Conformal quantum mechanics and quantum cosmology

Boris Pioline

LPTHE, Paris

Leuven

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w/ D. Kazhdan, A. Waldron, hep-th/0107222

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Conformal symmetry and universality

Conformal symmetry plays a crucial role in describing universal phases in many physics problems:

- At large distances, physics is either trivial or described by a non-trivial conformal fixed point of the renormalization group. This fixed point is stable under irrelevant perturbations, hence within large universality classes of theories.

- This is particularly powerful in two dimensions, where the conformal group $SO(2,2)$ extends to the infinite dimensional Virasoro group. Critical exponents can often be computed by identifying "phenomenologically" the right conformal fixed point. *Ising, percolation...*

- Gauge theories in 4 dimensions are classically conformally, some however remain conformal quantum mechanically: *$N = 4$ SYM, finite $N = 2$ theories.* The AdS/CFT correspondance makes the conformal symmetry geometrical.

- Non-trivial conformal fixed points in gauge theories in other dimensions surprisingly exist: *(2,0) theory, membranes, five-branes...*

Conformal symmetry in the UV

If conformal symmetry arises naturally in the infrared, it may be desirable to a have a fundamental theory conformal in the ultraviolet:

- A conformally invariant fundamental theory would ensure a vanishing cosmological constant, even after phase transitions.

- The $SO(2,10)$ structure of the 10D superalgebra is maybe rather an hint at conformal symmetry rather than at two-times physics...

- A cosmological singularity corresponds to an ultraviolet regime: in this talk we will show that gravity becomes a 0+1-dimensional conformal system, or rather an infinite family thereof.

Conformal quantum mechanics

Conformal quantum mechanics has been considered in other settings:

- Introduced as a toy model of conformal gauge theories

De Alfaro Fubini Furlan

- The near-horizon geometry of charged (Reissner-Nordström) black holes, $AdS_2 \times S_2$, exhibits conformal invariance, the dynamics of test particles is invariant under the 1-dimensional conformal group, $SO(2,1)$.

Claus Derix Kallosh Kumar Townsend Van Proeyen

- By the AdS/CFT correspondence, string theory on AdS_2 should be dual to a conformal quantum mechanical system in 0+1 dimensions.

Maldacena; Michelson Strominger

Outline

1. Conformal quantum mechanics and black-holes
2. Quantum cosmology at a spacelike singularity
3. Generalized conformal quantum mechanics
4. Applications to the BPS membrane
5. Outlook

Conformal quantum mechanics

- Conformal quantum mechanics was first introduced in 1976 by de Alfaro, Fubini and Furlan (DFF) as an attempt to understand soft breaking of conformal invariance:

$$L = \frac{1}{2}\left(\frac{dq}{dt}\right)^2 - \frac{g}{q^2}\ , \qquad g > 0$$

- The dynamics are invariant under conformal transformations of the time axis,

$$t \to \frac{at+b}{ct+d}\ , \quad q(t) \to \frac{q(t)}{ct+d}\ , \quad ad - bc = 1$$

- The Noether charges generating these transformations at $t = 0$ read

$$E_+ = \frac{1}{2}\left(p^2 + \frac{g}{q^2}\right) = H\ , \quad D = -\frac{1}{4}(pq+qp)\ , \quad E_- = \frac{1}{2}q^2$$

- They represent the conformal group $SO(2,1) = Sl(2)$ in 0+1 dimensions,

$$\{E_+, E_-\} = 2D, \quad \{D, E_\pm\} = \pm E_\pm$$

- Conformal invariance fixes ordering ambiguities.

- A superconformal version of this model can also be constructed, based on $SU(1,1|1) = OSp(2|2)$.

Fubini Rabinovici; Akulov Pashnev

- The coupling constant g is related to the quadratic Casimir of $SO(2,1) = Sl(2)$:

$$\Delta = \frac{1}{2}(E_+E_- + E_-E_+) - D^2 = \frac{g}{4} - \frac{3}{16}$$

which we can parameterize as $\Delta = r(r-1)$.

