} + 2\text{Re}(F_{GW})$$

where

$$\hat{C}(p) = C(p) + c_A p^A \,, \quad C(p) = C_{ABC} p^A p^B p^C \,, \quad C_{AB}(p) = C_{ABC} p^C$$

- LET US DROP F_{GW} and compute the Laplace transform

$$\Omega_{OSV}(p^A, q_A) = \int d\phi^0 d\phi^A \exp\left(\mathcal{F}(p,\phi) + \pi\phi^A q_A\right)$$

The ϕ^A integral is Gaussian, with saddle at $\phi^A_* = -C^{AB}(p) q_B \phi^0$:

$$\Omega_{OSV}(p^A, q_A) = \int d\phi^0 \phi_0^{(n_V-1)/2} \det[C_{AB}(p)]^{-1/2} \exp\left(-\frac{\pi}{6}\frac{\hat{C}(p)}{\phi^0} + \pi\phi^0 \hat{q}_0\right)$$

with $\hat{q}_0 = q_0 + \frac{1}{2} q_A C^{AB}(p) q_B$.

- The ϕ^0 integral is now of Bessel type, with saddle at $\phi^0_* = \pm\sqrt{-\hat{C}(p)/6\hat{q}_0}$. For an appropriate contour, we find

$$\Omega_{OSV}(p^A, q_A) = \det[C_{AB}(p)]^{-1/2} [\hat{C}(p)]^{(n_V+1)/2} \hat{I}_{(n_V+1)/2}\left[2\pi\sqrt{\hat{C}(p)\hat{q}_0/6}\right]$$

- Using the asymptotics

$$\hat{I}_\nu(z) \sim z^{-\nu-\frac{1}{2}} e^z \left(1 + a/z + b/z^2 + \dots\right)$$

we deduce the micro-canonical entropy predicted by the OSV formula:

$$S_{OSV}(p^A, q_A) \sim 2\pi\sqrt{\hat{C}(p)\hat{q}_0/6} - \frac{n_V + 2}{2}\log[\hat{C}(p)\hat{q}_0] + \dots$$

- The leading square-root term reproduces the tree-level Bekenstein-Hawking entropy $S_{BH} = 2\pi\sqrt{C(p)\hat{q}_0}$ at large charges.
- The replacement $C(p) \to \hat{C}(p) = C(p) + C_A p^A$ takes into account the large-volume limit of F_1, and reproduces the MSW correction:

$$2\pi\sqrt{\hat{C}(p)\hat{q}_0/6} = 2\pi\sqrt{(D_{ABC}p^A p^B p^C + c_{2A}p^A/6)q_0}$$

As a result, "small black holes" which were singular at tree-level ($C(p) = 0$) acquire a smooth horizon due to R^2 interaction.
- The logarithmic correction is purely an effect of changing from mixed to micro-canonical ensemble.

More comments

- Integrals have been carried out somewhat formally. Since $C_{AB}(p)$ in general has signature $(1, n_V - 2)$, the gaussian integral needs to be computed by rotating the contour for ϕ^A to the imaginary axis.
- In addition to the Bessel $\hat{I}$ function, the OSV integration measure predicts extra p-dependent factors, which, if taken literally, contradict T-duality on the heterotic side. We shall consider ratios $\Omega_{OSV}(p,q)/\Omega_{OSV}(p',q)$ only.
- Non-degenerate GW instantons contributions are exponentially suppressed in the large charge limit, and can be consistently neglected if $(p^a)^2 q_0 \gg C(p)$.
- Unless $\chi = 0$, the series of point-like instantons is strongly coupled when $\hat{q}_0 \gg \hat{C}(p)$ (which is the regime of applicability of the Cardy formula). Fortunately, they can be resummed into the Mac-Mahon function $\prod_{k=1}^{\infty}(1-e^{-k\lambda})^k$, and reabsorbed into a redefinition $e^{F_{top}} \to \lambda^{\chi/24} e^{F_{top}}$ of the topological amplitude.

