The Quantum Attractor Mechanism

Boris Pioline

Caltech, Nov 11, 2005

based on hep-th/0506228 and work in progress
with M. Gunaydin, A. Neitzke and A. Waldron

slides available from
`http://www.lpthe.jussieu.fr/~pioline/seminars.html`

Black hole thermodynamics and microscopic counting

- In general relativity, one associates to a macroscopic black hole with mass M, horizon area A and surface gravity κ an entropy $S_{BH} = A/4G_N$ and temperature $T = \kappa/2\pi$ such that the standard laws of thermodynamics are obeyed.

  Christodoulou, Bekenstein, Hawking

- String theory is famously known to provide a microscopic description of black hole microstates, reproducing the Bekenstein-Hawking entropy. This is especially successful for generic ("4-charge", no D6-brane) BPS black holes in N=2 4D SUGRA.

  Strominger Vafa; Maldacena Strominger; Maldacena Witten Strominger

- This agreement relies on the "thermodynamical" limit where $A \gg G_N$, or $Q \gg 1$, and classical gravity can be trusted. Can we test this beyond leading order ?
- General AdS/CFT arguments suggest that 4D black holes micro-states may be described by 0+1 conformal quantum mechanics. Can we find the dual description, and understand why the horizon area is a good measure of the number of states ?

Recent progress

- By better understanding the 5D M-theoretical origin of 4D black holes, it has been possible to compute the microscopic degeneracies of 4D black holes to high accuracy.

  Gaiotto Strominger Shih Yin

- An interesting relation has been proposed between black hole degeneracies and the topological string amplitude, which in principle allows for a detailed comparison of macroscopic Bekenstein-Hawking-Wald entropy and microscopic degeneracies.

  Ooguri Strominger Vafa; Cardoso de Wit Mohaupt

- The rationale behind this relation has been traced to channel duality in AdS_2 geometry: rather than computing the black hole spectrum by diagonalizing the (CFT1) Hamiltonian for time evolution, one may relate it to a wave function overlap in radial quantization. The topological amplitude is believed to be the “Hartle-Hawking” wave function for radial evolution.

  Ooguri Verlinde Vafa

Outline of the talk

- Our goal is to try and clarify these ideas, by considering situations with higher symmetry: $N = 8$ and $N = 4$ SUGRA, or "very special" $N = 2$ SUGRA. The complexity of CY geometry is jettisoned in favor of representation theory.
- In particular, we'll show how the 4D/5D lift generalizes to these models including all charges; we'll determine the wave functions which control the leading order degeneracies in these models; and most importantly, we'll study in depth the radial quantization of BPS black holes.
- Our main message is that, beyond the expected U-duality symmetry in 4 dimensions, under which black hole degeneracies ought to be invariant, there is a larger "spectrum generating" symmetry, namely the 3-dimensional U-duality group, which controls the black hole wave function, and probably the degeneracies themselves.
- Some important aspects have in fact appeared in some prescient works:

  Breitenlohner Gibbons Maison, Breitenlohner Hellmann

  Gutperle Spalinski, Gross Wallach, (Kazhdan) BP Waldron
- Warning: most of this is work in progress, and many loose ends remain to be tied up.

Black hole degeneracies and higher SUSY : $N = 4$

- In situations with high sypersymmetry $N \geq 4$, we may hope to use U-duality invariance to pin down black hole degeneracies: For $N = 4$, the U-duality group is

$$Sl(2, \mathbb{Z}) \times SO(6, n_v, \mathbb{Z})$$

  where n_v is the number of $N = 4$ vector multiplets: $n_v = 22$ for the simplest IIA$/K3 \times T^2$ - Het $/T^6$ model.
- Electric and magnetic charges transform like a doublet of $SO(6, n_v)$ vectors. The Bekenstein-Hawking entropy is given by

$$S_{BH} = 2\pi\sqrt{I_4}\,, \quad I_4 = \det \begin{pmatrix} \vec{p}^2 & \vec{p}\cdot\vec{q} \\ \vec{p}\cdot\vec{q} & \vec{q}^2 \end{pmatrix}$$

Counting $N = 4$ dyons

- Dijkgraav Verlinde Verlinde have made a conjecture for the 1/4-BPS black hole degeneracies in $n_v = 22$ model,

$$\sum_{p^I, q_I} \Omega(p^I, q_I) e^{i(\rho \vec{p}^2 + \sigma \vec{q}^2 + (2\nu - 1)\vec{p}\cdot\vec{q})} = \frac{1}{\Phi(\omega)}, \quad \omega = \begin{pmatrix} \rho & \nu \\ \nu & \sigma \end{pmatrix} \in \frac{Sp(4)}{U(4)}$$

  where Φ is the unique weight 10 cusp form of $Sp(4, \mathbb{Z})$. The S-duality group $Sl(2, \mathbb{Z})$ is realized as a subgroup of the “genus 2” modular group $Sp(4, \mathbb{Z})$.
- This conjecture is supported by the recent 4D/5D lift, using the elliptic genus of $Hilb(K3)$.
- Note however that p, q enter only via their inner products: they could exist more subtle invariants under T-duality.

Black hole degeneracies and higher SUSY : $N = 8$

- For $N = 8$, much less was known until recently. The U-duality group E_7 acts linearly on the 56 electric and magnetic charges, and the Bekenstein-Hawking entropy is

$$S = 2\pi \sqrt{I_4}$$

Kallosh Kol

where I_4 is the E_7 quartic invariant:

$$Q = \begin{pmatrix} D2^{ij} & [F1]^i & [kkm]^i \\ -[F1]^i & 0 & [D6] \\ -[kkm]^i & -[D6] & 0 \end{pmatrix} , \quad P = \begin{pmatrix} D4_{ij} & [NS]_i & [kk]_i \\ -[NS]_i & 0 & [D0] \\ -[kk]_i & -[D0] & 0 \end{pmatrix} ,$$

$$\begin{aligned} I_4(P,Q) &= -\mathrm{Tr}(QPQP) + \frac{1}{4}\left(\mathrm{Tr}QP\right)^2 - 4\left[\mathrm{Pf}(P) + \mathrm{Pf}(Q)\right] \\ &= 4p^0 I_3(q_A) - 4q_0 I_3(p^A) + 4\frac{\partial I_3(q_A)}{\partial q_A}\frac{\partial I_3(p^A)}{\partial p^A} - (p^0 q_0 + p^A q_A)^2 \end{aligned}$$

and I_3 is the cubic invariant of E6.

