Exceptional Theta series and the quantum BPS membrane

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Quantizing the membrane ?

- The fundamental degrees of freedom of M-theory are still elusive: D0-branes ? open strings ? Yang-Mills fields ? or membranes ?

- Membranes are attractive candidates from several viewpoints: reduce to strings after double reduction, 11D Poincaré invariance, kappa symmetry reproduces the 11D SUGRA tree-level equations of motion, M(atrix) regularization reproduces SUGRA graviton scattering.

- Yet there is no evidence that a consistent theory of quantum membranes exists: no genus expansion (the membrane topology itself is ill-defined due to spikes), no tunable coupling, continuous spectrum...

- Under simplifying circumstances, one might be able to quantize the membrane: e.g. open membranes in the large H limit: $S = \int d^2\sigma X^1 dX^2 \wedge dX^3$, a possible path towards the non-Abelian five-brane ?

  BP, hep-th/0201257

- Here, we consider BPS saturated scattering amplitudes such as R^4 in M-theory/T^d: one may expect that the bosonic and fermionic quantum fluctuations cancel, leaving BPS zero-modes whose quantization should reproduce the exact R^4 amplitude.

Automorphic forms and the quantum BPS membrane

- U-duality and supersymmetry predict the value of the four-graviton R^4 coupling for all values of T^d moduli (i.e. arbitrary string coupling), as an automorphic form of the U-duality group:

$$f_{R^4} = \mathrm{Eis}^{E_d(Z)}_{string;s=3/2}(G, C, \dots)$$

resumming string tree-level, one-loop and D-instantons – or alternatively, regularized 11D SUGRA tree-level, one-loop and toroidal membrane instantons.

Green,Gutperle,Vanhove;Kiritsis,Obers,BP

- In particular, the one-loop string amplitude contains no oscillator, has $Sl(2, Z)$ modular and $D_{d-1}(Z)$ T-duality invariance, and yields regularized 10D SUGRA tree-level, one-loop and toroidal worldsheet instantons:

$$\int_{U(1)\backslash Sl(2)/Sl(2,Z)} \frac{d^2\tau}{\tau_2^2} Z_{d-1}(\tau; g, B) = \mathrm{Eis}^{D_{d-1}(Z)}_{spinor;s=1}(g, B)$$

- Q: can one construct a partition function for the zero-modes of a toroidal membrane, having $Sl(3, Z)$ modular and $E_d(Z)$ U-duality invariance, reproducing the R^4 coupling after integration over membrane moduli ?

$$f_{R^4} =? \int_{SO(3)\backslash Sl(3)/Sl(3,Z)\times R^+} d\gamma \;\; Z_d(\gamma; G, C, \dots)$$

This is a topic in the theory of automorphic forms...

The one-loop string amplitude forever

- Recall the one-loop R^4 amplitude for type II on T^n:

$$Z = V_n \sum_{m^i, n^i \in Z} \exp\left(-\pi \frac{|m^i + n^i \tau|^2}{\tau_2} + 2\pi i m^i B_{ij} n^j \right)$$

  where the argument is the Polyakov action for the zero-mode configuration

$$X^i = m^i \sigma_1 + n^i \sigma_2 \ , \qquad \gamma_{\alpha\beta} = \frac{1}{\tau_2} \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}$$

  Modular invariance is manifest.

- An Hamiltonian interpretation is obtained after Poisson resummation over m^i:

$$Z = \tau_2^{d/2} \sum_{m_i, n^i} \exp\left(-\pi \tau_2 [M^2] + 2\pi i \tau_1 m_i n^i \right)$$

  where τ_2 is now the Schwinger time, τ_1 a Lagrange multiplier enforcing the BPS (level-matching) condition $m_i n^i = 0$, and $M^2 = (m_i + B_{ij} n^j)^2 + (n^i)^2$ the mass of BPS states in the loop:
  T-duality is manifest.