More details on point-like instantons

- The point-like instantons with $\beta' = 0$ lead to $n_0^0 = -\chi/2$ (χ=Euler number of CY). They contribute an infinite series of higher-genus contributions to the topological amplitude:

$$F_{point} = -\frac{\chi}{2}\left[\frac{\zeta(3)}{\lambda^2} + A + \sum_{h=2}^{\infty} \lambda^{2h-2} \frac{(2h-1)B_{2h}B_{2h-2}}{(2h-2)(2h)!}\right]$$

- The $\zeta(3)$ term follows from the tree-level R^4 amplitude in 10D, the terms with $h \geq 2$ are proportional to the Euler number of the moduli space of genus-h Riemann surfaces without punctures, and A is a naively divergent quantity, but, when properly regulated

$$A = \frac{1}{12}\log(2\pi/\lambda) + \text{finite}$$

- This asymptotic expansion is valid at $\lambda \ll 1$. If $\lambda \gg 1$, one may resume the series into the Mac Mahon function,

$$F_{point} = -\chi/2 \sum_{n=0}^{\infty} n \log(1-q^n) \,, \qquad q = e^{-\lambda}$$

  This is now exponentially suppressed at strong coupling, but F_{point} differs from the usual amplitude by a term $\chi/24 \log \lambda$.

Testing OSV: small black holes

- Our goal is to test the OSV conjecture in cases where black holes degeneracies are exactly known. For this, restrict to K_3-fibered CY, which admit a dual description as heterotic / $K^3 \times T^2$.
- The heterotic string admits a class of perturbative BPS states, known as Dabholkar-Harvey states:

$$|osc, N\rangle_L \otimes |osc, 0\rangle_R \otimes |n_i, w^i\rangle$$

  satisfying the matching condition $N - 1 = n_i w^i$. They preserve 8 SUSY, and carry purely electric charge, in the natural heterotic polarization. It is easy to count them exactly by using simple modular forms.
- At strong coupling, these states remain stable and become black holes, carrying both electric and magnetic charges, in the natural type II polarization. In contrast to the general "4-charge" black holes, they are singular at tree-level, but acquire a smooth horizon due to R^2 interactions.

Sen 95; Dabholkar 04; Kallosh Maloney Dabholkar; Hubeny Maloney Rangamani; Bak Kim Rey

OSV prediction for small black holes

- For a K3-fibered CY 3-fold, the Kähler moduli split into the modulus X^1/X^0 of the base, and the moduli X^a/X^0 of the fiber ($a = 2, \ldots n_V - 1$). The intersection form decomposes into

$$C_{ABC}X^A X^B X^C = X^1 C_{ab} X^a X^b + C_{abc} X^a X^b X^c$$

- Further consider a state whose only non-vanishing magnetic charge is p^1 (D4/K3):

$$C(p) = 0 \,, \quad \hat{C}(p) = 24p^1 \,, \quad C_{AB}(p) = \begin{pmatrix} 0 & 0 \\ 0 & p^1 C_{ab} \end{pmatrix} \,, \quad \hat{q}_0 = q_0 + \frac{1}{2} C^{ab} q_a q_b / p_1$$

- The dependence on ϕ^1 now disappears from the integrand. Since F_{top} is invariant under monodromies $\phi_1 \to \phi_1 + \phi_0$, it is natural to restrict the integration range to $[0, \phi_0]$:

$$\Omega_{OSV}(p^1, q_A) = \int d\phi^0 \, \phi_0^{n_V/2} \exp\left(-\frac{4\pi p_1}{\phi^0} + \pi \phi^0 \hat{q}_0 \right) \sim \hat{I}_{(n_V+2)/2}\left[4\pi \sqrt{p^1 \hat{q}_0} \right]$$

- Caveat: when $p^a = 0$, the Kähler classes vanish at the saddle point. Strictly speaking, for such states the OSV formula is meaningless...

A benchmark case: $II/K3 \times T^2$ vs Het/T^6

- On the macroscopic side: thanks to $N = 4$, $F_{h>1} = 0$. F_1 can be extracted from R^2 coupling,

$$f_{R^2} \sim \log T_2 |\eta(T)|^4 \ \Rightarrow\ F_1 = \log \eta^{24}(T)\ , \quad T = X_1/X_0$$

- The gauge group is $U(1)^6 \times U(1)^{22}$, but 4 $U(1)$ are part of $N = 2$ gravitino multiplets, hence $n_V = 24$. Accordingly,the OSV prediction for small BH degeneracies is

$$\Omega_{OSV}(p^1, q_0) = \hat{I}_{13}\left[4\pi\sqrt{p^1 \hat{q}_0}\right]$$

- On the heterotic side, these small BPS BH are dual to Dabholkar Harvey states, enumerated by

$$\frac{1}{\eta^{24}} := \frac{1}{q\prod_{k=1}^{\infty}(1-q^k)} = \sum_{N=0}^{\infty} p_{24}(N) q^{N-1}\ , \quad N-1 = p^1 q_0$$

- The leading exponential behavior is given by Cardy's formula $\log p_{24} = 2\pi\sqrt{24(N-1)/6}$. Subleading corrections can be extracted using the Rademacher formula...