Counting $N = 8$ dyons

- By studying the elliptic genus of $Hilb(T^4)$, Maldecena Moore Strominger conjectured (and partially prove) that degeneracies of 5D BPS black holes in type II on T^5 were given by

$$\Omega_{5D}(N, Q_1, Q_5, \ell) = \sum_{s|(NQ_1, NQ_5, Q_1Q_5, \ell); s^2|NQ_1Q_5} s\ N(s)\ \hat{c}\left(\frac{NQ_1Q_5}{s^2}, \frac{\ell}{s}\right)$$

where $\hat{c}(n, l)$ are the Fourier coefficients of the Jacobi form

$$-\frac{\theta_1^2(z,\tau)}{\eta^6} := \sum \hat{c}(n,l) q^n y^l\ , \quad \hat{c}(n,l) = \hat{c}(4n - l^2)$$

and $N(s)$ is the number of divisors of $N, Q_1, Q_5, s, \frac{NQ_1}{s}, \frac{NQ_5}{s}, \frac{Q_1Q_5}{s}, \frac{NQ_1Q_5}{s^2}$

- By using the same 4D-5D lift, one may show that the exact number of micro-states is equal to

$$\Omega(p^I, q_I) = \hat{c}(I_4)$$

at least for black holes U-dual to a D0-D4-D6 bound state with $p^0 = 1$, and with all charges coprime. Again, there probably exist more subtle U-duality invariants than I_4.

Shih Strominger Yin; BP

Black hole degeneracies for $N = 2$

- For $N = 2$, the situation is much less understood. For vanishing D6-brane charge, Maldecena Strominger Witten have shown how black hole degeneracies could be extracted from the D=1+1 “black string” (0,4) conformal field theory describing an M5-brane wrapped on a 4-cycle in CY. Using the Cardy formula,

$$S_{micro} = 2\pi\sqrt{(D_{ABC}p^A p^B p^C + c_{2A}p^A/6)q_0}$$

- This agrees with the macroscopic Bekenstein-Hawking entropy, upon incorporating the leading R^2 correction.
- Unfortunately, this is a “singular” CFT, with non-compact target space, and it is hard (although maybe not impossible) to get subleading corrections.

Dabholkar Denef BP Moore

- There is no proper U-duality group here. However, we expect that the monodromy group of the CY puts severe constraints on the BH degeneracies.

Very special supergravities

- There is an interesting class of $N=2$ supergravities where the moduli space is a symmetric space. Although they still possess 8 SUSY, their extended symmetries facilitate the analysis greatly, and we shall see that some of them are related to $N=4$ and $N=8$ theories by analytic continuation.
- Their prepotential is purely cubic

$$F = N(X)/X^0 = C_{ABC} X^A X^B X^C / X^0$$

where $N(X)$ is the norm of a degree 3 Jordan algebra J. The moduli space is a symmetric space

$$M_4 = \frac{\text{Conf}(J)}{\text{Struc}^c(J) \times U(1)}$$

where $\text{Struc}^c(J)$ is the reduced structure group of J (in its compact form), while $\text{Conf}(J)$ is the conformal group leaving the cubic light-cone $N(X)=0$ invariant.

Gunaydin Sierra Townsend

Very special supergravities

- Depending on the choice of J, this leads to two generic families

$$\frac{SU(n,1)}{SU(n)\times U(1)}\,, \quad \frac{SO(n,2)}{SO(n)\times SO(2)}\times\frac{Sl(2)}{U(1)}$$

and a number of exceptional cases,

$$\frac{Sl(2)}{U(1)}\,, \frac{Sp(6)}{SU(3)\times U(1)}\,, \frac{SU(3,3)}{SU(3)\times SU(3)\times U(1)}\,, \frac{SO^*(12)}{SU(6)\times U(1)}\,, \frac{E_{7(-25)}}{E_6\times U(1)}$$

corresponding to $N = X^0 Q_2, X^1 Q_2, (X^1)^3, \det(3x3_s), \det(3x3), \mathrm{Pf}(6\wedge 6), I_3(27)$ respectively.

- Although these may not exist as consistent string theories, they arise in the untwisted sector of type II orbifolds, or in heterotic string at tree-level.

A remark on Legendre invariance

- An important property following from the axiom of Jordan algebras

$$X^{\sharp\sharp} = N(X) X \ , \quad X^\sharp_A := C_{ABC} X^B X^C$$

is that F is invariant under Legendre transform in all variables:

$$\langle N(X)/X^0 + X^0 Y_0 + X^A Y_A \rangle_{X^I} = -N(Y)/Y^0$$

Proof: saddle point at

$$Y_A = X^\sharp_A / X^0 \, , \quad Y_0 = -N(X)/(X^0)^2$$

hence

$$\begin{aligned} N(X) X^A &= (X^0 Y_A)^\sharp = (X^0)^2 (Y^A)^\sharp \Rightarrow X^A = -Y^\sharp_A / Y^0 \\ N(Y) Y_A &= (-X^A Y_0)^\sharp \Rightarrow X^0 = N(Y)/(Y_0)^2 \end{aligned}$$

- This will play an important role in the sequel. In particular, this is at the heart of the construction of the minimal representation of $QConf(J)$, the 3-dimensional group for these supergravities.

The OSV Conjecture

- Based on a re-interpretation of earlier results by Cardoso, De Wit and Mohaupt (CDM), Ooguri, Strominger and Vafa (OSV) have proposed simple relations between micro-canonical degeneracies $\Omega(p^A, q_A)$ and the topological string amplitude:

$$\sum_{q_A \in \Lambda_{el}} \Omega(p^A, q_A) e^{-\pi \phi^A q_A} \quad \sim \quad |\exp\left(\frac{i\pi}{2}\mathrm{Im}F(p^A + i\phi^A, 2^8)\right)|^2 \qquad (*)$$

$$\Omega(p^A, q_A) \quad \sim \quad \int d\phi^A \left|\exp\left(\frac{i\pi}{2}F(p^A + i\phi^A)\right)\right|^2 e^{\phi^A q_A} \qquad (**)$$

- The lhs of (*) can be viewed as a partition function $Z(p^A, \phi^A)$ of BPS black holes in a "mixed" thermodynamical ensemble at fixed magnetic charge p^A and fixed electric potential ϕ^A. More precisely, it should be a suitable "supersymmetric index", robust under deformations.
- The rhs of (*) encodes the tree-level SUGRA lagrangian together with an infinite series of "BPS-saturated" gravitational corrections $\sum_{h=0}^{\infty} F_h(X^A) R_+^2 F_+^{2h-2}$, computed by the topological string.

Berschadsky Cecotti Ooguri Vafa;Antoniadis Gava Narain Taylor

- Semi-classically, the integral in (**) (or the sum in *) is dominated by a saddle point $(X, \bar{X})$ such that
$$\mathrm{Re}(X^A) = p^A, \quad Re(F_A) = q_A\ , \quad W^2 = 2^8$$
These are the (generalized) attractor equations, which determine the values of the scalar fields at the horizon in terms of the electric and magnetic charges. At the saddle,
$$S_{BHW}(p^A, q_A) = \mathsf{L}egendre\left[\mathcal{F}\right]\ , \quad \mathcal{F} = \pi \mathsf{Im} F(p^A, \phi^A)\Big]$$
in accord with CDM. The Bekenstein-Hawking-Wald entropy is thus understood as the entropy in the mixed ensemble, and differs from the "true" micro-canonical entropy $\log \Omega(p^A, q_A)$ due to statistical corrections around the saddle point.
- The $\sim$ sign in (**) allegedly denotes an equality to all orders in an expansion at large charges $(\lambda p^A, \lambda q_A)$, $\lambda \to \infty$. A non-perturbative generalization might be obtained upon completing the perturbative topological amplitude and specifying a contour.
- Since $Z(p^A, \phi_A)$ is manifestly periodic under integer imaginary shifts of ϕ_A, a more plausible version of (*) is
$$\sum_{q_A \in \Lambda_{el}} \Omega(p^A, q_A) e^{-\pi \phi^A q_A} \sim \sum_{k^A \in \Lambda^*_{el}} |\exp\left(\frac{i\pi}{2} \mathsf{Im} F(p^A + i\phi^A + k^A)\right)|^2$$
which has a strong smell of theta series.

