D-effects in String Theory

Boris Pioline
CERN & Ecole Polytechnique

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based on work in collaboration with **E. Kiritsis**,
hep-th/9707018 and hep-th/9710078,
I. Antoniadis and **T. Taylor,**
hep-th/9707222

transparencies available from
`http://wwwinfo.cern.ch/~bpioline/academic.html`

Beyond String Perturbation Theory

- String Theory is defined as an **asymptotic series** in the string coupling $g \to 0$.

- **String dualities** extend this domain to $g \to \infty$ but one is still exploring a *measure zero* domain of the moduli space. Only a few quantities are completely fixed by dualities.

- Attempts for a **non-perturbative** *ab initio* formulation of String Theory are still uncertain.

- A less ambitious approach would be to extend String Theory to the **semiclassical regime**:
  invent the **instanton calculus** of String Theory.

- The Detective approach: gather all hints from the exact results to reconstitute the plot.

Non-perturbative Effects in String Theory

- Perturbative amplitudes are computed from a **genus expansion**:

$$\mathcal{A} = \sum_{n=0}^{\infty} \int_{\mathcal{M}(\Sigma_n)} d\mu \quad \mathcal{Z}_n \; g^{\chi(\Sigma_n)} \ ,$$

  where Riemann surfaces are weighted according to their Euler Number:

  $\chi(\Sigma) = 2\#\text{handle} - 2 - \#\text{boundaries} - \#\text{crosscaps}$

- Field Theory Instantons $\subset$ String Theory Instantons

- The *large order behaviour* differs drastically from the usual field theory case:

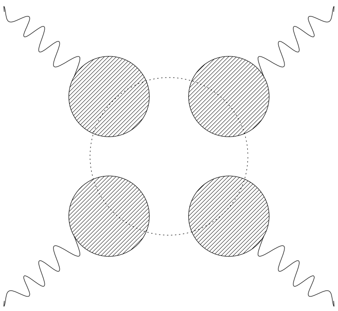
$$\mathcal{A}^{F.T} \sim \sum C^{-l} l! \; g^{2l} \sim e^{-C/g^2}$$

  whereas

$$\mathcal{A}^{S.T} \sim \sum C^{-2l} (2l)! \; g^{2l} \sim e^{-C/g}$$

  Shenker

- **D-branes** provide a source for such **D-effects**:



The sum over connected disks exponentiates to yield a $e^{-O(1/g)}$ contribution.

Polchinski

- Instantons effects are only useful insofar as they are not dominated by perturbative effects: non-renormalization theorems.

- In 4D, holomorphy can prevent such $O(e^{-1/g} = e^{\sqrt{S_2}})$ effects: *vector geometry* of $N = 2$ backgrounds.

- The first effects of this kind were computed in the context of corrections to the *hypermultiplet geometry* in type II $N = 2$ Calabi-Yau compactifications:

$$\Delta_{D2} R_{IJKL} \propto e^{-\frac{1}{g}e^{-K}\left|\int \Omega\right| - i\int C} \int d_I \int d_J \int d_K \int d_L$$

Becker,Becker,Strominger

and the resolution of the conifold singularity:

$$V = \frac{1}{2} \log \frac{\mu^2}{|\int \Omega|^2} + \sum_{m \neq 0} K_0 \left(2\pi m \frac{1}{g} \left| \int \Omega \right| \right) e^{2\pi i m x}$$

Ooguri,Vafa

Soliton Loops $\Leftrightarrow$ Instanton Effects

- Based on dualities, exact results involving infinite sums of instanton effects have been obtain for various physical couplings.

Harvey,Moore
Green,Gutperle,Vanhove;Kiritsis,P.
Bachas,Fabre,Kirisis,Obers,Vanhove
Antoniadis,P.,Taylor
...