The Rademacher expansion

Consider a vector-valued modular form $f_{\mu=1..r}(\tau)$ of weight $w < 0$,

$$f_\mu(\tau+1) = e^{2\pi i \Delta_\mu} f_\mu(\tau) \ , \quad f_\mu(-1/\tau) = (-i\tau)^w S_{\mu\nu} f_\nu(\tau)$$

with Fourier expansion $f_\mu(\tau) = q^{\Delta_\mu} \sum_{m=0}^{\infty} \Omega_\mu(m) q^m$

Theorem: the Fourier coefficients can be expressed as a convergent series

$$\Omega_\nu(n) = \sum_{s=1}^{\infty} \sum_{\mu=1}^{r} \sum_{m+\Delta_\mu<0} s^{w-2}\, \mathrm{Kl}(n,\nu;m,\mu;s) |m+\Delta_\mu|^{1-w}$$

$$\times \Omega_\mu(m) \times \hat{I}_{1-w}\left[\frac{4\pi}{s}\sqrt{|m+\Delta_\mu|(n+\Delta_\nu)}\right]$$

where $\mathrm{Kl}(n,\nu;m,\mu;s)$ are generalized Kloosterman sums, equal to $S^{-1}_{\nu\mu}$ for $s=1$ and $\hat{I}_\nu(z)$ is a modified, modified Bessel function of the 1st kind,

$$\hat{I}_\nu(z) = 2\pi \left(\frac{z}{4\pi}\right)^{-\nu} I_\nu(z) \sim z^{-\nu-\frac{1}{2}} e^z (1 + a/z + b/z^2 + \dots)$$

The Rademacher expansion (cont.)

- All $s > 1$ contributions are exponentially suppressed wrt to $s = 1$, yet they are exponentially large in an absolute sense.
- The Hardy-Ramanujan-Cardy formula emerges by keeping the leading term $s = 1, m = 0$, using $\Delta = c/24, w = -c/2$:

$$\begin{aligned} \log \Omega(n) &\sim 4\pi \sqrt{|\Delta|(n+\Delta)} + \frac{1}{2}(w - \frac{3}{2}) \log(n+\Delta) + \ldots \\ &= 2\pi \sqrt{\frac{c(n+\Delta)}{6}} - \frac{1}{4}(c+3) \log(n+\Delta) + \ldots \end{aligned}$$

- The Rademacher expansion depends only on the polar part

$$f_\mu^- = \sum_{m+\Delta_\mu<0} \Omega_\mu(m) q^{m+\Delta_\mu}$$

(and modular data). Indeed, one proof is to represent $f_\mu(\tau)$ (or rather its Farey transform $q\partial_q^{1-w} f$) as the Poincaré series (i.e. sum over $Sl(2,Z)$ images) of f_μ^-.

Back to the bench

- In particular, for the inverse of the Dedekind function, $w = -12$, $\Delta = -1$,$\Omega(0) = 1$ hence
$$p_{24}(N) = \hat{I}_{13}\left[4\pi\sqrt{p^1\hat{q}_0}\right] + 2^{-14}\hat{I}_{13}\left[2\pi\sqrt{p^1\hat{q}_0}\right] + \ldots$$
- Comparing to the OSV prediction, we find agreement to ALL orders in $1/(p^1 q_0)$! However, the OSV formula fails to reproduce subleading corrections which grow like $e^{2\pi\sqrt{p^1 q_0}}$.
- In order to obtain matching, it was crucial to drop non-holomorphic contributions from f_{R^2}, and consider the degeneracies of states with arbitrary angular momentum J.
$$\Omega(N; J = 0) \sim \hat{I}_{29/2}\left[4\pi\sqrt{N - 7/8}\right]$$
- Note also that the matching relies on little data: only the large volume limit of 1-loop f_{R^2} (which is a universal tree-level term on heterotic side), the number of vector multiplets and the modular weight.
- This is NOT another test of het/type II duality: we did not really need the heterotic string to count 1/2 BPS states in type II on K_3...