Checks on the OSV conjecture

- The proposal has been tested in the case of non-compact CY: $O(-m) \oplus O(m) \to T^2$: BPS states are counted by topologically twisted SYM on N D4-brane wrapped on a 4-cycle $O(-m) \to T^2$, which is equivalent to 2D Yang Mills. At large N, this "factorizes" into $\sum_l \Psi_{top}(t + mlg_s)\Psi_{top}(\bar{t} - mlg_s)$.

  Vafa

- This was generalized for $O(-m) \oplus O(2g - 2 + m) \to \Sigma_g$, whose topological amplitude is related to q-deformed 2D Yang-Mills. The agreement with OSV for genus $g > 1$ however requires modular properties of YM_q which are less than obvious.

  Aganagic Ooguri Saulina Vafa

- Exact degeneracies are known in a class of "small black holes", dual to perturbative heterotic states. The OSV formula works beautifully in all $N = 4$ models, with some important subtleties in $N = 2$ orbifold models.

  Dabholkar Denef Moore Pioline

- Using the previous formulae for 1/4-BPS black hole degeneracies in $N = 4$ and $1/8$-BPS in $N = 8$, the OSV formula is again warranted, with some "volume factor corrections".

  Shih Yin

BH entropy in very special SUGRA

- As an application, let us compute the tree-level entropy of a black hole with arbitrary charges in exceptional SUGRA. The free energy is

$$\mathcal{F} = \frac{\pi}{(p^0)^2 + (\phi^0)^2} \left\{ p^0 \left[\phi^A p^\sharp_A - I_3(\phi) \right] + \phi^0 \left[p^A \phi^\sharp_A - I_3(p) \right] \right\}$$

- In order to eliminate the quadratic term in ϕ^A, change variables to

$$x^A = \phi^A - \frac{\phi^0}{p^0} p^A \,, \quad x^0 = [(p^0)^2 + (\phi^0)^2]/p^0$$

and, so as to eliminate the square root in $q_0 \phi^0$, introduce an auxiliary variable t,

$$\mathcal{S} = \pi \langle -\frac{I_3(x)}{x^0} + \frac{p^\sharp_A + p^0 q_A}{p^0} x^A - \frac{t}{4} \left(\frac{x^0}{p^0} - 1 \right) - \frac{(2 I_3(p) + p^0 p^I q_I)^2}{t \, (p^0)^2} \rangle_{\{x^I, t\}}$$

BH entropy, 4D and 5D

- Using the Legendre invariance of $N(X)/X^0$, we find

$$\begin{aligned}
\mathcal{S} &= \pi \langle 4 \frac{I_3[p^\sharp_A + p^0 q_A]}{(p^0)^2 t} - \frac{[2 I_3(p) + p^0 p^I q_I]^2}{t\,(p^0)^2} - \frac{t}{4} \rangle_t \\
&= \frac{\pi}{p^0} \sqrt{4 I_3[p^\sharp_A + p^0 q_A] - [2 I_3(p) + p^0 p^I q_I]^2} \\
&= \pi \sqrt{4 p^0 I_3(q) - 4 q_0 I_3(p) + 4 q^A_\sharp p^\sharp_A - (p^0 q_0 + p^A q_A)^2}
\end{aligned}$$

- By Freudenthal's triple system construction, the quartic polynomial is recognized as the quartic invariant under the 4-dimensional U-duality group.
- The intermediate equation also has an interesting expression: it is $1/p^0$ times the entropy of a 5-dimensional black hole with electric charge and angular momentum

$$\begin{aligned}
Q_A &= p^0 q_A + C_{ABC} p^B p^C \\
2 J_L &= (p^0)^2 q_0 + p^0 p^A q_A + 2 I_3(p)
\end{aligned}$$

consistent with the 4D/5D lift, generalized to include all charges.

$N = 8$ and $N = 4$ topological amplitudes

- In particular, this holds in the very special $N = 2$ supergravity with $F = I_3(27)/X^0$, and leads to a $E_{7(-25)}$ invariant entropy formula. By analytic continuation, the same computation tells that the $E_{7(7)}$ invariant entropy of 1/8-BPS black holes in $N = 8$ can be obtained by pretending that the $N = 8$ topological amplitude is

$$\Psi_{N=8} = e^{i\frac{\pi}{2}I_3(27)/X^0}$$

  and describes all 56 electric-magnetic charges. $I_3(27)$ is now the cubic invariant of the $E_{6(6)}$ 5-dimensional U-duality group.
- Similarly, the $Sl(2) \times SO(6, n_v)$ invariant entropy of 1/4-BPS black holes in $N = 4$ with n_v multiplets can be obtained by analytic continuation from the very special $N = 2$ supergravity with $Sl(2) \times SO(2, n_v + 4)$ invariance, i.e. by pretending that the $N = 4$ topological amplitude is

$$\Psi_{N=4} = e^{i\frac{\pi}{2}X^1 X^a Q_{ab} X^b/X^0}$$

  where Q_{ab} is a signature $(5, n_v - 1)$ quadratic form. Note that $SO(5, n_v - 1)$ is the 5-dimensional U-duality group.

OSV formula and Wigner distribution

- As explained by Witten in relation with the holomorphic anomaly equations, the topological amplitude $\Psi(X) = e^{i\frac{\pi}{2}F}$ is best viewed as a quantum mechanical wave function, which transforms by Fourier transform under changes of polarization.
- The OSV relation (**) can be suggestively rewritten, upon setting $\phi^A = i\chi^A$ as

$$\Omega(p,q) \sim \int d\chi \, \Psi^*(p+\chi)\Psi(p-\chi)e^{iq\chi}$$

recognized as the Wigner distribution associated to the state $\Psi(p)$. Even more suggestively, defining

$$\Psi_{p,q}(\chi) = e^{iq\chi}\Psi(\chi - p) := V_{p,q}\Psi(\chi)$$

and assuming $\Psi(\chi) = \Psi(-\chi)$, it becomes an overlap

$$\Omega(p,q) \sim \langle \Psi_{p,q} | \Psi_{p,q} \rangle$$

OSV conjecture and channel duality

- This is reminiscent of open/closed duality on the cylinder,

$$\text{Tr}(-)^F e^{-\pi t H_{open}} = \langle B | e^{-\frac{\pi}{t} H_{closed}} | B \rangle$$

  where B is a closed string boundary state.
- Indeed, the near-horizon geometry $AdS_2 \times S^2$ has the topology of a cylinder, and $\Psi_{p,q}$ may be viewed as a quantum state for the radial evolution, while $\Omega(p, q)$ counts the number of states for time evolution. The respective Hamiltonians ought to vanish due to the Wheeler-DeWitt constraints of diffeomorphism invariance.