Some Wisdom about Instanton Effects

- Instantons are **saddle points** of the euclidean fundamental action. The **integration measure** for the collective coordinates is obtained by a change of variables from the canonical microscopic fields. Gaussian fluctuations around the instanton background give rise to the **fluctuation determinant**:

$$\langle \quad \rangle = \sum_{\text{sectors}} \int d\mu \langle \quad \rangle_I \frac{1}{\sqrt{\det Q}} e^{-S_I}$$

- Part of the collective coordinates ϕ are generated by the *symmetries broken by the instanton background* (translations, dilations). Integration over ϕ restores the broken symmetries.

- In the presence of SUSY, the bosonic and fermionic oscillators cancel, leaving the zero-modes only. The SUSY generators broken by the instanton background generate **fermionic zero-modes** to be saturated by the vertex insertions $\Rightarrow$ non-renormalization theorems

- When the supersymmetry is extended, several SUSY breaking patterns are possible. 1/2-BPS instantons would contribute to:

N=8	32 SUSY	ψ^{16}	R^4
N=4	16 SUSY	ψ^8	R^2, F^4
N=2	8 SUSY	ψ^4	$(\partial\phi)^2$
N=1	4 SUSY	ψ^2	$V(\phi)$

Instanton Configurations in String Theory

The low energy Supergravity action displays a variety of 1/2 BPS solutions:

- the NS 5-brane charged under $B_{\mu\nu}$ (Het,IIA/B):

$$S \sim \frac{1}{g^2}$$

- the Dp-branes (IIA p even, IIB p odd, I $p = 1,5$):

$$S_{BI} = \int \frac{1}{g}\sqrt{\det(\hat{G} + \hat{B} + F)} + i e^{\hat{B}+F} \wedge \underbrace{\sum \mathcal{A}}_{\text{RR potentials}}$$

  induced by fluctuations of open strings attached to the brane.

D-branes whose **euclidean** world-volume is wrapped around a **supersymmetric** cycle of the compactification manifold yield **1/2 BPS** D-instantons.

T-duality and D-brane dynamics

- Open strings in an electric background field A_μ:

$$
\begin{aligned}
Z(A_\mu) =& \langle e^{i\int d\tau A_\mu(X)\partial_\tau X^\mu}\rangle_{\text{disk}} \\
&= \int d^D x \frac{1}{g}\sqrt{-\det\left(\eta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}\right)}
\end{aligned}
$$

$\Rightarrow$ the dynamics of the gauge field is given at tree level by **Born-Infeld electrodynamics.**

Fradkin, Tseytlin

- Under T-duality along circle i,

$$
\begin{cases} \partial_\sigma A_i = 0 (\mathrm{N}) \\ A_i(X)\partial_\tau X^i \end{cases} \longrightarrow \begin{cases} \partial_\tau A_i = 0 (\mathrm{D}) \\ A_i(X)\partial_\sigma X^i \end{cases}
$$

A_i now couples to the momentum flow across the boundary, and therefore describes the *transverse fluctuations of the brane*. $A_{j\neq i}$ corresponds to the *gauge degrees of freedom on the brane worldvolume*. The dynamics of the brane is given by the **Dirac-Born-Infeld action**:

$$
S_{BI} = \int \frac{1}{g}\sqrt{\det(\widehat{G} + F)} \ + \ \text{RR couplings}
$$

where $\widehat{G}, \widehat{B}$ are the target space metric and $B_{\mu\nu}$ field pulled back to the worldvolume with the embedding coordinates.

- The BI limiting electric field now turns into the relativistic limiting velocity.

Bachas

- In the presence of a background $B_{\mu\nu}$ field, gauge invariance under $B \to B + d\Lambda$ requires that F should be replaced by $F + \widehat{B}$.

- N branes interact through open strings, resulting in a $U(N)$ **non-abelian gauge theory**. The D-brane coordinates become *non-commutative*, and the notion of the D-brane positions is only recovered at large separations, when $U(N)$ is broken to $U(1)^N$. Supersymmetric vacua of the gauge theory correspond to **D-brane bound states**. There has to be a single bound state of N D0-branes for each N.