- After integration over the fundamental domain, we get the one-loop SUGRA contribution, plus world-sheet instantons corresponding to saddle points in τ plane ($m^{ij} = m^i n^j - m^j n^i$),

$$\int_F Z = \dots + \sum_{m^{ij}} \mu(m^{ij}) \frac{\exp\left(-2\pi \sqrt{(m^{ij})^2} + 2\pi i m^{ij} B_{ij} \right)}{\sqrt{(m^{ij})^2}}$$

The membrane one-loop partition function (1st pass)

- By analogy with the string theory case, we may construct the one-loop BPS membrane amplitude by summing over zero-mode configurations $X^i = m^i_\alpha \sigma^\alpha$ and constant γ in $Sl(3)/SO(3)$ with Polyakov action

$$S = \frac{1}{l_p^3} \int d^3\sigma \sqrt{\gamma} \left(\gamma^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j G_{ij} - 1 \right) +$$

$$+ i\epsilon^{\alpha\beta\gamma} \partial_\alpha X^i \partial_\beta X^j \partial_\gamma X^k C_{ijk}$$

  and unit weight.

- The resulting partition function for $T^3 \to T^d$ maps,

$$Z = \sum_{m^i_\alpha} \exp\left(\sqrt{\gamma}\ \mathrm{Tr}[m\gamma m^t G] - 1) + iC \cdot (m \wedge m \wedge m) \right)$$

  after integration over the volume $u = \det(\gamma)$ and the fundamental domain of $SO(3)\backslash Sl(3)$ reproduces the correct membrane instanton saddle points and one-loop spectrum, but the summation measure and spectrum degeneracy are off.

  Nicolai, BP, Plefka, Waldron

- This partition function is by construction modular invariant $Sl(3)$ on the wv and $Sl(d)$ in target-space, but not under the full U-duality group $E_d(Z)$. E.g., for $d = 3$, not under modular transforms of $T = C_{123} + iV_3$.

Partition functions and correspondences

- Going back to the string partition function, the way modular $Sl(2,Z)$ and T-duality $SO(n,n,Z)$ are simultaneously realized is that they are maximal commuting subgroups of a larger group

  $$Sl(2)\times SO(n,n) \subset Sp(2n) : T \to (AT+B)(CT+D)^{-1}$$

  leaving the *symplectic theta series* invariant:

  $$Z_n(g,B;\tau) = \theta_{Sp}(T) := \sum_{m\in Z^{2n}} \exp\left(2\pi i\ m^I T_{IJ} m^J\right)$$

  thanks to Gaussian Poisson resummation.

- By integrating wrt to one of the 2 factors, Z_n furnishes a *correspondence* between automorphic forms of worldsheet $Sl(2,Z)$ and target space $SO(n,n,Z)$.

- In order to quantize the BPS membrane, one should therefore construct an *automorphic correspondence* between $Sl(3,Z)$ and $E_d(Z)$. For this, we need to find a group containing the two as (maximal) commuting factors), and generalize Gaussian Poisson resummation to cubic characters...

Membrane correspondences

- For compactifications on T^d, the U-duality groups can be unified with the modular $Sl(3)$ group as

$$
\begin{array}{lll}
d=2 & Sl(2)\times R^+\times Sl(3) & \subset Sl(5) \\
d=3 & Sl(3)\times Sl(2)\times R^+\times Sl(3)\subset (Sl(3)^3) & \subset E_6 \\
d=4 & Sl(5)\times R^+\times Sl(3)\subset Sl(6)\times Sl(3) & \subset E_7 \\
d=5 & SO(5,5)\times R^+\times Sl(3)\subset SO(5,5)\times Sl(4) & \subset E_8 \\
d=6 & E_6\times Sl(3) & \subset E_8
\end{array}
$$

- For $d \geq 3$ (no membrane instantons in $d < 3$) one is thus left to construct Theta series for exceptional groups: mathematicians know them in principle, but in practice...

- One should also require that after integrating over $R^+\times Sl(3)$, one gets an eigenmode of Δ_{E_d} with the correct eigenvalue. So before integration:

$$\left(\Delta_{E_d}+\Delta_{Sl(3)}+\partial_u^2+cste\right) Z_d=0$$

  This will be satisfied by construction...

- For $d = 6$, no integration on a volume factor is required: the integral will be manifestly finite. In addition to membrane instantons, the R^4 coupling contains M5-brane instantons: membranes and fivebranes will be unified in this framework.