$N = 4$ CHL strings

- More general $N = 4$ models with $0 \leq k \leq 22$ vector multiplets of $N = 4$ can be constructed, either as orbifolds of type II/ $K3 \times T^2$ by an Enriques involution, or as freely acting asymmetric orbifolds of Het/T^6.
- In the untwisted sector of the orbifold, the BPS states are a projection of the DH states in the Het/T^6 model. Their degeneracies are now counted by a modular form

$$Z_{untw} = \frac{1}{2}\left(\frac{\theta}{\eta^{24}} + \psi\right)$$

  where θ is a partition function for the lattice of electric charges under the $22 - k$ gauge fields which have been projected out, and ψ enforces the projection. Modular weight:

$$w = \frac{1}{2}(22 - k) - 12 = -1 - k/2 \qquad \Rightarrow 1 - w = (k + 4)/2 = (n_V + 2)/2$$

  Degeneracies are dominated by θ/η^{24}, and are in agreement with the OSV prediction.
- In addition, there are BPS states in the twisted sectors, which are counted by modular forms related to ψ by modular transformation. Their asymptotics appear to be equal to that of the untwisted, unprojected sector, again vindicating OSV.

$N=4$ CHL strings (a case study)

- Consider the simplest case:

$$\Gamma_{6,22}=E_8(-1)\oplus E_8(-1)\oplus II^{1,1}\oplus II^{5,5}$$

orbifolded by $g|P_1,P_2,P_3,P_4\rangle = e^{2\pi i\delta\cdot P_3}|P_2,P_1,P_3,P_4\rangle$ This projects out the $U(1)$ associated to P_1-P_2, leaving only the physical electric charges $Q=(P_1+P_2,P_3,P_4)$.

- DH states arise in the untwisted sector by taking the ground state on the right, an arbitrary, orbifold invariant excitation of the 24 oscillators on the left, and level-matched internal momentum:

$$Z_{untw}=\frac{1}{2}\left(\frac{Z_{6,6}[{}^0_0]\theta^2_{E_8[1]}(\tau)}{\eta^{24}(\tau)}+\frac{Z_{6,6}[{}^0_1]\theta_{E_8[1]}(2\tau)}{\eta^8(\tau)\eta^8(2\tau)}\right)$$

- From this we need to extract the number of states with given $Q=(P_1+P_2,P_3,P_4)$. For this, change basis from (P_1,P_2) to

$$P_1+P_2=2\Sigma+\wp\ ,\quad P_1-P_2=2\Delta-\wp$$

where S, Δ take values in the E_8 root lattice, and $\mathcal{P}$ is an element of the finite group $Z=\Lambda_r(E_8)/2\Lambda_r(E_8)$.

$N = 4$ CHL strings (cont)

- In order to sum over the "unphysical charges" Δ, introduce E_8 level-2 theta functions with characteristics:

$$\Theta_{E_8[2],\wp}(\tau) := \sum_{\Delta \in E_8(1)} e^{2\pi i \tau (\Delta - \frac{1}{2}\wp)^2}$$

and use

$$\theta^2_{E_8[1]}(\tau) = \sum_{\mathcal{P} \in E_8/2E_8} \theta_{E_8[2],\mathcal{P}}(\tau)\theta_{E_8[2],\mathcal{P}}(\tau) \ , \quad \theta_{E_8[1]}(2\tau) = \theta_{E_8[2],0}(\tau)$$

hence

$$Z_u = \frac{\theta^2_{E_8[2],\mathcal{P}}(\tau)}{\eta^{24}(\tau)} \pm \frac{1}{\eta^8(\tau)\eta^8(2\tau)} := q^{\Delta\pm} \sum_{N=0}^{\infty} d^u_\pm(N) q^N$$

CHL strings, cont.