  Ooguri, Vafa, Verlinde
- The topological wave function Ψ is naturally interpreted as a wave function for the attractor flow which controls the radial evolution of the moduli and geometry. One of the goals in this talk is to try and make this idea more precise.
- There are indications that degeneracies of multi-centered black holes may be obtained by second quantization of Ψ, in agreement with exponentially suppressed corrections originating from the mixing of two Fermi seas in 2D YM

  Dijkgraaf Gopakumar Ooguri Vafa
- This may be generalized beyond mini-superspace.

  Gukov Saraikin Vafa

The attractor mechanism

- Consider a general ansatz for a static, spherically symmetric BH in type IIA/CY:

$$ds^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}\left(dr^2 + r^2 d\Omega_2^2\right) + ds_{CY}^2$$

The shape of the CY is parameterized by Kähler moduli $z^A(r)$ (vectors) and complex structure moduli (hypers). The latter decouple and can be taken to be constant.

- The SUSY variations of gravitino and gauginos imply the "attractor flow equations"

$$\begin{aligned} dU/d\tau &= -e^U\,|Z| \\ dz^i/d\tau &= -2e^U\,g^{i\bar{j}}\partial_{\bar{j}}|Z| \end{aligned}$$

where $\tau = 1/r$, Z is the central charge,

$$Z = e^{K/2}W\,, \quad W = q^I F_I - p_I X^I\,, \quad K = -\log\left[i(\bar{X}^I F_I - X^I \bar{F}_I)\right]$$

and F is the tree-level prepotential,

$$F(X^A) = -\frac{1}{6} C_{ABC} \frac{X^A X^B X^C}{X^0} + \sum_{\beta \in H_2(Y,Z)} N_{0,\beta}\, e^{2\pi i \beta_A X^A / X^0}$$

- At $\tau = +\infty$, the moduli z^i settle to the minimum of $|Z|$. Integrating the 1st equation leads to
$$e^{-U} \sim |Z|_* \tau = |Z|_* / r$$
so that the near-horizon geometry is $AdS_2 \times S^2$, with horizon area $A = |Z|_* \propto \sqrt{I_4}$.
- By fixing the homogeneous gauge so that $i(\bar{X}^I F_I - X^I \bar{F}_I) = e^{-2U}$ and W be real, the equations combine into the "large phase space" attractor equations,

$$Re\left(\frac{d}{d\tau} X^A\right) = p^A\,, \quad Re\left(\frac{d}{d\tau} F_A\right) = q_A$$

which can be integrated right away. At the horizon, the moduli are fixed by the homogeneous attractor equations

$$Re\left(X^A\right) = p^A\,, \quad Re\left(F_A\right) = q_A$$

The attractor flow, revisited

- To shed light on the attractor flow, return to the basics: stationary solutions in 4D can be parameterized in the form

$$ds_4^2 = -e^{2U}(dt+\omega)^2 + e^{-2U} ds_3^2\ , \quad A_4^I = \zeta^I(t)dt + A_3^I$$

  where U and A^I are scalars on the t-independent 3D slice ds_3, and (ω, A_3^I) are a one-forms. This amounts to a Kaluza-Klein reduction on a time-like direction.

- In 3D, the one-forms A^I and ω can be dualized into pseudo-scalars $\tilde{\zeta}_I$ and the “NUT potential” A. For a regular compactification on a space-like circle, the 4D Einstein-Maxwell equations reduces to a non-linear sigma-model with a Riemannian target-space M_3, coupled to 3D gravity.

- Importantly, M_3 always has $2n+1$ isometries corresponding to the gauge symmetries of A^I and ω. These satisfy a Heisenberg algebra,

$$[\mathcal{L}_{\partial_{\zeta^I}}, \mathcal{L}_{\partial_{\tilde{\zeta}_J}}] = 2\delta^I_J \mathcal{L}_{\partial_a}$$

  so that $\zeta, \tilde{\zeta}, a)$ enter only through $d\zeta^I\ , d\tilde{\zeta}_I\ , da + \zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I$.

KK reduction on time

- For compactification on a time-like direction, the 4D equations still reduce to a non-linear sigma model on M_3^* bu M_3^* now has indefinite signature: it can be obtained from M_3 by Wick rotating $(\zeta^I, \tilde{\zeta}_I) \to i(\zeta^I, \tilde{\zeta}_I)$ (but NOT the scalar a dual to ω). If $M_3 = G/K$ was a Riemannian symmetric space, $M_3 = G/K'$ is still symmetric but K' is no longer the maximal compact subgroup of G.

Breitenlohner Gibbons Maison; Hull Julia

- For generic $N = 2$ SUGRA, the reduction to 3D on a space-like circle leads to a non-linear sigma model on a quaternionic-Kahler manifold known as the "s-map" of M_4. The time-like reduction leads to analytic continuation of this model, known as "para-quaternionic-Kahler manifold"

Ferrara Sabharwal

Attractor flow in higher (super)symmetry

- For $N = 8$ SUGRA,

$$M_3 = E_{8(8)}/SO(16) \ , \quad M_3^* = E_{8(8)}/SO^*(16)$$

The 70 moduli in 4D split into 15 vectors and 10 hypers. Only vectors are attracted.

Andrianopoli, D'Auria, Ferrara

- For $N = 4$,

$$M_3 = SO(8, n_v{+}2)/SO(8){\times}SO(n_v{+}2) \ , \quad M_3^* = SO(8, n_v{+}2)/SO(6, 2){\times}SO(2, n_v)$$

Again, only vectors are attracted.

- For very special $N = 2$ SUGRA, M_3 is a symmetric quaternionic-Kahler manifold again obtained from Jordan algebra technology:

$$M_3 = \frac{\text{QConf}(J)}{\text{Conf}^c(J) \times SU(2)}$$

where QConf(J) is the "quasi-conformal group" leaving the quartic light-cone $I_4 - y^2 = 0$ invariant, and Conf$^c(J)$ is the compact form of Conf$^c(J)$.