- Since D and E_- do not commute with the Hamiltonian, they evolve in time, but following the simple law,

$$\begin{aligned} dg/dt &= [h, g] \ , \\ g(t) &= \begin{pmatrix} D & E_- \\ -E_+ & -D \end{pmatrix} (t) \in sl(2)^* \ , \\ h &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in Sl(2) \end{aligned}$$

- The motion thus takes place on a coadjoint orbit of $Sl(2)$, flowing along the action of the nilpotent generator $h = E_+$,

$$G(t) = e^{th} G(0) e^{-th}$$

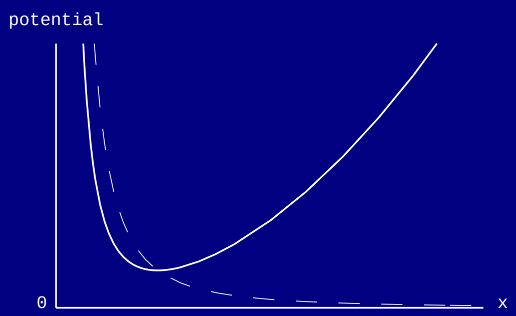
- Classically, the coupling constant is given by the invariant of the orbit, $\Delta \sim \det(G)$.

Mass can preserves conformal invariance !

- The Hamiltonian $H = E_+$ is a parabolic element of $SO(2,1)$. It has a delta-normalizable continuous positive spectrum starting at 0, with eigenfunctions

$$\psi_E(q) = q^{1/2} J_{2r-1}(x\sqrt{2E}) \to q^{2r-1/2} \text{ as } E \to 0$$

- The spectrum may be rendered discrete by deforming the Hamiltonian into $H = E_+ + \Lambda^2 E_-$ where $1/\Lambda$ is a new length scale, which can however be changed by acting with D.



- The Hamiltonian is now a compact (elliptic) element of $SO(2,1)$, with discrete normalizable spectrum, generated by the rising and lowering operators,

$$L_\pm = E_+ - E_- \pm iD \ , \quad [H, L_\pm] = \pm L_\pm$$

acting on the vacuum,

$$L_-\psi_0 = 0 \quad \Rightarrow \quad \psi_0(q) = q^{2r-\frac{1}{2}} e^{-q^2\Lambda^2/2}$$

- We thus have an evenly integer spaced spectrum, with eigenmodes

$$\psi_n(x) = (q\Lambda)^{2r-\frac{1}{2}}\, e^{-q^2\Lambda^2/2} L_n^{2r-1}(q^2\Lambda^2)$$

where L_n are Laguerre polynomials.

CQM and RN black holes

- Reissner-Nordström black holes have a near-horizon geometry given by $AdS_2 \times S_2$,

$$\begin{aligned} ds^2 &= -(2M/r)^4 dt^2 + (2M/r)^2 dr^2 + M^2 d\Omega^2 \,, \\ A &= (2M/r)^2 dt \end{aligned}$$

- The Hamiltonian of a free particle of mass m and charge q in static gauge is

$$H = \frac{p_r^2}{2f} + \frac{mg}{2r^2 f} \,,$$

where f is the function

$$f = \frac{1}{2}\left[\sqrt{m^2 + (r^2 p_r^2 + 4L^2)/4M^2} + q\right] ,$$

and g the effective coupling constant

$$g = 4M^2(m^2 - q^2)/m + 4L^2/m \,.$$

Claus Derix Kallosh Kumar Townsend Van Proeyen

- This gives a "relativistic" generalization of conformal mechanics, with generators

$$E_+ = H = \frac{1}{2f} p_r^2 + \frac{g}{2r^2 f} \,, \quad E_- = -\frac{1}{2} f r^2 \,, \quad D = \frac{1}{2} r p_r$$

near-horizon limit of relativistic CQM

- Upon taking the limit $M \to \infty$ with $M^2(m-q)$ fixed, $f \to m$, hence one recovers the DFF conformal quantum mechanics,

$$H = \frac{p_r^2}{2m} + \frac{g}{2r^2},$$

with

$$g = 8M^2(m-q) + 4\ell(\ell+1)/m .$$

- A superconformal version of this model can also be found by considering a superparticle on the near-horizon geometry.

