Wall-crossing from quantum multi-centered BPS black holes

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based on work with J. Manschot and A. Sen,

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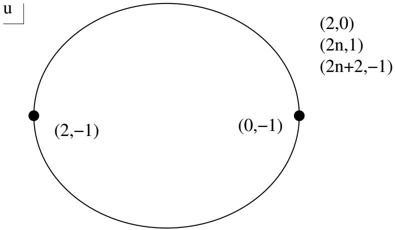
Introduction I

- In $D = 4$, $N = 2$ supersymmetric field and string theories, the exact spectrum of BPS states can often be determined at weak coupling, and extrapolated to strong coupling.
- E.g., in pure $SU(2)$ Seiberg-Witten theory,



Seiberg Witten; Bilal Ferrari

Introduction II

- In following the BPS spectrum from weak to strong coupling, one must be wary of two issues:
  - short multiplets may pair up into a long multiplet,
  - single-particle states may decay into multi-particle states.
- The first issue can be avoided by considering a suitable index $\Omega(\gamma, t)$, designed such that contributions from long multiplets cancel. $\Omega(\gamma, t)$ is then a piecewise constant function of the charge vector γ and couplings/moduli t.
- To deal with the second issue, one must understand how $\Omega(\gamma, t)$ changes across a wall of marginal stability W, where a single-particle state with charge γ can decay into a multi-particle state with charges $\{\alpha_i\}$, such that $\gamma = \sum_i \alpha_i$.

Introduction III

- Initial progress came from physics, by noting that single-particle states (in a certain limit) can be represented by multi-centered solitonic solutions. Those exist only on one side of the wall and decay into the continuum of multi-particle states on the other side.
- The simplest decay $\gamma \to \gamma_1 + \gamma_2$, where γ_1, γ_2 are primitive charge vectors, involves only two-centered configurations, whose index is easily computed:

$$\Delta\Omega(\gamma \to \gamma_1 + \gamma_2) = (-1)^{\gamma_{12}+1}\, |\gamma_{12}|\, \Omega^+(\gamma_1)\, \Omega^+(\gamma_2)$$

Denef Moore

- In the non-primitive case $\gamma = M\gamma_1 + N\gamma_2$ where $M, N > 1$, many multi-centered configurations in general contribute, and computing their index is non-trivial.

Introduction IV

- A general answer to this problem has come from the mathematical study of the wall-crossing properties of (generalized) Donaldson-Thomas invariants for Calabi-Yau three-folds, believed to be the mathematical translation of the BPS index $\Omega(\gamma)$ in type IIA CY vacua.
- Notably, Kontsevich & Soibelman (KS) and Joyce & Song (JS) gave two different-looking formulae for $\Delta\Omega(\gamma \to M\gamma_1 + N\gamma_2)$.
- The KS formula has already been derived/interpreted in several ways, e.g. by considering instanton corrections to the moduli space metric in 3D after compactification on a circle.

Gaiotto Moore Neitzke; Alexandrov BP Saueressig Vandoren

- Our goal will be to derive new wall-crossing formulae, based on the quantization of multi-centered solitonic configurations.

Outline

Outline

Preliminaries I

- We consider $\mathcal{N}=2$ supergravity in 4 dimensions (this includes field theories with rigid $\mathcal{N}=2$ as a special case). Let $\Gamma=\Gamma_e\oplus\Gamma_m$ be the lattice of electric and magnetic charges, with antisymmetric (Dirac-Schwinger- Zwanziger) integer pairing

$$\langle\gamma,\gamma'\rangle=\langle(p^\Lambda,q_\Lambda),(p'^\Lambda,q'_\Lambda)\rangle\equiv q_\Lambda p'^\Lambda-q'_\Lambda p_\Lambda\in\mathbb{Z}$$

- BPS states preserve 4 out of 8 supercharges, and saturate the bound $M\geq|Z(\gamma,t^a)|$ where $Z(\gamma,t^a)=e^{\mathcal{K}/2}(q_\Lambda X^\Lambda-p^\Lambda F_\Lambda)$ is the central charge/stability data.
- We are interested in the index $\Omega(\gamma;t^a)=\mathrm{Tr}_{\mathcal{H}'_\gamma(t^a)}(-1)^{2J_3}$ where $\mathcal{H}'_\gamma(t^a)$ is the Hilbert space of one-particle states with charge $\gamma\in\Gamma$ in the vacuum with vector moduli t^a.