Q	$D=5$	$D=4$	$D=3$	$D=3^*$
8		$\frac{SU(n,1)}{SU(n)\times U(1)}$	$\frac{SU(n+1,2)}{SU(n+1)\times SU(2)\times U(1)}$	$\frac{SU(n+1,2)}{SU(n,1)\times Sl(2)\times U(1)}$
8	$\mathbb{R}\times\frac{SO(n-1,1)}{SO(n-1)}$	$\frac{SO(n,2)}{SO(n)\times SO(2)}\times\frac{Sl(2)}{U(1)}$	$\frac{SO(n+2,4)}{SO(n+2)\times SO(4)}$	$\frac{SO(n+2,4)}{SO(n,2)\times SO(2,2)}$
8		$\frac{Sl(2)}{U(1)}$	$\frac{SU(2,1)}{SU(2)\times U(1)}$	$\frac{SU(2,1)}{Sl(2)\times U(1)}$
8	$\varnothing$	$\frac{Sl(2)}{U(1)}$	$\frac{G_{2(2)}}{SO(4)}$	$\frac{G_{2(2)}}{SO(2,2)}$
8	$\frac{Sl(3)}{SO(3)}$	$\frac{Sp(6)}{SU(3)\times U(1)}$	$\frac{F_{4(4)}}{USp(6)\times SU(2)}$	$\frac{F_{4(4)}}{Sp(6)\times Sl(2)}$
8	$\frac{Sl(3,C)}{SU(3)}$	$\frac{SU(3,3)}{SU(3)\times SU(3)\times U(1)}$	$\frac{E_{6(+2)}}{SU(6)\times SU(2)}$	$\frac{E_{6(+2)}}{SU(3,3)\times Sl(2)}$
24	$\frac{SU^*(6)}{USp(6)}$	$\frac{SO^*(12)}{SU(6)\times U(1)}$	$\frac{E_{7(-5)}}{SO(12)\times SU(2)}$	$\frac{E_{7(-5)}}{SO^*(12)\times Sl(2)}$
8	$\frac{E_{6(-26)}}{F_4}$	$\frac{E_{7(-25)}}{E_6\times U(1)}$	$\frac{E_{8(-24)}}{E_7\times SU(2)}$	$\frac{E_{8(-24)}}{E_{7(-25)}\times Sl(2)}$
10			$\frac{Sp(2n,4)}{Sp(2n)\times Sp(4)}$	
12			$\frac{SU(n,4)}{SU(n)\times SU(4)}$	
16	$\mathbb{R}\times\frac{SO(n-5,5)}{SO(n-5)\times SO(5)}$	$\frac{Sl(2)}{U(1)}\times\frac{SO(n-4,6)}{SO(n-4)\times SO(6)}$	$\frac{SO(n-2,8)}{SO(n-2)\times SO(8)}$	$\frac{SO(n-2,8)}{SO(n-4,2)\times SO(2,6)}$
18			$\frac{F_{4(-20)}}{SO(9)}$	
20		$\frac{SU(5,1)}{SU(5)\times U(1)}$	$\frac{E_{6(-14)}}{SO(10)\times SO(2)}$	
32	$\frac{E_{6(6)}}{USp(8)}$	$\frac{E_{7(7)}}{SU(8)}$	$\frac{E_{8(8)}}{SO(16)}$	$\frac{E_{8(8)}}{SO^*(16)}$

Attractor flow and geodesic motion

- Now, restrict to spherically symmetric solutions:

$$ds_3^2 = N^2(\rho)d\rho^2 + r^2(\rho)d\Omega_2^2$$

The sigma-model action becomes, up to a total derivative (g_{ij} is the metric on M_3^*):

$$S = \int d\rho \left[N + \frac{1}{N}\left(\dot{r}^2 - r^2 g_{ij}\dot{\phi}^i\dot{\phi}^j\right)\right]$$

- The Lagrange multiplier N imposes the Wheeler-DeWitt Hamiltonian constraint

$$H = (p_r)^2 - \frac{1}{r^2}g^{ij}p_i p_j = N^2$$

which can be set to $N = 1$ by a gauge choice. Solutions are thus massive geodesics on the cone $\mathbb{R}^+ \times M_3^*$. This separate into geodesic motion on M_3^*, times radial motion.

Attractor flow and geodesic motion

- The isometries translate into conserved Noether charges. In particular, the Noether charges corresponding to the Heisenberg algebra are the electric and magnetic charges, together with the NUT charge k:

  $$[p^I, q_J] = 2\delta^I_J k$$

  This is in fact a general feature of flux compactifications.

  Moore Freed

- If $k \neq 0$, regularity at the horizon requires that the time coordinate t is compact. Genuine asymptotically flat black holes in 4 dimensions are obtained only if $k = 0$.
- $k \to 0$ is a kind of classical limit. With hindsight, it suggests that the Wigner function will appear naturally when discussing 4D black holes.

Geodesic flow on special quaternionic Kahler manifolds

- Supersymmetry implies that the 3D slices have to be flat. Hence $r(\rho) = \rho$, and the geodesic motion on M_3^* has to be massless. In general, there are additional conditions, depending on the number of supersymmetries to be preserved.
- In particular, we can reproduce the attractor flow equations of BPS black holes in $N = 2$ SUGRA by studying geodesic flow on the on the (analytically continued) s-map from the special Kahler manifold M_4: ($\sigma = -2U$)

$$
\begin{aligned}
ds^2 =& \frac{1}{2}(d\sigma)^2 + g_{i\bar{j}}(z,\bar{z}) dz^i dz^{\bar{j}} + \frac{1}{2} e^{2\sigma} \left(da + \zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I \right)^2 \\
& - e^{\sigma} \left[(\mathrm{Im}\mathcal{N})_{IJ} d\zeta^I d\zeta^J + (\mathrm{Im}\mathcal{N}^{-1})^{IJ} \left(d\tilde{\zeta}_I + (\mathrm{Re}\mathcal{N})_{IK} d\zeta^K \right) \left(d\tilde{\zeta}_J + (\mathrm{Re}\mathcal{N})_{JL} d\zeta^L \right) \right.
\end{aligned}
$$

$$
\begin{aligned}
q_I &= -2e^{\sigma} \left[(\mathrm{Im}\mathcal{N})_{IJ} \partial\zeta^J + (\mathrm{Re}\mathcal{N})_{IJ} (Im\mathcal{N}^{-1})^{JL} \left(d\tilde{\zeta}_L + (\mathrm{Re}\mathcal{N})_{LM} d\zeta^M \right) \right] + 2\, k \tilde{\zeta}_I \\
p^I &= -2e^{\sigma} (\mathrm{Im}\mathcal{N}^{-1})^{IL} \left(d\tilde{\zeta}_L + (\mathrm{Re}\mathcal{N})_{LM} d\zeta^M \right) - 2\, k \zeta^I \\
k &= e^{2\sigma} \left(da + \zeta^I d\tilde{\zeta}_I - \tilde{\zeta}^I d\zeta_I \right)
\end{aligned}
$$

Quaternionic viel-bein

- The quaternionic geometry can be exposed by defining a $SU(2) \times Sp(n_v)$ quaternionic vielbein, i.e. a $2 \times n_v$ pseudo-real matrix

$$V^{\alpha\Gamma} = \begin{pmatrix} u & v \\ e^A & E^A \\ -\bar{v} & \bar{u} \\ -\bar{E}^A & \bar{e}^A \end{pmatrix} = \left[\epsilon_{\alpha\beta}\rho_{\Gamma\Gamma'}V^{\beta\Gamma'}\right]^*$$

so that the three Kahler forms and metric are

$$\Omega^i = \epsilon_{\alpha\beta}\,(\sigma^i)^\beta_\gamma\rho_{\Gamma\Gamma'}V^{\alpha\Gamma}\wedge V^{\gamma\Gamma'}\,, \quad ds^2 = \epsilon_{\alpha\beta}\rho_{\Gamma\Gamma'}V^{\alpha\Gamma}\otimes V^{\beta\Gamma'}$$