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de Azcarraga Izquierdo Perez Bueno Townsend

- This can be generalized to the motion of N charged BPS black holes in the moduli space approximation. In a near-horizon-like limit $|x - x'| \ll l_p$, the moduli space turns out to have an homothetic closed Killing vector, hence the quantum mechanics is conformal. The spectrum of $H + K$ is now discrete, where

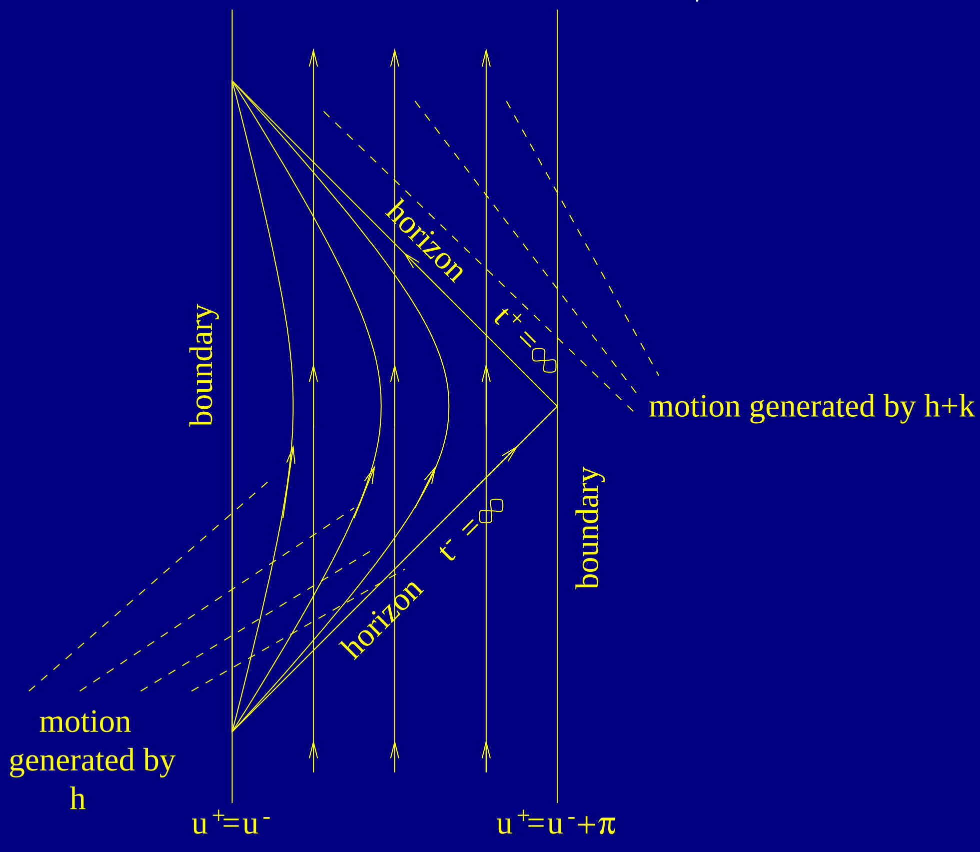
$$K = \sum_{A<B} \frac{Q_A^2 Q_B}{|x_A - x_B|^2}$$

Michelson Strominger

Britto Michelson Strominger Volovich

From parabolic to elliptic, now justified

- Instead of working with asymptotic time ∂_t with has a degenerate Killing horizon, one may choose instead a global time, e.g. $(u+v)$ in coordinates where the AdS^2 metric is $ds^2 = dudv/\sin^2(u-v)$:



- The Hamiltonian wrt to the new Killing vector is just the combination introduced by DFF,

$$\partial_u + \partial_v = E_+ + E_- = \frac{p_r^2}{2} + \frac{1}{2r^2} + \frac{r^2}{2}$$

yielding a discrete spectrum of normalizable states.

Claus Derix Kallosh Kumar Townsend Van Proeyen; Kallosh

2. Spacelike singularity and CQM

- As one approaches a cosmological (spacelike) singularity, the dynamics of points separated by more than a cosmological horizon $\sim cT$ decouple. As $T \to 0$, this reduces to a set of decoupled 0+1-dimensional (quantum) mechanical systems at each point on the spacelike slice !

  Belinskii Khalatnikov Lifschitz; Misner

- In this limit, a minisuperspace ansatz is legitimate,

  $$ds^2 = -\alpha^2 dt^2 + g_{ij}(t) dx^i dx^j$$

  with analogous ansatz for gauge fields. Evaluating Einstein's action on this configuration, we obtain the motion of a fictitious particle on the moduli space of (spatially constant) metrics and gauge fields.

- "Integrating out" off-diagonal dof yields potential terms, which become reflection walls towards the singularity: this leads to an hyperbolic billiard in the Weyl chamber of a Kac-Moody algebra. Motion consists of Kasner flights separated by bounces. The apparition of chaos is related to hyperbolicity of the algebra.

  Damour Henneaux + de Buyl Schomblond Julia Nicolai . . .