Preliminaries II

- The BPS invariants $\Omega(\gamma; t^a)$ are locally constant functions of t^a, but may jump across codimension-one subspaces

$$W(\gamma_1, \gamma_2) = \{t^a \,/\, \arg[Z(\gamma_1)] = \arg[Z(\gamma_2)]\}$$

  where γ_1 and γ_2 are two primitive (non-zero) vectors such that $\gamma = M\gamma_1 + N\gamma_2$, $M, N \geq 1$. Assume for definiteness that $\gamma_{12} < 0$.
- We choose γ_1, γ_2 such that $\Omega(\gamma; t^a)$ has support only on the positive cone (root basis property)

$$\tilde{\Gamma} : \quad \{M\gamma_1 + N\gamma_2, \quad M, N \geq 0, \quad (M, N) \neq (0, 0)\} \, .$$

- Let $c_\pm$ be the chamber in which $\arg(Z_{\gamma_1}) \gtrless \arg(Z_{\gamma_2})$. Our aim is to compute $\Delta\Omega(\gamma) \equiv \Omega^-(\gamma) - \Omega^+(\gamma)$ as a function of $\Omega^+(\gamma)$ (say).

Wall-crossing from semi-classical solutions I

- Assume that $M(\gamma_1), M(\gamma_2) \gg \Lambda, m_P$. Single-particle states which are potentially unstable across W are described by classical configurations with n centers of charge $\alpha_i = M_i\gamma_1 + N_i\gamma_2 \in \tilde{\Gamma}$, satisfying $(M, N) = \sum_i (M_i, N_i)$.
- Such bound states exist only on one side of the wall, and the distances r_{ij} diverge at the wall. Across the wall, the single-particle bound state has decayed into the continuum of multi-particle states. $\Delta\Omega(\gamma)$ is given by the index of such configurations.
- *In addition, in either chamber, there may be multi-centered configurations whose charge vectors do not lie in $\tilde{\Gamma}$. However, they remain bound across W and do not contribute to $\Delta\Omega(\gamma)$.*

Wall-crossing from semi-classical solutions II

- In $\mathcal{N}=2$ supergravity (and presumably also in $\mathcal{N}=2$ Abelian gauge theories), the locations of the centers are constrained by

$$\sum_{j=1\dots n, j\neq i}^{n} \frac{\alpha_{ij}}{|\vec{r}_i - \vec{r}_j|} = c_i \,, \qquad \begin{cases} c_i = 2\,\mathrm{Im}\left[e^{-i\phi} Z(\alpha_i, t^a)\right] \\ \phi \equiv \arg[Z(\alpha_1 + \cdots \alpha_n, t^a)] \\ \alpha_{ij} \equiv \langle \alpha_i, \alpha_j \rangle \end{cases}$$

Denef

  If all $\alpha_i \in \tilde{\Gamma}$, the constants c_i are given by $c_i = \Lambda \sum_{i\neq j} \alpha_{ij}$, with $\Lambda \to \infty$ near the wall.

- After factoring out an overall translational mode, the solution space is (generically) a $(2n-2)$-dimensional symplectic manifold $(\mathcal{M}_n(\alpha_{ij}, c_i), \omega)$, with $\omega = \frac{1}{2} \sum_{i<j} \alpha_{ij} \sin\theta_{ij}\, \mathrm{d}\theta_{ij} \wedge \mathrm{d}\phi_{ij}$.

de Boer El Showk Messamah Van den Bleeken

- Up to issues of statistics, $\Delta\Omega(\gamma)$ is equal to the index of the SUSY quantum mechanics on $\mathcal{M}_n$, multiplied by the index $\Omega(\gamma_i)$ of the internal d.o.f. carried by each center.