Witten

U-duality in N=8 type II theories

Type II theories compactified on a N torus give $N=8$ maximally supersymmetric theories in dimension $D = 10-D$, with U-duality groups $E_{11-D}(\mathbb{Z})$:

D	*Scalar manifold*	*U-duality*
9	$Sl(2,\mathbb{R})/U(1)\times\mathbb{R}_+$	$Sl(2,\mathbb{Z})$
8	$Sl(2,\mathbb{R})/U(1)\times Sl(3,\mathbb{R})/SO(3)$	$Sl(2,\mathbb{Z})\times Sl(3,\mathbb{Z})$
7	$Sl(5,\mathbb{R})/SO(5)$	$Sl(5,\mathbb{Z})$
6	$SO(5,5)/SO(5)\times SO(5)$	$SO(5,5,\mathbb{Z})$
5	$E_6/Usp(8)$	$E_6(\mathbb{Z})$
4	$E_7/SU(8)$	$E_7(\mathbb{Z})$
3	$E_8/SO(16)$	$E_8(\mathbb{Z})$

Cremmer,Julia

- The **perturbative T-duality group** is $\mathrm{S}O(N,N,\mathbb{Z})_T$. It leaves the D-dimensional dilaton $e^{-2\phi_D}=V\ e^{-2\phi}$ invariant. The extra $\mathbb{Z}_2$ in $O(N,N,\mathbb{Z})$ sends type IIA to type IIB. $Sl(N,\mathbb{Z})\subset O(N,N,\mathbb{Z})$ reparametrizes the torus. RR scalars form a spinorial representation under $SO(N,N,\mathbb{R})$.

- From the type IIA point of view, $Sl(N,\mathbb{Z})$ is enhanced to the **M-theory reparametrization group** $\mathrm{S}l(N+1,\mathbb{Z})_M$. The $N+1$-torus has metric

$$ds^2_{11}=R^2_{11}(dx^{11}+\mathcal{A}_i dx^i)^2+\frac{1}{R^{11}}dx^i g_{ij}dx^j,$$

whre $R_{11}=e^{2\phi/3}$,and $B_{\mu\nu}$ corresponds to $C_{11;\mu\nu}$.

Witten

- From the type IIB point of view, the **D=10 symmetry** $\mathrm{S}l(2,\mathbb{Z})_B$ subsists in lower dimensions, acting as a modular transformation on $\tau = a + ie^{-\phi}$, and leaving $\nu = e^{\phi}/V^{4/N}$ invariant. The $Sl(2,\mathbb{Z})_B$ symmetry can be encoded in a 12-dimensional *F-theory* description, but there is no resulting $Sl(N+2,\mathbb{Z})$ symmetry.

  Vafa

- The groups $SO(N,N,\mathbb{Z})_T$,$Sl(N+1,\mathbb{Z})_M$ and $Sl(2,\mathbb{Z})_B$ generate the full $E_{11-D}(\mathbb{Z})$ U-duality group (Conjecture/definition).

Non-perturbative R^4 couplings in $D = 10, N = 8$ type IIB String Theory

$$S_{R^4} = \mathcal{N}_{10} \int d^{10}x \sqrt{-g_E}\ f_{10}^B\ (t_8 t_8\ R^4 + \frac{1}{8}\epsilon_{10}\epsilon_{10} R^4)$$