2+1-gravity at a spacelike singularity

- For simplicity, we consider 2+1 dimensional Einstein gravity, dimensionally reduced to 0+1 at a spacelike singularity:

$$ds^2 = -\alpha^2 dt^2 + e^{\phi}\frac{[(dx_1 + U_1 dx_2)^2 + U_2^2 dx_2^2]}{U_2}$$

  where e^{ϕ} is the volume and $U = U_1 + iU_2 \in Sl(2)/U(1)$ the “complex structure” of the spatial slice. We refrain from integrating U_1 out.

- The Einstein-Hilbert action becomes, after integrating by part,

$$\begin{aligned} S &= \int dt\ \sqrt{-g}(R - 2\Lambda) \\ &= \int dt \left[\frac{1}{2\alpha}e^{\phi}\left(-\dot{\phi}^2 + \frac{\dot{U}_1^2 + \dot{U}_2^2}{U_2^2}\right) - 2\alpha e^{\phi}\Lambda\right] \end{aligned}$$

  This action is invariant by under general time reparameterization, keeping αdt fixed.

- The variation wrt α imposes the Hamiltonian constraint (in $\alpha = 1$ gauge)

$$H_{WDW} := \frac{1}{2}e^{\phi}\left(\dot{\phi}^2 - \frac{\dot{U}_1^2 + \dot{U}_2^2}{U_2^2} - 4\Lambda\right) \equiv 0$$

Moving on the cone

- Changing variables to $V = e^{\phi}, \eta = \alpha e^{\phi}$, we get
$$S = \int dt \left[\frac{1}{2\eta} \left(-\dot{V}^2 + V^2 \frac{\dot{U}_1^2 + \dot{U}_2^2}{U_2^2} \right) - 2\eta\Lambda \right]$$
- The dynamics is therefore given by the motion of a free particle of mass $m^2 = 4\Lambda$ on the Lorentzian cone with metric
$$ds^2 = -dV^2 + V^2 \frac{dU_1^2 + dU_2^2}{U_2^2}$$
Note this is flat $R^{2,1}$ in polar coordinates. For $\Lambda < 0$ the particle is tachyonic.
- The volume V appears with a negative signature: it can be chosen as a reference time, against which to measure other phenomena.
DeWitt
- The motion is now easily integrated: in the gauge $\eta = V^2$, the motion of U decouples from V, hence U follows geodesics in the upper half plane.
- The charge p_1 associated to the isometry $U_1 \to U_1 + c$ is conserved. The motion of U_2 effectively receives an harmonic potential $p_1^2 U_2^2$: for $p_1 \neq 0$, this prevents U_2 from reaching $+\infty$: trajectories are half circles centered on the boundary of the upper half plane.

- Now put $V = \rho^2$. Going to momentum variables
  $p = -4\rho\dot{\rho}/\eta, \quad p_1 = \rho^4\dot{U}_1/(\eta U_2^2), \quad p_2 = \rho^4\dot{U}_2/(\eta U_2^2),$
  we get the Hamiltonian
$$H = \frac{\eta}{\rho^2}\left[-\frac{p^2}{8} - \frac{\Delta}{2\rho^2} + \frac{1}{8}\Lambda\rho^2\right]$$
- The Hamiltonian constraint $\delta H/\delta\eta \equiv 0$ reads
$$H_{WDW} = \frac{1}{2}p^2 + \frac{2\Delta}{\rho^2} - \frac{1}{2}\Lambda\rho^2 \equiv 0$$
- This is nothing but the Hamiltonian of conformal mechanics, upon identifying $g = 4\Delta$, where Δ is the angular momentum on $Sl(2)/U(1)$. The sign of g depends on boundary conditions on the upper half plane (square integrable modes have $\Delta < 0$)

- The quadratic potential is provided by the cosmological constant. For $\Lambda < 0$, we get an operator with discrete normalizable states.

- Even so, we are looking for a zero energy state, which will not be normalizable.

- For $\Lambda < 0$, we are looking for a state which is invariant under the compact generator $\mathcal{E}_+ + E_-$: the wave function of the Universe is therefore the spherical vector of the representation.

DFF vs WDW

Despite formal identity between the two problems, there are some important differences:

- The WDW equation picks out zero-energy states only. So boundedness from below of H is no longer a requirement. Indeed, the sign of g depends on boundary conditions on S (square integrable wave functions have $g < 0$), and the sign of m^2 depends on Λ (discrete spectrum for $\Lambda < 0$)

- Usual quantum mechanics analysis requires wave functions to be square integrable. Here ρ should be thought as a time variable, square integrability along ρ should not be imposed. Instead perhaps, use a Klein-Gordon type norm on spacelike slices (and "third" quantize the system in order to get rid of negative norm states)

Those are problems in any quantum cosmology investigation, so we proceed anyway.