The Jacobi Theta series torn to pieces

- Let us go back to the simplest theta series

$$\theta = \sum_{m \in Z} e^{i\pi\tau m^2} = \langle \delta, \rho(g_\tau) f \rangle$$

where $f(x) = e^{-x^2/2}$,

$$\delta_Z(x) = \sum_{m \in Z} \delta(x - m) \ , \quad g_\tau = \begin{pmatrix} 1 & \tau_1 \\ 0 & \tau_2 \end{pmatrix} / \sqrt{\tau_2}$$

and $\rho(g)$ is a representation of $Sl(2)$ on functions of one variable:

$$\rho \ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : \phi(x) \to e^{i\pi t x^2} \phi(x) \, ,$$

$$\rho \ \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} : \phi(x) \to e^{t/2} \phi(e^t x) \, ,$$

$$\rho \ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : \phi(x) \to e^{i\pi/4} \widetilde{\phi}(-x) \, .$$

- Infinitesimally, this is

$$E_+ = i\pi\, x^2 \, , \quad H = \frac{1}{2} (x\partial_x + \partial_x x) \ , \quad E_- = \frac{i}{4\pi} \partial_x^2 \, ,$$

satisfying the $Sl(2,R)$ algebra,

$$[H, E_\pm] = \pm 2\, E_\pm \ , \quad H = [E_+, E_-] \ .$$

Spherical vector and summation measure

- In order to get a function on $Sl(2)/U(1)$, the kernel f has to be (quasi) invariant under the compact generator,
$$(E_+ - E_-)f = \frac{i}{2}f \tag{1}$$
so that f is the ground state of the harmonic oscillator.

- The distribution δ (i.e. the summation measure) has to be invariant under $Sl(2,Z)$. The non-trivial requirement is the invariance under the Weyl generator: this is guaranteed by the Poisson resummation formula
$$\sum_{m\in Z} f(m) = \sum_{\tilde{m}\in Z} \tilde{f}(\tilde{m})$$

- ** The distribution δ can be rewritten as a product over primes p,
$$\delta(x) = \prod_p \gamma_p(x)$$
where $\gamma_p(x)$ is 1 on p-adic integers, 0 elsewhere. γ_p is also invariant under p-adic Fourier transform: the summation measure δ can therefore be gotten as a product of p-adic spherical vectors for all p !

- *** This also works for the Eisenstein series $E_s^{Sl(2)}$,
$$\mu_s(m) = \sum_{d|m} d^{-(2s-1)} = \prod_p \frac{1 - p^{-2s+1}|m|_p^{2s-1}}{1 - p^{-2s+1}}$$

Min.rep and conformal quantum mechanics

- The first ingredient in constructing the theta series is the (infinite dimensional) unitary irreductible representation ρ of G . The smallest representation can be constructed by *quantizing the coadjoint orbit O of the highest root of* G.

- *(Any coadjoint orbit carries a canonical symplectic structure, hence is a phase space of a classical mechanical system. In order to quantize that system, split coordinates into momenta and positions, and consider wavefunctions of the position variables.* ρ *is then the action of* G *on this Hilbert space.)*

- The coadjoint orbit O can be parameterized by positions $(y, x_0, x_{i=1\ldots N-1})$ and their associated momenta. The generators of G on $\mathcal{O}$ can be gotten by performing the canonical treatment of the *conformal* dynamical system

$$\mathcal{L} = \dot{x}_0 \dot{y} + 2x_0 \sqrt{I_3(\dot{x}_i)} + \frac{d}{dt}\left(\frac{x_0 I_3(x_i)}{y}\right)$$

where I_3 is a cubic form invariant under the linear action of a subgroup H_0 of G. The remaining generators of G act by conformal transformations of the time $t \to (at+b)/(ct+d)$.

Quantization of the minimal nilpotent orbit

- The quantization of this system goes by replacing $p_i \to id/dx_i$ and adding normal ordering terms so that the generators still close.