Wall-crossing from semi-classical solutions III

- For primitive decay $\gamma \to \gamma_1 + \gamma_2$, the quantization of the phase space $(\mathcal{M}_2, \omega) = (S^2, \frac{1}{2}\gamma_{12} \sin\theta \, \mathrm{d}\theta \mathrm{d}\phi)$ reproduces the primitive WCF

$$\Delta\Omega(\gamma \to \gamma_1 + \gamma_2) = (-1)^{\gamma_{12}+1} \, |\gamma_{12}| \, \Omega^+(\gamma_1) \, \Omega^+(\gamma_2) \; ,$$

  where $(-1)^{\gamma_{12}+1} \, |\gamma_{12}|$ is the index of the angular momentum multiplet of spin $j = \frac{1}{2}(\gamma_{12} - 1)$.
- This generalizes to semi-primitive wall-crossing $\gamma \to \gamma_1 + N\gamma_2$: unstable configurations consist of a halo of n_s particles of charge $s\gamma_2$, with total charge $\sum s n_s \gamma_2 = n\gamma_2$, orbiting around a core of charge $\gamma_1 + (N - n)\gamma_2$. The phase space is $\mathcal{M}_n = \prod_s (\mathcal{M}_2)^{n_s} / S_{n_s}$.

Wall-crossing from semi-classical solutions IV

- Taking into account the Bose/Fermi statistics of the n_s identical particles, one arrives at a Mac-Mahon type partition function,

$$\frac{\sum_{N\geq 0} \Omega^-(1,N)\, q^N}{\sum_{N\geq 0} \Omega^+(1,N)\, q^N} = \prod_{k>0} \left(1-(-1)^{k\gamma_{12}} q^k\right)^{k\,|\gamma_{12}|\;\Omega^+(k\gamma_2)} .$$

Denef Moore

- E.g. for $\gamma \mapsto \gamma_1 + 2\gamma_2$,

$$\begin{aligned}\Delta\Omega(1,2) = & (-1)^{\gamma_{12}} \gamma_{12} \Omega^+(0,1)\, \Omega^+(1,1) + 2\gamma_{12}\, \Omega^+(0,2)\, \Omega^+(1,0) \\ & + \frac{1}{2}\gamma_{12}\, \Omega^+(0,1) \left(\gamma_{12}\Omega^+(0,1)+1\right) \Omega^+(1,0) .\end{aligned}$$

In particular, the term $\frac{1}{2}d(d+1)$ with $d = \gamma_{12}\Omega^+(0,1)$, reflects the projection on (anti)symmetric wave functions.

Wall-crossing from semi-classical solutions V

- It is instructive to rewrite the semi-primitive WCF using the rational BPS invariants, related to the usual integer invariants via

$$\bar{\Omega}(\gamma) \equiv \sum_{d|\gamma} \Omega(\gamma/d)/d^2 , \qquad \Omega(\gamma) = \sum_{d|\gamma} \mu(d)\, \bar{\Omega}(\gamma/d)/d^2$$

  where $\mu(d)$ is the Möbius function.
- Using the identity $\prod_{d=1}^{\infty}(1-q^d)^{\mu(d)/d} = e^{-q}$, or working backwards, one arrives at

$$\frac{\sum_{N\geq 0} \bar{\Omega}^{-}(1,N)\, q^N}{\sum_{N\geq 0} \bar{\Omega}^{+}(1,N)\, q^N} = \exp\left[\sum_{s=1}^{\infty} q^s (-1)^{\langle\gamma_1, s\gamma_2\rangle} \langle\gamma_1, s\gamma_2\rangle \bar{\Omega}^{+}(s\gamma_2)\right] .$$

- The same result follows by treating particles in the halo as distinguishable (satisfying Boltzmann statistics), and attaching an effective index $\bar{\Omega}(s\gamma_2)$!