- Let us know consider the reduction of $n+1$-dim Einstein gravity: The metric ansatz is

$$ds^2 = -\frac{\eta(t)}{V(t)}\,dt^2 + V^{2/n}(t)\,\widehat{g}_{ij}(t)\,dx^i dx^j\,,$$

where V is the spatial volume and $\det(\widehat{g}) = 1$.

- The Einstein-Hilbert action reduces to

$$\int dt \left\{ \frac{1}{2\eta} \left[-\frac{2(n-1)}{n}\dot{V}^2 + V^2\,\dot{U}^M G_{MN} \dot{U}^N \right] - 2\Lambda\eta \right\}$$

Here U^M coordinatize the negative curvature symmetric space $S = Sl(n)/SO(n)$ describing all spatially constant unit volume metrics $\widehat{g}$.

- One recognizes the Lagrangian for a free particle propagating on the Lorentzian cone

$$d\sigma^2 = -\frac{2(n-1)}{n}\,dV^2 + V^2 dU^M G_{MN} dU^N\,.$$

- Change vars to $\rho = \sqrt{8(n-1)V/n}$ and go to canonical coordinates. The Hamiltonian now reads

$$H = \frac{\eta}{V}\left[\frac{1}{2}p^2 + \frac{4(n-1)}{n\rho^2}\Delta - \frac{n\Lambda}{4(n-1)}\rho^2\right]$$

The eom for η is again the DFF Hamiltonian, at zero energy, with $g = 8(n-1)\Delta/n$ related to the Laplacian on S.

- The conformal symmetry is a direct consequence of the conical structure of moduli space.

Dimensional reduction of supergravity

- In addition to the graviton, supergravity also contains scalar and gauge fields. Upon dimensional reduction, we still obtain the geodesic motion of a free particle on a Lorentzian cone with negatively curved sections G/K. E.g: gravity+dilaton+B yields a cone over $SO(n,n)/SO(n) \times SO(n)$.

- The positive roots in G correspond to off-diagonal metric and gauge fields; they can be eliminated by using the associated conserved Noether charges, producing a potential for the Cartan degrees of freedom, aka dilatonic scalars: as in the 2+1 case, these yield reflection walls keeping $R_1 \leq R_2 \leq \ldots$ in a fundamental chamber.

- In addition, there are potential terms originating from spatial gradients of the metric; these could be incorporated using a "dual" description of the graviton (e.g: in $11 \to 3$ reduction, $g_{\mu i}$ are 8 vectors fields dual to 8 scalars, hence yield a scalar field in $(8,1)$ Young tableau).

  Obers BP Rabinovici;Boulanger et al

- Duality implies an infinite set of such domain walls, corresponding to a roots of an hyperbolic Kac-Moody algebra E_{10}. Upon introducing these new degrees of freedom, one would expect to still have a conformal symmetry.

  Damour Henneaux Julia Nicolai

CQM and coadjoint orbits, in detail

- We have already mentioned that DFF conformal mechanics correspond to free motion on a coadjoint orbit of $Sl(2)$. In more detail: let us consider the coadjoint orbit of a generic hyperbolic element of $sl(2)$:

$$\Omega = \{g^{-1}Jg \ , g \in Sl(2)\} \ , \quad J = \begin{pmatrix} \lambda & \\ & -\lambda \end{pmatrix}$$

- The orbit Ω can be viewed as $\Omega = Stab\backslash G$ where $Stab = \{g, g^{-1}Jg = J\}$ is the stabilizer of J. A gauge slice can be chosen as

$$\Omega = \{g = \begin{pmatrix} 1 & \\ \gamma & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \beta \\ & 1 \end{pmatrix}\}$$

  $G = Sl(2)$ acts from the right on Ω, hence on (β, γ).