- The quantum generators can be systematically constructed by starting from the standard representation of an Heisenberg subalgebra $E_{\beta_i} = y\partial_i, E_{\gamma_i} = ix_i$, and acting with the Weyl generators

$$(Sf)(y, x_0, \ldots, x_{N-1}) = \int \frac{\prod_{i=0}^{d-1} dp_i}{(2\pi y)^{N/2}} f(y, p_0, \ldots, p_{N-1})\, e^{\frac{i}{y}\sum_{i=0}^{N-1} p_i x_i}$$

$$(Af)(y, x_0, x_1, \ldots, x_{d-1}) = \exp\left(-\frac{iI_3}{x_0 y}\right) f(-x_0, y, x_1, \ldots, x_{d-1})$$

- The fact that these generators satisfy the Weyl group relation $(AS)^3 = (SA)^3$ follows from the invariance of the cubic character under Fourier transform,

$$\int dp_0 \prod dp_i \exp\left(-\frac{iI_3(x_i)}{\hbar x_0} + ip_0 x_0 + ip_i x_i\right) = [\cdot] \exp\left(\frac{i\hbar I_3(p_i)}{p_0}\right)$$

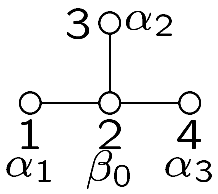
  up to a one-loop determinant factor. Those are examples of non-Gaussian free theories, unfortunately with a finite number of degrees of freedom.

Minimal Nilpotent Orbit

$$\begin{array}{ccc}
Sl(n) & \supset & Sl(2)\times Sl(n-2)\times R^+ \\
adj & = & (3,1,0)\oplus[(2,n-2,1)\oplus(2,n-2,-1)]\oplus(1,adj,0) \\
 & = & 1\oplus 2(n-2)\oplus[1\oplus adj]\oplus 2(n-2)\oplus 1 \\
SO(2n) & \supset & Sl(2)\times Sl(2)\times SO(2n-4) \\
adj & = & (3,1,1)\oplus(2,2,2n-4)\oplus(1,3,1)\oplus(1,1,adj) \\
 & = & 1\oplus(2,2n-4)\oplus[1\oplus adj]\oplus(2,2n-4)\oplus 1 \\
E_6 & \supset & Sl(2)\times Sl(6) \\
78 & = & (3,1)\oplus(2,20)\oplus(1,35) \\
 & = & 1\oplus 20\oplus[1\oplus 35]\oplus 20\oplus 1 \\
E_7 & \supset & Sl(2)\times SO(6,6) \\
133 & = & (3,1)\oplus(2,32)\oplus(1,66) \\
 & = & 1\oplus 32\oplus[1\oplus 66]\oplus 32\oplus 1 \\
E_8 & \supset & Sl(2)\times E_7 \\
248 & = & (3,1)\oplus(2,56)\oplus(1,133) \\
 & = & 1\oplus 56\oplus[1\oplus 133]\oplus 56\oplus 1
\end{array}$$

G	dim	H_0	G_1^*	I_3
$Sl(n)$	$n-1$	$Sl(n-3)$	$[n-3]$	0
$SO(n,n)$	$2n-3$	$SO(n-3,n-3)$	$1\oplus[2n-6]$	$x_1(\sum x_{2i}x_{2i+1})$
E_6	11	$Sl(3)\times Sl(3)$	$(3,3)$	det
E_7	17	$Sl(6)$	15	Pf
E_8	29	E_6	27	$27^{\otimes_s 3}\|_1$

Example: $D4 = SO(4,4)$



$$\begin{aligned} \beta_i &= \beta_0 + \alpha_i \ , \quad \gamma_i = \beta_0 + \alpha_j + \alpha_k \ , \\ \gamma_0 &= \beta_0 + \alpha_1 + \alpha_2 + \alpha_3 \ , \quad \omega = \beta_0 + \gamma_0 \end{aligned}$$

$$I_3 = x_1 x_2 x_3$$

$$\begin{array}{ll} E_{\beta_0} = y\partial_0 & E_{\gamma_0} = ix_0 \\ E_{\beta_1} = y\partial_1 & E_{\gamma_1} = ix_1 \\ E_{\beta_2} = y\partial_2 & E_{\gamma_2} = ix_2 \\ E_{\beta_3} = y\partial_3 & E_{\gamma_3} = ix_3 \end{array}$$