Wall-crossing from semi-classical solutions VI

- One advantage is that $\Delta\bar\Omega(\gamma)$ takes a simpler form, and makes charge conservation manifest. E.g for $\gamma \mapsto \gamma_1 + 2\gamma_2$,

$$\begin{aligned}\Delta\bar\Omega(1,2) =& (-1)^{\gamma_{12}}\gamma_{12}\bar\Omega^+(0,1)\,\bar\Omega^+(1,1) + 2\gamma_{12}\,\bar\Omega^+(0,2)\,\bar\Omega^+(1,0)\\ &+ \frac{1}{2}\gamma_{12}\,\bar\Omega^+(0,1)^2\,\bar\Omega^+(1,0)\ .\end{aligned}$$

- The rational DT invariants $\bar\Omega(\gamma)$ appear in the JS formula, in constructions of modular invariant black hole partition functions, and in instanton corrections to hypermultiplet moduli spaces.

Joyce Song; Manschot; Alexandrov BP Saueressig Vandoren

The main conjecture I

- In general, we expect that the jump to be given by a finite sum

$$\Delta\bar{\Omega}(\gamma) = \sum_{n\geq 2} \sum_{\substack{\{\alpha_1,\dots\alpha_n\}\in\tilde{\Gamma} \\ \gamma=\alpha_1+\dots+\alpha_n}} \frac{g(\{\alpha_i\})}{|\mathrm{Aut}(\{\alpha_i\})|} \prod_{i=1}^{n} \bar{\Omega}^{+}(\alpha_i)\ ,$$

  over all unordered decompositions of the total charge vector γ into a sum of n vectors $\alpha_i \in \tilde{\Gamma}$. The symmetry factor $|\mathrm{Aut}(\{\alpha_i\})|$ reflects Boltzmannian statistics.

- $g(\{\alpha_i\})$ are universal factors depending only on the charges α_i, which should be given by the index of the supersymmetric quantum mechanics on $\mathcal{M}_n$.
- The KS and JS formulae give a mathematical prediction for these coefficients $g(\{\alpha_i\})$, which we shall compare with the index.

Outline

The Kontsevich-Soibelman formula I

- Consider the Lie algebra $\mathcal{A}$ spanned by abstract generators $\{e_\gamma, \gamma \in \Gamma\}$, satisfying the commutation rule

$$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \langle \gamma_1, \gamma_2 \rangle \, e_{\gamma_1+\gamma_2} \ .$$

- For a given charge vector γ and value of the VM moduli t^a, consider the operator $U_\gamma(t^a)$ in the Lie group $\exp(\mathcal{A})$

$$U_\gamma(t^a) \equiv \exp\left(\Omega(\gamma; t^a) \sum_{d=1}^{\infty} \frac{e_{d\gamma}}{d^2} \right)$$

- The operators e_γ / U_γ can be realized as Hamiltonian vector fields / symplectomorphisms of a twisted torus.

Gaiotto Moore Neitzke

The Kontsevich-Soibelman formula II

- The KS wall-crossing formula states that the product

$$A_{\gamma_1,\gamma_2} = \prod_{\substack{\gamma=M\gamma_1+N\gamma_2,\\ M\geq 0, N\geq 0}} U_\gamma \ ,$$

ordered so that $\arg(Z_\gamma)$ decreases from left to right, stays constant across the wall. As t^a crosses W, $\Omega(\gamma;t^a)$ jumps and the order of the factors is reversed, but the operator A_{γ_1,γ_2} stays constant. Equivalently,

$$\prod_{\substack{M\geq 0, N\geq 0,\\ M/N\downarrow}} U^{+}_{M\gamma_1+N\gamma_2} = \prod_{\substack{M\geq 0, N\geq 0,\\ M/N\uparrow}} U^{-}_{M\gamma_1+N\gamma_2} \ ,$$