In terms of the conserved charges, the one-forms entering V are

$$u = -\frac{i}{2}e^{(K-\sigma)/2}X^I\left[q_I - 2k\tilde{\zeta}_I - \mathcal{N}_{IJ}(p^J + 2k\zeta^J)\right]\,, \quad v = \frac{1}{2}d\sigma + \frac{i}{2}e^{-\sigma}k$$

$$e^A = e^A_i\,dz^i\,, \quad E^A = -\frac{i}{2}e^{-\sigma/2}e^{Ai}g^{i\bar{j}}\bar{f}^I_{\bar{j}}\left[q_I - 2k\tilde{\zeta}_I - \mathcal{N}_{IJ}(p^J + 2k\zeta^J)\right]$$

SUSY Geodesic flow and attractor equations

- The supersymmetry constraints are given by requiring that the SUSY variation

$$\delta\chi^{\Gamma} = V_{\mu}^{\alpha\Gamma}\sigma_{\alpha}^{\mu\beta}\epsilon_{\beta} = V^{\alpha\Gamma}\tilde{\epsilon}_{\alpha}$$

vanishes. Equivalently, the matrix V has a zero eigenvector, which can be taken to be $(1, \lambda)$:

$$\begin{aligned}
\frac{1}{2}d\sigma + \frac{i}{2}e^{-\sigma}k &= -\frac{i}{2}\lambda e^{(K-\sigma)/2}X^{I}\left(q_I - k\tilde{\zeta}_I - \mathcal{N}_{IJ}(p^J + k\zeta^J)\right) \\
dz^i &= -\frac{i}{2}\lambda e^{-\sigma/2}g^{i\bar{j}}\bar{f}_{\bar{j}}^{I}\left(q_I - k\tilde{\zeta}_I - \mathcal{N}_{IJ}(p^J + k\zeta^J)\right)
\end{aligned}$$

Generalized attractor equations with NUT charge

- Using standard special geometry formulae this can be rewritten as

$$\begin{aligned} \frac{1}{2}d\sigma + \frac{i}{2}e^{-\sigma}k &= -\frac{i}{2}\lambda e^{-\sigma/2}Z \\ dz^i &= -i\lambda\frac{|Z|}{Z}e^{-\sigma/2}g^{i\bar{j}}\partial_{\bar{j}}|Z| \end{aligned}$$

where

$$Z = e^{K/2}\left[(q_I - 2k\tilde{\zeta}_I)X^I - (p^I + 2k\zeta^I)F_I\right]$$

For $k=0$, and choosing λ so that σ be real, this reproduces the standard attractor flow equation.

Black holes and D-instantons

- This coincidence was in fact first observed by Gutperle and Spalinski in their study of D-instanton solutions in $N = 2$ SUGRA in 5 dimensions: p^I and q_I are M2-brane instanton charge, while k is the M5-brane instanton charge. In fact, such instantons are T-dual to the stationary black holes of interest to us, and this coincidence is a reflection of mirror symmetry.
- This in fact suggests how to incorporate higher-derivative corrections: by mirror symmetry, the $F_h R^2 F^{2h-2}$ corrections in 4D are mapped to

$$\sum_{h=1}^{\infty} \tilde{F}_h \partial^2 S \partial^2 S (\partial C)^{2h-2}$$

which depend on the hypers only. Their reduction to 3D gives rise to higher derivative corrections to the particle action.

Antoniadis Gava Narain Taylor

The universal $SU(2,1)$ sector

- It is instructive to investigate the "universal sector", which encodes the scale U, the graviphoton electric and magnetic charges, and the NUT charge k (this amounts to truncating all moduli away). The Hamiltonian is

$$H = \frac{1}{8}(p_U)^2 - \frac{1}{4}e^{2U}\left[(p_{\tilde{\zeta}} - k\zeta)^2 + (p_\zeta + k\tilde{\zeta})^2\right] + \frac{1}{2}e^{4U}k^2$$

  Gauge conditions are $U = \zeta = \tilde{\zeta} = a = 0$ at $\tau = 0$.

- The motion in the $(\tilde{\zeta}, \zeta)$ plane is that of a charged particle in a constant magnetic field. The electric, magnetic charges are the generators of translations; together with the angular momentum

$$p = p_{\tilde{\zeta}} + \zeta k \; , \quad q = p_\zeta - \tilde{\zeta} k \; , \quad J = \zeta p_{\tilde{\zeta}} - \tilde{\zeta} p_\zeta$$

  they satisfy the usual magnetic algebra

$$[p,q] = k \; , [J,p] = q \; , [J,q] = -p$$

- The motion in the U direction is governed effectively by

$$H = \frac{1}{8}(p_U)^2 + \frac{1}{2}e^{4U}k^2 - \frac{1}{4}e^{2U}\left[p^2 + q^2 - 4kJ\right]$$

- At spatial infinity, p_U becomes equal to the ADM mass, and J vanishes; hence the BPS mass relation

$$M^2 + k^2 = p^2 + q^2$$

- At the horizon $U \to -\infty$, $\tau \to \infty$, the last term is irrelevant and one recovers $AdS2 \times S_2$ geometry (with compact time if $k \neq 0$) with area

$$A = p^2 + q^2 = \sqrt{(p^2 + q^2)^2}$$

Spectrum generating symmetry

- Since the space is symmetric, there is in fact an whole $su(2,1)$ matrix Q of conserved charges,

$$Q = \begin{pmatrix} -m & -(p-iq) & 2ik \\ -(p+iq) & 0 & p+iq \\ -2ik & p-iq & m \end{pmatrix}$$

with

$$H = \mathrm{Tr}(Q^2) \ , \quad \det(Q) = 0$$

The last condition can be checked explicitely, and is necessary in order for the motion not to be over-determined. Note in particular that the ADM mass does NOT commute with p,q,k:

$$[m,p] = p \ \ , [m,q] = q \ , \ [m,k] = 2k \ .$$

SUSY geodesic motion on SU(2,1)

- SUSY is equivalent to $H = 0$ in this simple case:

$$H = \frac{1}{2}\left|p_U + ike^{2U}\right|^2 - \frac{1}{4}e^{2U}\left|p + iq|\right|^2$$

From the Cayley-Hamilton theorem for 3x3 matrices

$$Q^3 - (\text{Tr}Q)Q^2 - \frac{1}{2}\left[\text{Tr}Q^2 - (\text{Tr}Q)^2\right]Q - \det Q = 0$$

we conclude that $Q^3 = 0$: Q is in the maximal nilpotent coadjoint orbit of $SU(2,1)$. This is the classical supersymmetric phase space for the generalized attractor flow in this case ! Its dimension is 6, compared to 8 for the general non SUSY motion. Its symplectic form may be obtained from the general theory of coadjoint orbits.