- A coadjoint orbit has a canonical invariant symplectic form, the Kirillov-Kostant symplectic form,

$$\omega = d\theta \ , \quad \theta = \mathrm{Tr}(Jdgg^{-1}) = -2\lambda\beta d\gamma$$

- The right action of $h \in G$ preserves ω, hence can be represented by its moment map E_h such that $i_h\omega = dE_h$. h then acts by Poisson bracket with E_h on functions of (β, γ). Here:

$$E_+ = 2\lambda\gamma \ , \quad D = 2\lambda(1+2\beta\gamma) \quad E_- = -2\lambda\beta(1+\beta\gamma)$$

Coadjoint orbits and unireps

- This can be recast in the conformal quantum mechanics form through a canonical transformation,

$$E_+ = y^2 \ , \quad D = 2py \ , \quad E_- = \frac{1}{4}p^2 + \frac{\lambda}{2y^2}$$

- Note that this construction is purely classical: the non-trivial part is to quantize the coadjoint orbit. This can be done by induced representation methods.

- One could have started from a nilpotent element of $Sl(2)$ instead:

$$J = \begin{pmatrix} 0 & \\ 1 & 0 \end{pmatrix} \ , \qquad g = \begin{pmatrix} \sqrt{t} & \\ & 1/\sqrt{t} \end{pmatrix} \cdot \begin{pmatrix} 1 & \beta \\ & 1 \end{pmatrix}$$

$$\theta = t d\beta \ , \quad E_+ = t \ , \quad D = 2\beta t \ , \quad E_- = \beta^2 t$$

Redefining $t = y^2$ and $\beta = p/(2y)$ we get

$$E_+ = y^2 \ , \quad D = py \ , \quad E_- = \frac{1}{4}p^2$$

This is the usual harmonic oscillator. Its quantization gives the metaplectic representation of $Sl(2)$.

- Kirillov's philosophy: there is a 1-1 correspondence between unireps and coadjoint orbits.

CQM from nilpotent orbits

- Quantization of coadjoint orbit of any group containing $Sl(2)$ will yield a conformal quantum mechanical model: simply need to find the right variables such that $D = pq$ etc.

- E.g, for $Sl(n)$, the coadjoint orbit of an element with $n-2$ coinciding eigenvalues, after symplectic reduction, yields the Conformal Calogero Model,

$$H = \sum p_i^2 + \sum_{i<j} \frac{g}{(q_i - q_j)^2}$$

Barucci Regge;Freedman Mende;Gibbons Townsend

- Generic coadjoint orbits have (even) dimension $n = \mathrm{dim}G - \mathrm{Rank}G$. Non-generic ones have a bigger stabilizer, hence correspond to a phase space of smaller dimension. They also have fewer parameters.

- The coadjoint orbit of minimal dimension is the orbit of any root (for $Sl(n)$: only one 2×2 Jordan block). Its quantization leads to the minimal representation of G, analogous to the metaplectic representation of $Sl(2)$.

- Motivated by a conjecture about the BPS quantum supermembrane, we turn to the quantization of the minimal nilpotent orbit of ADE groups.

The minimal nilpotent orbit of ADE groups

- We consider the coadjoint orbit of the lowest root, $E_{-\omega}$. Under $D_\omega = [E_\omega, E_{-\omega}]$, the algebra of G decomposes into the 5-grading

$$G = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_1 \oplus G_2$$

  where $G_{\pm 2} = \{E_{\pm\omega}\}$ are one-dimensional.

- The stabilizer of $E_{-\omega}$ consists of $G_{-2} \oplus G_{-1} \oplus G_0^\perp$, where $\perp$ is the component of G_0 orthogonal to D_ω. The coadjoint orbit can then be parameterized by

$$Stab \backslash G = \{D_\omega\} \oplus G_1 \oplus E_\omega$$

- G_1 is an Heisenberg algebra,

$$[E_\beta, E_\gamma] = E_\omega \text{ iff } \beta + \gamma = \omega$$

- A Lagrangian subspace can be chosen as $\{E_{\beta_0}, E_{\beta_i}\}$ where β_0 is the the next-to-affine-root, and β_i the positive roots such that $\langle \beta_0, \beta_i \rangle = 1$.

- We represent the β_i and γ_i as conjugate positions and momenta:

$$E_{\beta_i} = y p_i, \quad E_{\gamma_i} = x_i \quad , E_\omega = y$$

- Generators in G_0 preserve the symplectic form of G_1, hence act by canonical transformations on positions and momenta. G_{-1} and G_{-2} are non-trivial.

Nilpotent Conformal Quantum Mechanics

- The action of G_{-1} and G_{-2} can be found using two Weyl generators, (i) Fourier transform on all positions x_i, (ii) the Weyl reflection wrt β_0,

$$W\psi(y,x_0,x_i)=e^{-\frac{I_3(x_i)}{2yx_0}}\psi(-x_0,y,x_i)$$

where I_3 is the cubic invariant of the positions under the linearly realized H_l. The Weyl group relation $(AS)^3=(SA)^3$ holds thanks to the invariance of the non-Gaussian character $\exp(iI_3(x_i)/x_0)$ under Fourier transform.