- In the twisted sector, the situation is simpler:

$$Z_t = \frac{1}{2}\left(\frac{1}{\eta^{12}\theta_4^4} \pm \frac{1}{\eta^{12}\theta_3^4}\right) := q^{\Delta_\pm}\sum_{N=0}^{\infty} d_\pm^t(N) q^N$$

- Using the Rademacher formula, we find

$$\begin{aligned}\dim \mathcal{H}_{BPS}(Q) \quad &= \quad 2^{-5}\hat{I}_9\left(4\pi\sqrt{Q^2/2}\right) \\ &\quad +\hat{I}_9\left(4\pi\sqrt{Q^2/4}\right)\begin{cases} 15\cdot 2^{-10} + 2^{-6}e^{2\pi i P\cdot\delta}\,, & \wp\in\mathcal{O}_1 \\ 2^{-10}\,, & \wp\in\mathcal{O}_{248} \\ -2^{-10}\,, & \wp\in\mathcal{O}_{3875} \\ 2^{-10}e^{i\pi Q^2}\,, & Q\in\Lambda_1 \end{cases} + \ldots\end{aligned}$$

Hence we have agreement to all orders with OSV in all sectors. Subleading terms however are not captured by OSV, and depend crucially on the sector.

Absolute degeneracies vs. helicity supertraces

- We obtained agreement to all orders between the OSV prediction (at strong gravitational coupling) and the absolute degeneracy of DH states (at weak coupling). In general however, we expect that only a suitable index can be trusted in comparing weak and strong coupling results.
- The natural indexes to invoke are helicity supertraces:

$$\Omega_n = \mathrm{Tr}(-1)^F J_3^n$$

where F is the target space fermion number, and J_3 one generator of the little group of a massive particle in D=3+1. For low n, and large supersymmetry, this index receives only contributions from short multiplets, while long (non BPS) multiplets cancel out.
- For $N = 4$ SUSY, the natural index for $1/2$ (resp. $1/4$) BPS states is Ω_4 (resp. Ω_6). In heterotic orbifold constructions, Ω_4 is in fact equal to the absolute degeneracy of 1/2-BPS states, "explaining" agreement.
- For $N = 2$ SUSY, the natural index is $\Omega_2 \sim N_V - N_H$. As we shall see, in heterotic orbifolds this can be much smaller than the absolute degeneracy !

A few words on $N = 2$ models

- A number of type II/CY - Het/$K3 \times T^2$ dual pairs are known, where OSV can be tested. While $F_{h>1}$ are now $\neq 0$, the degeneracies of small BH predicted by OSV, to all orders in $1/p^1 q_0$, at small p^1/q_0 are universally given by

$$\Omega_{OSV} = \hat{I}_{(n_V+2)/2}(4\pi\sqrt{Q^2/2})$$

- For heterotic asymmetric orbifolds with $N = 2$ supersymmetry, the DH states can be counted as before. In contrast to $N = 4$, in the untwisted sector DH states typically come in vector/hyper pairs, and the helicity supertrace Ω_2 is exponentially smaller than the OSV prediction. The absolute degeneracies agree with Ω_{OSV} at leading order only.
- In contrast, twisted states are all hypers, and have $\Omega_{abs} = \Omega_2$ in agreement to Ω_{OSV} to all orders in 1/Q.
- In a class of models such as Het/K3 with standard embedding, untwisted and twisted states cannot be distinguished, hence OSV gives the correct result to all orders.
- In other models such as FHSV, untwisted and twisted states can be distinguished by the modding of their charges, and OSV appears to fail in reproducing either Ω_{abs} or Ω_2, unless some coarse-graining is made.

An $N = 2$ example: the FHSV model

- Consider a Z_2 orbifold of type II/$K_3 \times T^2$, by an Enriques involution of K_3 times a shift of T^2. This is dual to a Z_2 orbifold of Het/T^6 by a reversal of T^4 times an exchange of the two E_8.
- The electric charges untwisted/twisted states take value in the lattices

$$M_0 = E_8(-1/2) \oplus II^{2,2} \, , \quad M_1 = E_8(-1/2) \oplus (II^{2,2} + \delta)$$

Define M_0' the sublattice of vectors $2P_1 \oplus P_2$ in M_0.

- Absolute degeneracies go as

$$\Omega_{abs}(Q) = \begin{cases} \hat{I}_\nu(4\pi\sqrt{\frac{1}{2}Q^2}) + O(e^{\pi\sqrt{Q^2/2}}) & Q \in M_0' \\ 0 & Q \in M_0 - M_0' \\ \hat{I}_7(4\pi\sqrt{\frac{1}{2}Q^2}) & Q \in M_1 \end{cases}$$

$\nu = 13$ for generic moduli, but can vary.