A (tree-level + one-loop) perturbative computation yields ($e^\phi = g$):

$$f_{10}^B = e^{\phi/2}\ \left(2\zeta(3) e^{-2\phi} + \frac{2\pi^2}{3}\right)$$

However, under the U-duality group $Sl(2, \mathbb{Z})$, the scalar $\tau = a + ie^{-\phi}$ transforms as a modular parameter. f_{10} should be an invariant function:

$$\begin{aligned} f_{10}^B &= \sum_{(m,n)\neq(0,0)} \left(\frac{\tau_2}{|m+n\tau|^2}\right)^{3/2} \\ &= 2\zeta(3) e^{-3\phi/2} + \frac{2\pi^2}{3} e^{\phi/2} + 0 \\ &+ 4\pi e^{-\phi/2} \hat{\sum_m}\hat{\sum_n} \left|\frac{m}{n}\right| K_1\left(2\pi e^{-\phi}|mn|\right) e^{2\pi i m n a} \\ &\sim 4\pi \hat{\sum_m}\hat{\sum_n} |mn^3|^{-1/2} e^{-2\pi |mn| (e^{-\phi} \pm ia)} \left(1 + O(e^\phi)\right) \end{aligned}$$

Green, Gutperle

This exhibits the tree-level and one-loop contributions, **no extra perturbative contribution**, and an infinite sum of non-perturbative effects imputable to the 10D type IIB (anti)**D-instantons**.

Non-holomorphic Modular Functions

Consider the modular invariant weight-s Eisenstein series:

$$E_s(\tau,\bar\tau) = \sum_{(m,n)\neq(0,0)} \left(\frac{\tau_2}{|m+n\tau|^2}\right)^s .$$

Small coupling means $\tau_2 = 1/g \to \infty$.

- The leading contribution comes from $n=0$:

$$E_s \sim 2\zeta(2s)\tau_2^s$$

- When $n \neq 0$, all m's contribute roughly the same: *Poisson resum* on m:

$$\sum_{m;n\neq 0} \left(\frac{\tau_2}{|m+n\tau|^2}\right)^s = \frac{\pi^s}{\Gamma(s)} \int \frac{dt}{t^{1+s}} \sum e^{-\frac{\pi}{t}\frac{|m+n\tau|^2}{\tau_2}}$$

$$= \frac{\pi^s\sqrt{\tau_2}}{\Gamma(s)} \int \frac{dt}{t^{s+1/2}} \sum_{\tilde m;n\neq 0} e^{-\pi\tau_2\left(t\tilde m^2+\frac{n^2}{t}\right)+2\pi i\tilde m n\tau_1}$$

- This is a saddle integral except when $\tilde m = 0$, which yield a second perturbative contribution:

$$2\sqrt{\pi}\zeta(2s-1)\frac{\Gamma(s-1/2)}{\Gamma(s)}\tau_2^{1-s}$$

- The remaining terms are given by Bessel functions:

$$\frac{\pi^s\sqrt{\tau_2}}{\Gamma(s)} \hat{\sum_{\tilde m}} \hat{\sum_{n}} \left|\frac{\tilde m}{n}\right|^{s-1/2} K_{s-1/2}\left(2\pi\tau_2|\tilde m n|\right) e^{2\pi i\tilde m n\tau_1}$$

- The two perturbative terms $O(e^{-s\phi})+O(e^{(s-1)\phi})$ are separated by l loops when $s=l+1/2 \Rightarrow E_{3/2}$ may be relevant for R^4 couplings.

- The overall power of the dilaton fits the dimension of the R^4 operator, and the perturbative coefficients as well.

- The instanton expansion defines E_s by analytic continuation to all $s \neq 1$, where there is a pole $\pi/(s-1)$.

- Eisenstein series are eigenvectors of the Laplacian:

$$\Delta E_s = s(s-1)E_s$$

  but only $E_{1/2}$ is square integrable.
  E_{1-s} is proportional to E_s.

- The Laplacian also has a discrete spectrum of **cusp forms** for $s \in 1/2 + i\,\mathbb{R}$, that would not affect the perturbative contributions.

Compactification on a circle

- Compactification of type IIB String Theory on a circle does not bring in additional D-instantons. The D=9 R^4 coupling is therefore the same as in D=10 modulo a volume factor (Just as fine since the duality group in D=9 is still $Sl(2,\mathbb{Z})$).