The Kontsevich-Soibelman formula III

- The algebra $\mathcal{A}$ is infinite dimensional but filtered. The KS formula may be projected to any finite-dimensional algebra

$$\mathcal{A}_{M,N} = \mathcal{A}/\{ \sum_{m>M \text{ or } n>N} \mathbb{R} \cdot e_{m\gamma_1+n\gamma_2} \} \,.$$

  This projection is sufficient to infer $\Delta\Omega(m\gamma_1 + n\gamma_2)$ for any $m \le M, n \le N$, e.g. using the Baker-Campbell-Hausdorff formula.
- For example, the primitive WCF follows in $\mathcal{A}_{1,1}$ from

$$\begin{aligned} &\exp(\bar\Omega^+(\gamma_1)e_{\gamma_1}) \exp(\bar\Omega^+(\gamma_1+\gamma_2)e_{\gamma_1+\gamma_2}) \exp(\bar\Omega^+(\gamma_2)e_{\gamma_2}) \\ = &\exp(\bar\Omega^-(\gamma_2)e_{\gamma_2}) \exp(\bar\Omega^-(\gamma_1+\gamma_2)e_{\gamma_1+\gamma_2}) \exp(\bar\Omega^-(\gamma_1)e_{\gamma_1}) \end{aligned}$$

  and the order 2 truncation of the BCH formula

$$e^X\, e^Y = e^{X+Y+\frac{1}{2}[X,Y]} \,.$$

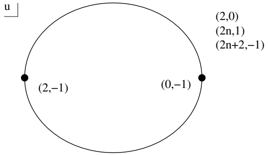
The Kontsevich-Soibelman formula IV

- In some simple cases, one may work in the full algebra $\mathcal{A}$, and use the “pentagonal identity"

$$U_{\gamma_2}\, U_{\gamma_1} = U_{\gamma_1}\, U_{\gamma_1+\gamma_2}\, U_{\gamma_2}\ , \qquad \gamma_{12} = -1$$

- Using this identity repeatedly, one can e.g. establish the wall-crossing identity in pure Seiberg-Witten $SU(2)$ theory,

$$U_{2,-1} \cdot U_{0,1} = U_{0,1} \cdot U_{2,1} \cdot U_{4,1} \dots U_{2,0}^{(-2)} \dots U_{3,-1} \cdot U_{2,-1}\, U_{1,-1}$$



Denef Moore; Dimofte Gukov Soibelman

The Kontsevich-Soibelman formula V

- Noting that the operators $U_{k\gamma}$ for different $k \geq 1$ commute, one may combine them into a single factor

$$V_\gamma \equiv \prod_{k=1}^{\infty} U_{k\gamma} = \exp\left(\sum_{\ell=1}^{\infty} \bar{\Omega}(\ell\gamma)\, e_{\ell\gamma} \right) , \qquad \bar{\Omega}(\gamma) = \sum_{d|\gamma} \Omega(\gamma/d)/d^2 \, .$$

and rewrite the KS formula as a product over primitive charge vectors only,

$$\prod_{\substack{M\geq 0, N\geq 0, \\ \gcd(M,N)=1, M/N\downarrow}} V^{+}_{M\gamma_1+N\gamma_2} = \prod_{\substack{M\geq 0, N\geq 0, \\ \gcd(M,N)=1, M/N\uparrow}} V^{-}_{M\gamma_1+N\gamma_2} \, ,$$

The Kontsevich-Soibelman formula VI

- Using the BCH formula, one easily derives the semi-primitive wcf formula, and generalizations to $\gamma \to 2\gamma_1 + N\gamma_2, \dots$.
- The fact that the algebra is graded by the charge lattice and the expression of V_γ guarantees that the jumps in the rational invariant will be of the form

$$\Delta\bar{\Omega}(\gamma) = \sum_{n\geq 2} \sum_{\substack{\{\alpha_1,\dots\alpha_n\}\in\tilde{\Gamma} \\ \gamma=\alpha_1+\dots+\alpha_n}} \frac{g(\{\alpha_i\})}{|\mathrm{Aut}(\{\alpha_i\})|} \prod_{i=1}^n \bar{\Omega}^+(\alpha_i) \,,$$

  with some universal coefficients $g(\{\alpha_i\})$.
- The Joyce-Song wall-crossing formula expresses $g(\{\alpha_i\})$ as a complicated sum over trees, permutations, etc.