An $N = 2$ example: the FHSV model (cont)

- Helicity supertraces are counted by

$$Z_u = \frac{2^6}{\eta^6 \vartheta_2^6}\,, \qquad Z_t^{\pm} = \frac{1}{2}\left(\frac{2^6}{\eta^6 \vartheta_4^6} \pm \frac{2^6}{\eta^6 \vartheta_3^6}\right)$$

- Using the Rademacher formula, they grow as

$$\Omega_2(Q) = \begin{cases} 2^{-8} e^{2\pi i Q\cdot\delta}(1 - e^{i\pi Q^2/2})\hat{I}_7(2\pi\sqrt{\tfrac{1}{2}Q^2}) + O(e^{\pi\sqrt{Q^2/2}}) & Q \in M_0' \\ 0 & Q \in M_0 - M_0' \\ -2^{-3}\hat{I}_7(4\pi\sqrt{\tfrac{1}{2}Q^2}) + 2^{-11} i e^{i\pi Q^2}\hat{I}_7(2\pi\sqrt{\tfrac{1}{2}Q^2}) + O(e^{\pi\sqrt{Q^2/2}}) & Q \in M_1 \end{cases}$$

- Compare to the OSV prediction ($\chi = 0$):

$$I_7\left(4\pi\sqrt{\frac{1}{2}Q^2}\right) \qquad \forall Q$$

Could the OSV formula have been exact ?

- Go back to the benchmark case: exact degeneracies can be extracted by a contour integral:

$$p_{24}(N) = \frac{1}{2\pi i}\oint q^{-N} dq/\Delta(q) = \int dt\ t^{-14}\,\frac{\exp\left(\frac{\pi(N-1)}{t}\right)}{\Delta\left(e^{-4\pi t}\right)}$$

- By contrast, the OSV formula can be rewritten as

$$\Omega_{OSV}(p^1, q_0) \sim \int d\tau_1\, d\tau_2\, \tau_2^{-14}\frac{\exp\left(\frac{\pi(N-1)}{\tau_2}\right)}{|\Delta\left(e^{-2\pi\tau_2+2\pi i\tau_1}\right)|^2}$$

- The two agree asymptotically when $\Delta(q) \sim q$, but the OSV formula does not appear to make sense non-perturbatively ! Furthermore the fact that Δ appears on both sides is a peculiarity of this model !

An $N=4$ exception to OSV

- Let us consider Type $IIA/K_3 \times T^2$ at the Z_2 orbifold point, and perform a further orbifold by the “quantum symmetry” acting as -1 on each twisted sector, combined with a shift along T^2: this gives a type II $N=4$ model with 6+6 gauge fields.
- The heterotic dual is unclear; however, another dual description can be obtained by making a Z_2 orbifold of type II/$T^4 \times T^2$ by $(-1)^{F_L}$ times a shift on T^2 This projects out all RR fields, leaving 6+6 vectors. In constrast to the previous (2,2) case, SUSY is realized as (4,0) on the worldsheet.

Vafa Sen

- The amplitude F_1 can be computed at one-loop on the (2,2) case: one finds $F_1 \sim \log \theta_4(T)$, which has no perturbative part but only instantons: thus small black holes remain small, even with R^2 corrections !

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- Just as in the heterotic case, the (4,0) model admits a spectrum of DH states, enumerated by θ_i^4/η^{12}. The microscopic degeneracies thus grow as $\hat{I}_5(2\pi\sqrt{2p^1 q_0})$, not matched by OSV !

Reverse engineering

- Rather than extracting BH degeneracies from the topological amplitude, one may try to construct the BH partition function from our partial knowledge of exact degeneracies.
- In type II/$K3 \times T^2$, the lattices of electric charges are

$$\begin{aligned} \Lambda^{IIA}_{elec} &= D0(q_0) \oplus D2/T2(q_1) \oplus D2/\gamma_2(q_a) \oplus \ldots \\ \Lambda^{IIA}_{mag} &= D6/K3 \times T^2(p^0) \oplus D4/K3(p^1) \oplus D4/T_2 \times \gamma_2(p^a) \oplus \ldots \end{aligned}$$

Exact degeneracies are known for purely electric heterotic states , i.e. for vanishing $D2/T2$, $D4/T^2 \times \gamma^2$, $D6/K3 \times T^2$.

- Setting $p^0 = p^a = 0$, the BH partition function includes terms with $q^1 = 0$:

$$Z'_{BH} = \sum_{q^0, q^a \in II^{3,19}} p_{24} \left(1 + p^1 q_0 + \frac{1}{2} q_a C^{ab} q_b \right) e^{-\pi(q_0 \phi^0 + q_a \phi^a)}$$

Reverse engineering (cont.)