- T-duality allows to derive the type IIA result from the type IIB:

$$f_9^A = e^{2\phi/7}\left(2\zeta(3) r_A e^{-2\phi} + \frac{2\pi^2}{3r_A} + 0 \right. \\ \left. + \; 4\pi e^{-\phi} \hat{\sum_m} \hat{\sum_n} \left|\frac{m}{n}\right| K_1\left(2\pi r_A e^{-\phi}|mn|\right) e^{2\pi i mn\mathcal{A}}\right)$$

  reproducing the tree-level and one-loop contributions together with the contributions of D0-brane instantons wrapped on the circle.

- Upon decompactifying the circle, the **D0-brane** contributions vanish, and one is left with the tree-level result only:

$$f_{10}^A = 2\zeta(3) e^{-3\phi/2}$$

  (but there is also a one-loop $(t_8 t_8 - \frac{1}{8}\epsilon_{10}\epsilon_{10})R^4$ coupling in type IIA D=10)

Compactification of type IIB on a 2-torus

- Upon compactification of type IIB String Theory on a 2-torus, we expect additional effects from **D-string instantons** to enhance the U-duality symmetry to $Sl(3,\mathbb{Z})_{T,\phi,\mathcal{B},a} \times Sl(2,\mathbb{Z})_U$.

- Up to one-loop order, the R^4 coupling reads

$$f_8^B = 2\zeta(3) T_2 e^{-2\phi} - 2\pi \log T_2 |\eta(T)|^4,$$

only invariant under $Sl(2,\mathbb{Z})_T \subset Sl(3,\mathbb{Z})$. It is however invariant (independent) under $Sl(2,\mathbb{Z})_U$.

- The definition of the Eisenstein series can be extended to

$$E_s^{Sl(3,Z)} = \hat{\sum_{m_1,m_2,m_3}} \left[m^t M m \right]^{-s}$$

where M is the det 1 symmetric 3x3 matrix parametrizing the $SL(3,\mathbb{R})/SO(3,\mathbb{R})$ scalar manifold:

$$m^t M m = \nu^{1/3} \left(\frac{|m_1 + \tau m_2 + (\mathcal{B} + \tau B) m_3|^2}{\tau_2} + \frac{m_3^2}{\nu} \right)$$

$$\nu = 1/(\tau_2 V^2)$$

- The natural conjecture is therefore:

$$f_8^B = E_{3/2}^{Sl(3,\mathbb{Z})}.$$

Kiritsis,P.

Upon expansion in weak coupling:

$$
\begin{aligned}
f_8^B &= 2\zeta(3) T_2 e^{-2\phi} - 2\pi \log T_2 |\eta(T)|^4 \\
&\quad + 4\pi V \tau_2 \hat{\sum_m} \hat{\sum_{n,q}} \frac{|m|}{\sqrt{(n+qB)^2 + V^2 p^2}} \cdot \\
&\qquad K_1 \left((2\pi |m| e^{-\phi} \sqrt{(n+qB)^2 + V^2 q^2} \right) e^{2\pi i m (q(\mathcal{B}+aB)+na)} \\
&\quad - \pi \log \left(\tau_2 \nu^{1/3} \right).
\end{aligned}
$$

This exhibits the tree-level and one-loop contributions, together with a sum of non-perturbative terms corresponding to **bound states of D-instantons and D-strings**:

- $q = 0, n \neq 0$: $S = |n|(e^{-\phi} \pm ia)$ corresponds to the good old D-instantons,

- $q \neq 0, n = 0$:

$$
\begin{aligned}
S =& |q| \left(e^{-\phi} \sqrt{V^2 + B^2} \pm i(\mathcal{B} + aB) \right) \\
=& \int e^{-\phi} \sqrt{\det(\hat{G} + \hat{B})} + i(\hat{\mathcal{B}} + a\hat{B})
\end{aligned}
$$

  corresponds to the D-strings wrapped q times on T^2,

- $q \neq 0, n \neq 0$ correspond to D-strings **with a non vanishing flux** $F = n$ of the $U(1)$ gauge field. $\Rightarrow$*D-instantons appear as fluxes on the D-string.*
  Douglas;Witten

- There is a non-perturbative logarithmic term $-\pi \log\left(\tau_2 \nu^{1/3}\right)$ coming from the regularization of the divergence of $E_{3/2}$.