Kazhdan Savin

- In order to bring the generator $E_\omega=y$ to standard form, we make a further canonical transformation,

$$y=\frac{\rho^2}{2},\quad x_i=\frac{\rho q_i}{2},\quad p=\frac{1}{\rho}p_\rho-\frac{1}{\rho^2}q_i\pi_i,\quad p_i=2\frac{\pi_i}{\rho}$$

- The generators of the conf. subalgebra now read

$$E_\omega=\frac{1}{2}\rho^2\ ,\quad D_\omega=2\rho p_\rho+1\ ,\quad E_{-\omega}=\frac{1}{2}\left(p^2+\frac{4\Delta}{\rho^2}\right)$$

where Δ is, up to an additive constant, the quadratic Casimir of $G_0^\perp$: it is the only quartic invariant of the coordinates and momenta $\{q_i,\pi_i\}$ under $G_0^\perp$.

- The spherical vector, ie the wave function invariant under all compact generators $E_\alpha+E_{-\alpha}$, has been computed for any G by solving the corresponding PDE's.

Kazhdan BP Waldron

D_4 *minimal orbit*

- For D_4, the 5-grading reads

$$\begin{array}{rcl} D_4 & \supset & Sl(2) \times Sl(2) \times Sl(2) \\ adj & = & [(1,1,1) + (3,1,1) + \mathrm{perm}]_0 \oplus (2,2,2)_1 \oplus 1_2 \end{array}$$

- The coordinates and momenta transform as a $(2,2,2)$, and satisfy the Heisenberg algebra

$$[q^{aA\alpha}, q^{bB\beta}] = \epsilon^{ab}\epsilon^{AB}\epsilon^{\alpha\beta}$$

- The actions of each $Sl(2)$ factor in H are represented by the angular momentum-like operators

$$\Sigma^\mu = \sigma^\mu_{\alpha\beta} \;\; \epsilon_{ab} \; \epsilon_{AB} \; q^{aA\alpha} \;\; q^{bB\beta} \;\; , \quad [\Sigma^\mu, \Sigma^\nu] = \epsilon^{\mu\nu\rho}\Sigma^\rho$$

  The quadratic Casimirs of all three $Sl(2)$'s are identical and equal to the unique quartic invariant I_4 of the $(2,2,2)$ representation.

- Quantization: choose $Q^{A\alpha} = q^{1A\alpha}$ as positions. $Q^{A\alpha}$ is a vector Q^I of $SO(2,2)$, parameterized in polar coordinates by

$$\Omega \in H_3 = \frac{SO(2,2)}{SO(2,1)} = SO(2,1) \;, \quad \kappa^2 = Q^I \eta_{IJ} Q^J$$

  The quadratic Casimir is the angular momentum squared on H_3, *i.e.* the Laplacian on $Sl(2)$.

D_4 *minimal orbit and q-cosmology*

- We thus obtain

$$H = E_\omega + E_{-\omega} = \frac{p^2}{2} + \frac{\Delta_{Sl(2)}}{\rho^2} + \frac{1}{2}\rho^2$$

- The Laplacian can be further reduced to the Laplacian on $Sl(2)/U(1)$ by focussing on modes invariant under the maximal compact $K \subset G$.

- This is the conformal mechanical model coming from dimensional reduction of $2+1$ dimensional gravity near a spacelike singularity, except for a decoupled degree of freedom κ.

- The spherical vector in this representation reads

$$\psi_{D_4} = \frac{\rho^{3/2} e^{-S}}{S} \ , \ S = \frac{1}{2}\sqrt{\rho^4 + \rho^2 \mathrm{tr}(Q^t Q) + \kappa^4}$$

  The coordinate $\kappa^2 = \det(Q^{A\alpha})$ appears here as a degeneracy label.

4. CQM and the quantum membrane

- It has been proposed that the 4-graviton amplitude at R^4 order in M-theory compactified on T^d, known exactly on the basis of SUSY and U-duality, be obtainable from a one-loop computation in the BPS supermembrane theory:

$$f_{R^4} = \int_{R^+ \times Sl(3,Z) \backslash Sl(3)/SO(3)} Z(\gamma; G, C, \dots)$$

- Z is the partition function of the membrane zero-modes, mapping $T^3 \to T^d$. It should be invariant under both U-duality $E_d(Z)$ and modular group $Sl(3,Z)$.