Geodesic motion in $N = 8$

- For $N = 8$, the SUSY variation is

$$\delta\lambda_A = \epsilon_I \Gamma^I_{A\dot{A}} P^{\dot{A}}$$

where ϵ_I is a vector of the R-symmetry group in 3 dimensions $SO^*(16)$, $P^{\dot{A}}$ is a 128 spinor of $SO^*(16)$ corresponding to the tangent space to $E_{8(8)}/SO^*(16)$, and λ_A is a conjugate spinor.

- This can be interpreted as a Dirac equation in 16 dimensions, where ϵ_I is the momentum, hence ϵ_I should be light-like. In order to have an ϵ_I such that (*) vanishes, $P^{\dot{A}}$ should be a special spinor. For example, 1/2-SUSY trajectories correspond to pure spinors of $SO(16)$, of dimension 58. This is the dimension of the minimal nilpotent orbit of $E_{8(8)}$.

Geodesic motion in $N = 4$

- For $N = 4$, the SUSY variation is

$$\delta\lambda^a_A = \epsilon_I \Gamma^I_{A\dot{A}} V^{\dot{A},a}$$

where ϵ_I is a vector R-symmetry group $SO(6,2)$, and $V^{\cdot A,a}$ ($a = 1 \ldots n_v$), corresponding to the tangent space of $SO(8,n_v)/SO(6,2) \times SO(2,n_v-2)$, is a collection of n_v spinors of $SO(8)$. Solutions can be obtained by requiring that $V^{\dot{A},a} = \lambda^{\dot{A}} v^a$. 1/2 SUSY trajectories correspond to pure spinors of $SO(8)$, hence the dimension is $n_v + 5$. This is the dimension of the minimal nilpotent orbit of $SO(8,24)$.

- The coincidence between the dimensions of the $K(\mathbb{C})$ orbits of elements in the tangent space P ($g = t + p$) and the dimensions of the orbits in $G(\mathbb{R})$ is a general consequence of the Kostant-Sekiguchi correspondence:

$$P \text{ : velocities } \leftrightarrow Q \text{ : Noether charges}$$

From classical to quantum mechanics

- A standard way to quantize geodesic motion of a particle on M_3^* is to replace the classical trajectories by a harmonic functions on M_3^*:

$$\left[-\frac{\partial^2}{\partial r^2} - \frac{\Delta}{r^2} \right] \Psi(r, U, z, \bar{z}, \zeta^I, \tilde{\zeta}_I, a) = 0$$

  where Δ is the Laplace-Beltrami operator on M_3^*.
- One may attempt to solve this problem by finding a maximal commuting set of observables. In particular, one may choose to diagonalize all p^I and q_J in the subspace $k = 0$, or all p^I and k. In addition, if M_3^* is an homogeneous space, there are additional conserved quantities, including higher order Casimirs of G.
- As a matter of fact, we have to deal with the geodesic motion of a superparticle, since it comes by reduction from SUGRA in 4D. The wave functions are therefore harmonic spinors, or equivalently, harmonic differential forms on M_3^*.

The BPS Hilbert space

- Our interest is in further restricting to supersymmetric geodesic motion of a superparticle on M_3^*. We should therefore impose further conditions, e.g. in $N = 2$

$$\exists \epsilon / \ \epsilon^\alpha \frac{\partial}{\partial X^A_\alpha} \Psi = 0$$

- At fixed (projective) ϵ, this implies that the function does not depend on half of the coordinates X^A. Ψ should be a holomorphic function with respect to the complex structure determined by ϵ^α. Better to say, Ψ should be a holomorphic function (or an element of the sheaf cohomology group $H_l(T, O(-k))$ for some l, k) on the twistor space T over the quaternionic-Kahler space M_3. This can be viewed as a higher dimensional, quaternionic version of the Penrose - Atiyah Hitchin Singer twistor tranform in 4 dimensions.
- More generally, it may be fruitful to consider the hyperkahler cone HKC over a quaternionic-Kahler manifold, by including the cone direction r and an extra conjugate variable together with the twistor directions. The minimal representation of G, relevant for BPS states with 16 supercharges, should then consist of tri-holomorphic functions on HKC.

Quantum SUSY motion on a symmetric space

- In the case where M_3^* is a symmetric space G/H, the Hilbert space H may be decomposed into unitary representation ρ_i of G. Furthermore their should exist a map between vectors of each representation and the unconstrained Hilbert space $L^2(G/K)$.
- This can be achieved if the representation admits a (preferably unique) vector f_H invariant under H. Then

$$\Psi(g) = \langle f_H, \rho(g) v \rangle$$

  is H-invariant for any choice of v. If such a "spherical vector" does not exist, any other finite-dim irrep of H will do, and give a section of some non-trivial bundle over G/H rather than a function.
- Supersymmetric geodesic motion should correspond to unitary representations of unusually small functional dimension. Let us see how to construct those.

Co-adjoint orbits as phase spaces

- Recall that the Noether charges take values in the dual of the Lie algebra g^*. This is foliated into orbits of the action of G. Each orbit is a symmetric space

$$\mathcal{O}_J = \{g^{-1} J g, g \in G\} = G/Stab(J)$$

  where Stab(J) is the stabilizer of J.
- Each orbit carries a natural G-invariant symplectic form, known as the Kirillov-Kostant symplectic form:

$$\omega(X, Y) = \mathrm{Tr}([X, Y] J)$$

  on the tangent space around at J. This is evidently non-degenerate (its kernel is given by the commutant of J, which is orthogonal to O_J) Globally,

$$\omega = d\theta \ , \quad \theta = \mathrm{Tr}(g^{-1} dg \ J)$$

  where g is a gauge-fixed element in $G/Stab$.
- Generic orbits correspond to orbits of semi-simple (=diagonalizable) elements, whose stabilizer is $U(1)^r$, where r is the rank. Their dimension is $\dim G -$ rank$G$ (an even number).

Nilpotent orbits as small phase spaces

- However, when J has a non-trivial nilpotent part (i.e. non diagonal Jordan form), the stabilizer is typically larger (and non semi-simple), hence the orbit is smaller. The smallest orbit is that of a root. More generally, nilpotent orbits are classified by homomorphisms of $Sl(2)$ into G.
- As an example, the minimal nilpotent orbit of $SU(2,1)$ has dimension 4. The maximal nilpotent orbit has dimension 6:

$$J = \begin{pmatrix} & & \\ 1 & & \\ & 1 & \end{pmatrix} \ , \quad Stab = \begin{pmatrix} x_0 & & \\ x_1 & x_0 & \\ x_2 & x_1 & x_0 \end{pmatrix}$$

The orbit method

- Since the action of G on $\mathcal{O}_J$ preserves the symplectic form, its action on functions on $\mathcal{O}_J$ may be expressed in terms of Poisson brackets. The moment maps form an element Q of the dual of the Lie algebra, in the orbit of J itself.
- The general "orbit method philosophy" indicates that (most of the) unitary representations of G may be obtained by quantizing the Hamiltonian action of G on such a phase space.
- For example, the representation of G on $L^2(G/K)$ at fixed values of the Casimirs (assuming that G is split and K is its maximal compact subgroup) is associated to the orbit of a generic semi-simple element:

$$\dim(G/\text{Stab}) = \dim G - \text{rank} G \, , \qquad \dim(G/K) = (\dim G + \text{rank} G)/2$$

  This is the quantum mechanics obtained by quantizing geodesic motion on G/K, at fixed values of the Casimirs !
- Similarly, nilpotent orbits are associated to "small representations" of G, which describe the Hilbert space of supersymmetric geodesic motion on G/K !