Generic decay I

- When α_i have generic phases, $g(\{\alpha_i\})$ can be computed by projecting the KS formula to the subalgebra spanned by $e_{\sum \alpha_j}$ where $\{\alpha_j\}$ runs over all subsets of $\{\alpha_i\}$.
- E.g., for $n = 3$, assuming that the phase of the charges are ordered according to

$$\alpha_1,\ \alpha_1+\alpha_2,\ \alpha_1+\alpha_3,\ \alpha_1+\alpha_2+\alpha_3,\ \alpha_2,\ \alpha_2+\alpha_3,\ \alpha_3\ ,$$

we find

$$g(\{\alpha_1,\alpha_2,\alpha_3\}) = (-1)^{\alpha_{12}+\alpha_{23}+\alpha_{13}}\,\alpha_{12}\,(\alpha_{13}+\alpha_{23})$$

As we shall see later, this fits the macroscopic index of 3-centered configurations !

The motivic Kontsevich-Soibelman formula I

- KS have proposed a quantum deformation of their formula, which governs wall-crossing properties of motivic DT invariants $\Omega_{\rm ref}(\gamma; y, t)$. Physically, these correspond to the "refined index"

$$\Omega_{\rm ref}(\gamma, y) = {\rm Tr}'_{\mathcal{H}(\gamma)}(-y)^{2J_3} \equiv \sum_{n\in\mathbb{Z}} (-y)^n\, \Omega_{{\rm ref},n}(\gamma)\,,$$

  where J_3 is the angular momentum in 3 dimensions along the z axis. As $y \to 1$, $\Omega_{\rm ref}(\gamma; y, t) \to \Omega(\gamma; t)$.

  Dimofte Gukov; D G Soibelman

- Caution: this index (rather, a variant of it using a combination of angular momentum and $SU(2)_R$ quantum numbers) is protected in $\mathcal{N} = 2, D = 4$ field theories, but not in supergravity/string theory, where $SU(2)_R$ is generically broken.

  Gaiotto Moore Neitzke

The motivic Kontsevich-Soibelman formula II

- To state the formula, consider the Lie algebra $\mathcal{A}(y)$ spanned by generators $\{\tilde{e}_\gamma, \gamma \in \Gamma\}$, satisfying the commutation rule

$$[\tilde{e}_{\gamma_1}, \tilde{e}_{\gamma_2}] = \kappa(\langle\gamma_1, \gamma_2\rangle)\, \tilde{e}_{\gamma_1+\gamma_2}\ , \qquad \kappa(x) = \frac{(-y)^x - (-y)^{-x}}{y - 1/y}\ .$$

- To any primitive charge vector γ, attach the operator

$$\hat{V}_\gamma = \prod_{k\geq 1} \hat{U}_{k\gamma} = \exp\left[\sum_{\ell=1}^{\infty} \bar{\Omega}_{\rm ref}(\ell\gamma, y)\, \tilde{e}_{\ell\gamma}\right]$$

where $\bar{\Omega}_{\rm ref}(\gamma, y)$ are the “rational motivic invariants", defined by

$$\bar{\Omega}^+_{\rm ref}(\gamma, y) \equiv \sum_{d|\gamma} \frac{(y - y^{-1})}{d(y^d - y^{-d})} \Omega^+_{\rm ref}(\gamma/d, y^d)\,.$$