- Inserting the unity

$$1 = \sum_N \delta\left[N - 1 - \frac{1}{2} q_a C^{ab} q_b\right] = \sum_N \sum_{k^0=0}^{p^1-1} \frac{1}{p^1} e^{2\pi i k^0 (N-1-\frac{1}{2} q_a C^{ab} q_b)/p^1}$$

inside the sum, the sum over N reconstructs the Dedekind function

$$Z'_{BH} = \frac{1}{p^1} \sum_{k^0=0}^{p^1-1} \frac{e^{-2\pi i \tau q_a C^{ab} q_b - \pi \phi^a q_a}}{\Delta(\tau)}\ , \quad \tau = \frac{i\phi^0 + 2k^0}{2p^1}$$

Doing a modular transformation on τ and a Poisson resummation on q_a gives

$$Z'_{BH} = \sum_{k_0=0}^{p^1-1} \sum_{k^a \in II^{19,3}} Z_0(\phi^A + 2ik^A)\ , \quad Z_0(\phi^A) = \frac{\exp\left[-\frac{\pi}{2} \frac{p^1 C_{ab} \phi^a \phi^b}{\phi^0}\right]}{(p^1)^2\ \Delta\left(\frac{2ip_1}{\phi^0}\right)}$$

Reverse engineering (cont.)

- While Z_0 looks close to the topological string amplitude, it is in fact different: no $|\Delta|^2$, and the argument has no ϕ^1 dependence !
- The sum over translations $\phi^A \to \phi^A + 2ik^A$ guarantees that the BH partition function has the expected periodicity due to the charge quantization. Yet much of the information in the topological string amplitude could be lost in the process of averaging !
- It is tempting to conjecture that the exact black hole partition function is a theta series whose general term is the topological string amplitude.
- Indeed, in a unrelated development, non-Gaussian theta series have been constructed based on cubic characters $\exp(I_3(X^A)/X_0)$ quite similar to CY prepotentials. It would be very interesting if invariance under monodromies in the CY moduli space could be realized in the same fashion.

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Discussion

- The OSV conjecture for the partition function of BPS black holes has passed several non-trivial tests, leading to agreement with microscopic degeneracies to all orders in $1/Q^2$.
- For this to hold, a number of ambiguities had to be lifted: integration contour, holomorphic anomalies, identification of Ω_{OSV} with helicity supertraces, count states with arbitrary J.
- OSV is very successful in $N = 4$ models, less so in some $N = 2$ models. When $\chi \neq 0$, the saddle point lies at strong coupling of the pointlike instanton series, requiring a non-perturbative completion of the topological amplitude in this sector.
- At the non-perturbative level, a relation like "$Z_{BH} = |e^F|^2$" cannot hold, if only because the rhs is not periodic in ϕ modulo $2i$. This suggests that the BH partition function may instead be a theta series built on e^F, possibly with interesting automorphic properties.
- Combining a recent analysis of the relation between 4D and 5D BH, the Gopakumar-Vafa relation between entropy of 5D BH and F_{top} and the OSV formula in 4D, leads to an unlikely relation $|e^{F(1/g_s)}|^2 = e^{F(g_s)}$... Understanding multi-centered configurations is probably important in solving this puzzle.

Open problems

- In principle, one expects the infinite series of tree-level higher derivative corrections to become important in the singular geometry of Het DH states. The amazing agreement at leading order suggests some kind of non-renormalization theorem.
- There are also DH states in type IIA/$K_3 \times T^2$, with a large entropy $2\pi\sqrt{2nw}$. In contrast to the Het DH states, they are 1/4-BPS, have zero helicity supertrace, but do not seem to be resolved by R^2 corrections.
- Similarly, there are 1/4-BPS DH states in type II/T^6, with the same entropy. The leading higher derivative corrections are the famous $\zeta(3)R^4$, but those are unlikely to give the correct entropy !
- In a rather orthogonal approach, Sen was able to reproduce the BH entropy to all orders using a different ensemble, with a chemical potential μ for Q^2 rather than Q, and keeping non-holomorphic corrections. It would be interesting to relate the two approaches...
- An outstanding challenge is to understand subleading corrections to large black holes. A somewhat naive analysis of the elliptic genus almost gives the right Bessel function...