$$E_\omega = iy \, .$$

$$\begin{array}{ll} E_{\alpha_1} = -x_0\partial_1 - \frac{ix_2x_3}{y} \ , & E_{-\alpha_1} = x_1\partial_0 + iy\partial_2\partial_3 \\ E_{\alpha_2} = -x_0\partial_2 - \frac{ix_3x_1}{y} \ , & E_{-\alpha_2} = x_2\partial_0 + iy\partial_3\partial_1 \\ E_{\alpha_3} = -x_0\partial_3 - \frac{ix_1x_2}{y} \ , & E_{-\alpha_3} = x_3\partial_0 + iy\partial_1\partial_2 \, . \end{array}$$

$$\begin{aligned} E_{-\beta_0} &= -x_0\partial + \frac{ix_1x_2x_3}{y^2} \\ E_{-\beta_1} &= x_1\partial + \frac{x_1}{y}(1 + x_2\partial_2 + x_3\partial_3) - ix_0\partial_2\partial_3 \\ E_{-\gamma_0} &= 3i\partial_0 + iy\partial\partial_0 - y\partial_1\partial_2\partial_3 + i(x_0\partial_0 + x_1\partial_1 + x_2\partial_2 + x_3\partial_3)\,\partial_0 \\ E_{-\gamma_1} &= iy\partial_1\partial + i(2 + x_0\partial_0 + x_1\partial_1)\,\partial_1 - \frac{x_2x_3}{y}\,\partial_0 \\ E_{-\omega} &= 3i\partial + iy\partial^2 + \frac{i}{y} + ix_0\partial_0\partial + \frac{x_1x_2x_3}{y^2}\,\partial_0 + \\ & + \frac{i}{y}(x_1x_2\partial_1\partial_2 + x_3x_1\partial_3\partial_1 + x_2x_3\partial_2\partial_3) \\ & + \ i(x_1\partial_1 + x_2\partial_2 + x_3\partial_3)\,(\partial + \frac{1}{y}) + x_0\partial_1\partial_2\partial_3 \, , \end{aligned}$$

Example: $D4 = SO(4,4)$ *(continued)*

$$
\begin{aligned}
H_{\beta_0} &= -y\partial + x_0\partial_0 \\
H_{\alpha_1} &= -1 - x_0\partial_0 + x_1\partial_1 - x_2\partial_2 - x_3\partial_3 \\
H_{\alpha_2} &= -1 - x_0\partial_0 - x_1\partial_1 + x_2\partial_2 - x_3\partial_3 \\
H_{\alpha_3} &= -1 - x_0\partial_0 - x_1\partial_1 - x_2\partial_2 + x_3\partial_3 \,,
\end{aligned}
$$

- Spherical vector: solve PDE $(E_\alpha - E_{-\alpha})f = 0$:

  $$f_{D_4} = \frac{4\pi}{|z|} K_0 \left(\frac{\sqrt{(|z|^2 + x_1^2)(|z|^2 + x_2^2)(|z|^2 + x_3^2)}}{|z|^2} \right) e^{-i \frac{x_0 x_1 x_2 x_3}{y|z|^2}}$$

  where $z = y + i x_0$. This is manifestly invariant under $SO(4,4)$ triality, permuting x_1, x_2, x_3.

- Rk: this minimal representation is equivalent to the one arising from the string worldsheet instantons on T^4: by Fourier transforming on x_3 and renaming variables, we can rewrite f as

  $$f = \frac{e^{-2\pi\sqrt{(m^{ij})^2}}}{\sqrt{(m^{ij})^2}} \,, \qquad \epsilon^{ijkl} m_{ij} m_{kl} = 0$$

  This implies that the one-loop BPS amplitude of Het/T^4 is invariant under $SO(4,4)$ triality, as predicted from Heterotic-Type II duality.

Kiritsis Obers BP

E_6 *theta series and Membrane*/T^3

- For E_6 the minimal nilpotent orbit is parameterized by 11 positions (y, x_0, M^i_α) transforming as $1 + 1 + (3,3)$ under the linearly represented subgroup $H_0 = Sl(3) \times Sl(3)$. The cubic form is simply $I_3 = \det(M)$.