- Upon Poisson resummation on the **D-instanton number** $n \to p$:

$$f_8^B = 2\zeta(3) T_2 e^{-2\phi} - 2\pi \sum_{(p,q)=1} \log T_2^{(p,q)} |\eta(T^{(p,q)})|^4 - \pi \log\left(\tau_2 \nu^{1/3}\right).$$

where $T^{p,q} = (q\mathcal{B} - pB) + i|p + q\tau| T_2$ is the T parameter seen by a (p,q) string with tension $\alpha^{(p,q)} = \alpha'/|p + q\tau|$, p and q coprime.

$\Rightarrow$ the D=8 R^4 coupling can be interpreted **either as a sum of instanton effects or a sum of soliton loops.**

D0-branes instantons in type IIA

In terms of type IIA variables, the $D = 8$ R^4 coupling reads:

$$f^A = 2\zeta(3) V e^{-2\phi} + 2V \hat{\sum_{n^i}} \frac{1}{n^i g_{ij} n^j}$$
$$+ 4\pi V \tau_2 \hat{\sum_{m}} \hat{\sum_{n^i}} \frac{|m|}{\sqrt{n^i g_{ij} n^j}} K_1 \left(2\pi |m| e^{-\phi} \sqrt{n^i g_{ij} n^j} \right) e^{2\pi i m n^i \mathcal{A}_i}$$

This result gives the tree-level, one-loop (degenerate) contributions together with the D0-brane instantons for any compactification. It has a nice interpretation from the D=11 M-theory point of view:

$$f^A = 2\pi V_{11} \int_0^\infty \frac{dt}{t^{5/2}} \hat{\sum_{n^I}} e^{-\frac{\pi}{t} n^I g_{IJ} n^J} = V_{11} \hat{\sum_{n^I}} \left(n^I g_{IJ} n^J \right)^{-3/2}$$

is the *one-loop D=11 SUGRA scattering amplitude of four gravitons*.

Green, Gutperle, Vanhove

It however has to be supplemented with **membrane and five-brane** contributions, in order to enhance the duality group from $Sl(11-D, \mathbb{Z})$ to $E_{11-D}(\mathbb{Z})$. The contribution of these objects can be obtained either by *postulating* a result invariant under the full duality group, or more conveniently by *applying a sequence of T, $Sl(11-D)_M$ and , $Sl(10-D)_B \times Sl(2,\mathbb{Z})_B$ dualities.*

P., Kiritsis

From D0-branes to Dp-branes

Start from

$$S_{D0} = e^{-\phi}\sqrt{n^i g_{ij} n^j} + i\ n^i \mathcal{A}_i.$$

T-dualize on one circle, by first decomposing the metric as

$$R^2(dx^1 + A_a dx^a)^2 + dx^a g_{ab} dx^b$$

and applying a duality

$$R \leftrightarrow 1/R, \qquad A_a \leftrightarrow B_{1a}, \qquad \mathcal{A}_a \leftrightarrow \mathcal{B}_{1a}, \qquad \mathcal{A}_1 \leftrightarrow a,$$
$$B_{ab} \leftrightarrow B_{ab} - A_a B_b + B_a A_b, \qquad e^{-2\phi} R = \text{const.}$$

The action becomes

$$S_{D0} \to e^{-\phi}\sqrt{(n^1 + B_{1a} n^a)^2 + n^a\ g_{11} g_{ab}\ n^b} + i\ (n^1 a + n^a \tilde{\mathcal{B}}_{1a}).$$