Quaternionic discrete series and very special SUGRA

- For very special supergravities, the moduli-space in 3 dimensions $M_3 = G/(M \times SU(2)$ is a symmetric quaternionic-Kahler space of dimension $4n_v$, where G is in its rank 4 "quaternionic" real form.
- Gross and Wallach have constructed unitary representations π_k of G by considering the sheaf cohomology group $H^1(T, O(-k))$ on the twistor space T over the quaternionic-Kahler space M_3. For $k \geq 2n_v + 1$, this representation is irreducible, lies in the discrete series and has functional dimension $2n_v + 1$: this can be viewed as the space of functions of p, q, k.
- For lower values of k, the representation becomes decomposable. It admits a unitarizable submodule π'_k of smaller functional dimension:

k	$\dim$	(p, q, k)	$Interpretation$
$\geq 2n_v + 1$	$2n_v + 1$	$I_4 \neq 0$	4charges
$n_v - 1$	$2n_v$	$I_4 = 0$	3charges
$(2n_v - 2)/3$	$(5n_v - 2)/3$	$\partial I_4(p, q) = 0$	2charges
$(n_v + 2)/3$	$n_v + 2$	$\partial\partial I_4(p, q) = 0$	1charges

These are relevant for the 4-, 3-, 2- and 1-charge black holes, respectively.

Quaternionic discrete series and N=4,8 SUGRA

- By analytic continuation of the $G = E_{8(-24)}$ case to $G = E_{8(8)}$, we expect those to be relevant to 1/8, 1/8 with zero entropy, 1/4, and 1/2 BPS black holes, respectively.
- The minimal representation has been constructed independently, and its spherical vector is known. Amazingly ($y^2 = k$),

$$M_{ADM} = p_y^2 + \frac{I_4(p,q)}{y^2} + y^2$$

is the Hamiltonian of conformal quantum mechanics, and

$$\lim_{\beta\to\infty} e^{\beta H_\omega} f_H = e^{iI_3(\chi^A)/\chi^0}$$

reproduces the tree-level topological amplitude !

Kazhdan Pioline Waldron, Gunaydin Koepsell Nicolai

Physical interpretation of the wave function

- As usual in diffeomorphism invariant theories (e.g. quantum cosmology), the wave function is independent of the "time" variable ρ, and observables need to be defined by correlating variables.
- It is natural to use e^U as the natural radial coordinate, since it goes from 0 at the horizon to ∞ at spatial infinity. One could also use the black hole area $A = e^{2U}/r^2$, although its classically its range depends on the charges. We expect the wave function to be peaked towards the attractor values of the moduli and the horizon area as $U \to -\infty$.
- The natural inner product is obtained by using the Klein-Gordon inner product (also known as Wronskian, or $U(1)$ charge) at fixed values of U. E.g, the mean value of the horizon area should be roughly

$$A \sim e^{2U} \int \frac{dr}{r^2} dz d\bar{z} \Psi^*_{p,q} \overleftrightarrow{\partial_U} \Psi_{p,q}$$

  If the wave function factorizes into $\theta(r) \times \Psi'(U, z, \bar{z})$, it may be given by the square norm of the state (in some normalization), as proposed by OVV.
- Unfortunately, this product is famously known to be not positive definite. A possible way out is "third quantization", where the wave function Ψ becomes itself an operator... this may describe the possible black hole fragmentation near the horizon...

Back to OSV

- We have started to clarify the meaning of the black hole wave function in radial quantization. OSV proposed that the degeneracies would be given by

$$\Omega(p,q) = \langle \Psi_{top} | V^{\dagger}_{p,q} \, V_{p,q} | \Psi_{top} \rangle$$

  According to OVV, this should be a consequence of channel duality in AdS_2 and of the fact that Ψ_{top} is a solution of the Wheeler-DeWitt equation. However, there is as yet compelling reason to choose Ψ_{top} over any other solution of the WdW equation !

- On the other hand, at least for cases with a non-trivial U-duality group in 3 dimensions, such as $N=8$, $N=4$ or very special $N=2$ SUGRA, there is a special wave function, known as the spherical vector f_H, invariant under the maximal compact subgroup H. (this may turn out to be the same as the topological amplitude)

An automorphic OSV-type formula

- Furthermore, although we have not expanded on this, there also exists a special vector (or rather, distribution) invariant under the U-duality group $f_{G(\mathbb{Z})}$: this is in fact the product of all spherical vectors for the associated representations over the p-adic fields (p prime)!
- Given the asymmetry between the horizon and infinity, the following modification of OSV is rather attractive:

$$\Omega(p, q, k) = \langle f_{G(\mathbb{Z})} | V_{p,q}^{\dagger} \, V_{p,q} | f_H \rangle$$

Said otherwise, the natural wave function at infinity (resp. at the horizon) is the real (resp. adelic) spherical vector ! This puts us firmly into the realm of automorphic forms, the natural habitat of black hole degeneracies. Indeed,

$$\theta_G(g) = \langle f_{G(\mathbb{Z})} | \rho(g) | f_H \rangle$$

is the general way of constructing an automorphic form, i.e. a function of $G(\mathbb{Z})\backslash G/H$!

Kazhdan BP Waldron

Very special black holes and Nahm equations

- We have seen that the black hole radial evolution is equivalent to geodesic motion on (the HKC over) a quaternionic Kahler manifold. For very special SUGRA, this is a symmetric space $G/M \times SU(2)$.
- Hyperkahler cones crop up in a completely different context, namely as moduli spaces of the Nahm equations on the semi-infinite line, or equivalently Dirac monopoles, or D1 strings attached to a D3 brane.

Kronheimer; Bachas Hoppe Pioline

- In the monopole context, the geodesic motion on moduli space describe low energy scattering, in particular *time* evolution. The Nahm equation on the other hand describes the radial evolution away from the D3-brane.
- Channel duality suggests that we should identify the time evolution for black holes with the radial evolution for monopoles. Hence one could think of the Nahm equations as a baby model for the conformal quantum mechanics describing the black hole !
- This is less crazy then it sounds: Recent work suggests that the CQM describing D0-D4 bound states on the quintic is a quiver quantum mechanics, not unlike Nahm !

Gaiotto Guica Huang Simons Strominger Yin

Open problems

- higher derivative corrections
- rotating black holes in 4D
- multi-centered black holes in 4D
- black holes and black rings in 5D and beyond
- automorphic wave functions, and relations to other counting formulae
- genuine N=2 theories and monodromy groups
- time-dependence and midi-superspace models