- The spherical vector invariant under the maximal compact $SU(8)$ can be obtained by integrating the PDE $(E_\alpha - E_{-\alpha})f = 0$:

$$f_{E_6} = \frac{e^{-(S_1 + iS_2)}}{|z|^2 S_1} \ , \qquad z = y + ix_0$$

$$S_1 = \frac{\sqrt{\det(MM^t + |z|^2 \mathbb{I}_3)}}{|z|^2} \ , \quad S_2 = \frac{x_0 \det(M)}{y|z|^2}$$

- The M variables can be identified with the winding numbers of the membrane $X^i = \mathcal{M}^i \alpha \sigma^\alpha$. z looks like a complex scalar on the worldvolume, or rather an $Sl(2)$ doublet of 3-form field strengths: no dof, cosmological constant on the worldvolume.

- The representation satisfies identically

$$\Delta_{Sl(3)_1} = \Delta_{Sl(3)_2} = \Delta_{Sl(3)_3}$$

  which guarantees the SUSY identity after integrating over wv Sl(3).

Exceptional theta series

- For E_7 the minimal nilpotent orbit is parameterized by 17 positions (y, x_0, M_{ij}) transforming as 1+1+15 under the linearly represented subgroup $H_0 = Sl(6)$. The cubic form is simply $I_3 = \mathrm{Pf}(M)$.

- For E_8 the minimal nilpotent orbit is parameterized by 29 positions (y, x_0, M_i) transforming as 1+1+27 under the linearly represented subgroup $H_0 = E_6$. The cubic form is the symmetric cubic invariant of 27^3.

- The spherical vector invariant under the maximal compact $USp(8)$ can be obtained by integrating the PDE $(E_\alpha - E_{-\alpha})f = 0$:

$$f_{E_n} = \frac{\mathcal{K}_{s/2}(S_1)}{|z|^{(s+1)}} e^{iS_2}, \qquad z = y + ix_0$$

$$S_1 = ||(X, \nabla_X\Big(\frac{I_3}{|z|}\Big))|| \ , \quad S_2 = \frac{x_0 I_3}{y|z|^2}$$

  where $X = (y, x_0, M_i)$, $\mathcal{K}_t(x) \equiv x^{-t} K_t(x)$ and $s = 1, 2, 4$ for $n = 6, 7, 8$.

- The same formula holds over the p-adic field, upon replacing the orthogonal norm by the p-adic norm $||(x_i)|| = \max(|x_i|_p)$, and the Bessel function by $\mathcal{K}_s(x) \to (1 - p^s x^{-s})/(1 - p^s)$.

Kazhdan Polischuk, to appear

Summary - prospects

- We have obtained explicit formulae for theta series for exceptional groups $E_{6,7,8}$. The real spherical vector yields the kernel, and the summation measure is given by the product of all p-adic spherical vectors. Can one find a simple combinatoric formula for this summation measure ? Can one understand degenerate contributions ?

- This construction is based on the quantization of the phase space of an exotic conformal quantum mechanical model. Any deeper meaning or application to motion on black hole moduli space ?

- The minimal representation relies on the invariance of the cubic character $\exp(iI_3(x)/x_0)$ under Fourier transform: a class of non-Gaussian cubic models which are nonetheless free. Can models be found with ∞ degrees of freedom ? The topological open membrane ? Quartic or higher interactions ?

Summary - prospects (continued)

- By restricting to maximal subgroups $E_d \times Sl(3)(\times R^+)$, one obtains theta correspondences with symmetries appropriate for the one-loop BPS membrane. Additional quantum numbers are required for the construction: what is their interpretation ? Can the zero-mode action S be generalized to include fluctuations while preserving duality ?

- What is the meaning of the extra symmetries mixing worldvolume and target space moduli ?

- Could one derive the summation measure from the SUSY index of the $U(N)$ superconformal field theory on N membranes ? Can the adelic formula be derived by considering a p-adic generalization of the supermembrane ?

- For the E_8 case (M theory on T^6), can one recover the five-brane instantons as well ? Insights on membrane/five brane duality ?

- Can these exceptional theta series describe physical amplitudes in M-theory on $T^{6,7,8}$?