Completing the $Sl(10-D)$ symmetry by introducing a **two-form integer charge** n^{ij} such that $n^{1a} = n^a$, and reinterpreting n^1 as a singlet, gives the **type IIB D-string** action:

$$S_{D1} = e^{-\phi}\sqrt{\left(n + \frac{1}{2} n^{ij} B_{ij}\right)^2 + \frac{1}{2} n^{ij}\ g_{ik} g_{jl}\ n^{kl}} + i\ (n a + \frac{1}{2} n^{ij} \mathcal{B}_{ij}).$$

n^{ij} describes the wrappings of the D-string world-sheet on the N-torus, generalizing the q integer of $N = 2$.

This game can be played *ad libitum* and gives the higher brane contributions:

$$
\begin{aligned}
S_{D2} =& e^{-\phi}\left[\left(n^i+\frac{1}{2}n^{ijk}B_{jk}\right)g_{il}\left(n^l+\frac{1}{2}n^{lmn}B_{mn}\right)\right.\\
&\left.+\frac{1}{6}n^{ijk}\ g_{il}g_{jm}g_{kn}\ n^{lmn}\right]^{1/2}+i\ \left(n^i\mathcal{A}_i+\frac{1}{6}n^{ijk}\mathcal{C}_{ijk}\right)\\
S_{D3} =& e^{-\phi}\left[\left(n+\frac{1}{2}n^{ij}B_{ij}+\frac{1}{8}n^{ijkl}B_{ij}B_{kl}\right)^2\right.\\
&+\frac{1}{2}\left(n^{ij}+\frac{1}{2}n^{ijkl}B_{kl}\right)g_{im}g_{jn}\left(n^{mn}+\frac{1}{2}n^{mnpq}B_{pq}\right)\\
&\left.+\frac{1}{24}n^{ijkl}g_{im}g_{jn}g_{kp}g_{lq}n^{mnpq}\right]^{1/2}\\
&+i\ \left(na+\frac{1}{2}n^{ij}\tilde{\mathcal{B}}_{ij}+\frac{1}{24}n^{ijkl}\tilde{\mathcal{D}}_{ijkl}\right)
\end{aligned}
$$

This is the Born-Infeld action

$$
S_{BI}=\int\frac{1}{g}\sqrt{\mathrm{det}(\widehat{G}+\widehat{B}+F)}+ie^{\widehat{B}+F}\wedge\mathcal{A}_{RR},
$$

evaluated on a 3-brane configuration wrapped on a N-torus:

$$
X^i(\sigma^\alpha)=N^i_\alpha\sigma^\alpha,\qquad n^{ijkl}=\epsilon^{\alpha\beta\gamma\delta}N^i_\alpha N^j_\beta N^k_\gamma N^l_\delta,
$$

$$
n^{ij}=\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}N^i_\alpha N^j_\beta F_{\gamma\delta},\qquad n=\frac{1}{8}\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F\gamma\delta
$$

The precise weighting of these states is also obtained.

U-duality completion

Up to now, the couplings are only guaranteed to be invariant under T-duality. Invariance under U-duality is best seen in the **M-theory representation**:

$$A_4^{D2} = V_{11} \hat{\sum} \left(\mathcal{M}^2 \right)^{-3/2}$$

with (taking only the D0 and D2 into account)

$$\begin{aligned}\mathcal{M}^2 =& R_{11}^2 \left(n^{11} + \mathcal{A}_i n^i + \frac{1}{6} n^{ijk} \tilde{\mathcal{C}}_{ijk} \right)^2 \\ &+ \left(n^i + \frac{1}{2} n^{ijk} B_{jk} \right) \frac{g_{il}}{R_{11}} \left(n^l + \frac{1}{2} n^{lmn} B_{mn} \right) \\ &+ \frac{R_{11}^2}{6} n^{ijk} \ \frac{g_{il} g_{jm} g_{kn}}{R_{11}^3} \ n^{lmn}\end{aligned}$$