BP Nicolai Plefka Waldron

- The simplest non-trivial case is for $d = 3$, which allows for membrane instantons. U-duality and modular can be embedded into a larger group,

$$E_6 \supset Sl(3)^3 \supset Sl(3)_{mod} \times R^+ \times [Sl(2) \times Sl(3)]_U$$

- In order to obtain an automorphic form of G, the partition function should be written as

$$Z(g) = \sum_{m \in Z^n} \mu(m) [R(g) \cdot f](m) \ , \quad \mu(m) =^{**} \prod_{p \text{prime}} f_p(m)$$

where R is a representation of G on functions $\phi(m)$ of n variables, $f(m)$ is the spherical vector of this representation *** over the real field, and f_p over the p-adic field.

The minimal rep of E_6

- The minimal orbit of E_6 has dimension 22. The 5-grading reads

$$\begin{array}{rcl} E_6 & \supset & Sl(6) \times Sl(2) \\ adj & = & 1 \oplus 20 \oplus [3 + 35] \oplus 20 \oplus 1 \end{array}$$

- A Lagrangian subspace of G_1 can be chosen by breaking $Sl(6)$ down to $Sl(3) \times Sl(3)$: The minimal representation therefore acts on

$$10 = 1 + 1 + (3,3) = y, x_0, Z_{a\alpha}$$

- From the membrane point of view, $Z_{a\alpha}$ can be thought as the 3×3 matrix of winding numbers, however (y, x_0) are two new quantum numbers: discrete fluxes ?

- The spherical vector has a Born-Infeld like form,

$$f_{E_6} = \frac{1}{|z|^2 S_1} \exp\left(-\frac{\sqrt{\det(ZZ^t + |z|^2 1_3)}}{|z|^2} + i\, \frac{x_0 \det(Z)}{y|z|^2} \right)$$

- Full verification of the conjecture still in progress.

BP Waldron, to appear soon

Summary

- Conformal symmetry arises in many different problems where a universal regime is reached: non-trivial infrared dynamics of gauge theories, near horizon limit of black holes, and here: gravity near a spacelike singularity.

- The appearance of conformal symmetry here is perhaps not surprising, since we are expanding around a solution with power-like behavior, $g_{\mu\nu}(t,x) = t^{\alpha} g^{0}_{\mu\nu}(x)$: at least scaling symmetry is guaranteed.

- From a mathematical viewpoint, conformal quantum mechanics can be understood as free motion on a coadjoint orbit of $Sl(2)$. It can be generalized to any group G containing an $Sl(2)$. This allows for a general quantization of these models.

- Nilpotent orbits are particularly interesting, since they have the smallest phase space and parameter space: the minimal orbit has no parameter at all.

- We have identified the minimal orbit of D_4 with the dimensional reduction of 2+1 gravity. How about other ADE groups ? ADE groups ?

Hyperbolic quantum chaos ?

- Our cosmological model has avoided the complications of the general hyperbolic Kac-Moody groups: can one reformulate the dimensional reduction of M-theory to 0+1 dimensions as the free motion on a (nilpotent) coadjoint orbit of an Hyperbolic Kac-Moody group ?

- Is there any remnant of U-duality at a cosmological singularity ? Can the automorphic theta series constructed by quantization of coadjoint orbits over the real and p-adic describe, say, the wave function of the universe ?

- The construction relied on the invariance of the cubic character $\exp(iI_3(x_i)/x_0)$ under Fourier transform: a class of non-Gaussian yet free cubic models. Can models be found with ∞ degrees of freedom ?

Poetry: gravity and fluid mechanics

- The dynamics of gravity at a spacelike singularity has a strong flavor of fully developped turbulency in fluid mechanics. Indeed, one may think of each of the fictitious particles as fluid elements moving on the moduli space, with the spatial position playing the role of the particle label in a Lagrangian description.

- Recall that Euler's perfect fluid equations can be thought of as a geodesic motion on the coadjoint orbit of volume preserving diffeomorphisms. Is there a similar picture for gravity ?

Arnold

- The chaotic behavior is reminiscent of the energy cascade in turbulency. The conformal symmetry that we argued for should correspond to Kolmogorov's "inertial range". Can quantum fluctuations and particle production provide a dissipation cut-off ?