$Sl(11-D, \mathbb{Z})$ invariance w.r.t the metric

$$ds_{11}^2 = R_{11}^2 (dx^{11} + \mathcal{A}_i dx^i)^2 + \frac{1}{R^{11}} dx^i g_{ij} dx^j$$

can be recovered by **extending the three-form $\mathbf{n}^{ijk}$ into a four-form $\mathbf{n}^{IJKL}$**:

$$\begin{aligned}\mathcal{M}^2 = & \left(n^I + \frac{1}{6} n^{IJKL} \mathcal{C}^{(11)}_{JKL} \right) g_{IM} \left(n^M + \frac{1}{6} n^{MNPQ} \mathcal{C}^{(11)}_{NPQ} \right) \\ & + \frac{1}{24} n^{IJKL} g_{IM} g_{JN} g_{KP} g_{LQ} n^{MNPQ} \ .\end{aligned}$$

- In terms of the type IIA theory, this means introducing an extra *four-form charge* n^{ijkl} when $D \leq 6$, contributing $O(1/g^4)$ to $\mathcal{M}^2$:

$$\frac{\mathcal{M}^2}{R_{11}^2} = \left(n^{11} + \mathcal{A}_i n^i + \frac{1}{6} n^{ijk} \tilde{\mathcal{C}}_{ijk} + \frac{1}{6} n^{ijkl} A_i C_{jkl}\right)^2 + \frac{1}{g^2}\left(n^i + \frac{1}{2} n^{ijk} B_{jk} + \frac{1}{2} n^{ijkl} A_j B_{kl}\right) g_{il} \left(n^l + \dots\right) + \frac{1}{6g^2} n^{ijk}\ g_{il} g_{jm} g_{kn}\ n^{lmn} + \frac{1}{24 g^4} n^{ijkl}\ g_{im} g_{jn} g_{kp} g_{lq}\ n^{mnpq}$$

  This implies $\mathbf{O}(1/g^2)$ **corrections** in the Eisenstein series expansion.

- In terms of the type IIB theory, this also introduces a four-form charge m^{ijkl} which, together with the D3-brane four-form charge n^{ijkl}, transforms as a doublet of $Sl(2,\mathbb{Z})_B$.

Unfortunately, although invariant under the duality group $SO(5,5,\mathbb{Z})$ the Eisenstein series seems to fail to reproduce the one-loop result for $D=6$. Moreover, duality in lower dimensions requires to introduce extra non-D-brane states at each step.

Obers,P.,work in progress

Conclusions

- R^4 couplings offer a wonderful window into the rules of semiclassical calculus in String Theory, and give an explicit realization of the idea that instanton sectors with any number of D-branes wrapped on any cycle should be incorporated in any type II computation.

- The interpretation is not univocal: it seems possible to interpret certain contributions both as instanton sums or soliton loops.

- A few surprises occur: states with any number of D-branes contribute, while they naively have more fermionic zero-modes. In addition, the non-abelian structure seems irrelevant.

  Bachas,Vanhove

- The exact identification of the instanton summation measure is still uncertain. Investigation of related couplings with the same type of instanton contributions should help desentangling this weight from the vertex contributions.

  Russo,Tseytlin
  Kehagias,Partouche
  Green,Gutperle,Kwon

- The derivation of R^4 couplings from a Yang-Mills computation is a major challenge to Matrix Theory. A first step has already been taken by identifying the $E_{11-D}(\mathbb{Z})$ symmetry at the level of the gauge theory.

  Elitzur,Giveon,Kutasov,Rabinovici

- Other 1/2BPS saturated terms in compactifications with lower supersymmetry (type II/K_3) are also accessible to analysis, and display sums of instantons wrapped on all cycles of the curved manifold.

  Antoniadis,P.,Taylor

- 1/4BPS saturated couplings (R^6 for $N=8$) should exhibit contributions from D-brane at angles.