The motivic Kontsevich-Soibelman formula III

- The motivic version of the KS wall-crossing formula states that

$$\prod_{\substack{M\geq 0,N\geq 0>0,\\ \gcd(M,N)=1,M/N\downarrow}} \hat{V}^{+}_{M\gamma_1+N\gamma_2} = \prod_{\substack{M\geq 0,N\geq 0>0,\\ \gcd(M,N)=1,M/N\uparrow}} \hat{V}^{-}_{M\gamma_1+N\gamma_2} ,$$

- $\Delta\bar{\Omega}_{\rm ref}(\gamma, y)$ can be computed using the same techniques as before, e.g. the primitive wcf read

$$\Delta\Omega_{\rm ref}(\gamma_1+\gamma_2, y) = \frac{(-y)^{\langle\gamma_1,\gamma_2\rangle} - (-y)^{-\langle\gamma_1,\gamma_2\rangle}}{y-1/y}\,\Omega_{\rm ref}(\gamma_1, y)\,\Omega_{\rm ref}(\gamma_2, y)$$

- The general formula for $\Delta\bar{\Omega}_{\rm ref}$ involves universal factors $g(\{\alpha_i\}, y)$, which reduce to $g(\{\alpha_i\})$ in the limit $y \to 1$. We expect that they are given by $\mathrm{Tr}'(-y)^{2J_3}$ in the corresponding SUSY quantum mechanics.

The Joyce-Song formula I

- In the context of the Abelian category of coherent sheaves on a Calabi-Yau three-fold, Joyce & Song have shown that the jump of (generalized, rational) DT invariants across the wall is given by

$$\Delta\bar\Omega(\gamma) = \sum_{n\geq 2} \sum_{\substack{\{\alpha_1,\dots\alpha_n\}\in\tilde\Gamma \\ \gamma=\alpha_1+\dots+\alpha_n}} \frac{g(\{\alpha_i\})}{|\mathrm{Aut}(\{\alpha_i\})|} \prod_{i=1}^n \bar\Omega^+(\alpha_i) \,.$$

  where $g(\{\alpha_i\})$ is a rather complicated sum over permutations, trees, etc.
- To formulate the JS formula, we need to introduce S, U and $\mathcal{L}$ factors, which are functions of an ordered list of charge vectors $\alpha_i \in \tilde\Gamma, i = 1 \dots n$.

The Joyce-Song formula II

- We define $S(\alpha_1,\dots,\alpha_n) \in \{0,\pm 1\}$ as follows. If $n=1$, set $S(\alpha_1)=1$. If $n>1$ and, for every $i=1\dots n-1$, either

$$
\begin{array}{llll}
(a) & \langle \alpha_i, \alpha_{i+1} \rangle \le 0 & \text{and} & \langle \alpha_1+\dots+\alpha_i, \alpha_{i+1}+\dots+\alpha_n \rangle < 0, \\
(b) & \langle \alpha_i, \alpha_{i+1} \rangle > 0 & \text{and} & \langle \alpha_1+\dots+\alpha_i, \alpha_{i+1}+\dots+\alpha_n \rangle \ge 0 ,
\end{array}
$$

  let $S(\alpha_1,\dots,\alpha_n) = (-1)^r$, where r is the number of times option (a) is realized; otherwise, $S(\alpha_1,\dots,\alpha_n)=0$.

The Joyce-Song formula III

- To define the U factor, consider all ordered partitions of the n vectors α_i into $1 \leq m \leq n$ packets $\{\alpha_{a_{j-1}+1}, \cdots, \alpha_{a_j}\}$, $j = 1 \dots m$, with $0 = a_0 < a_1 < \cdots < a_m = n$, such that all vectors in each packet have the same phase $\arg Z(\alpha_i)$. Let

$$\beta_j = \alpha_{a_{j-1}+1} + \cdots + \alpha_{a_j}, \qquad j = 1 \dots m$$

  be the sum of the charge vectors in each packet.
- Next, consider all ordered partitions of the m vectors β_j into $1 \leq l \leq m$ packets $\{\beta_{b_{k-1}+1}, \cdots, \beta_{b_k}\}$, with $0 = b_0 < b_1 < \cdots < b_l = m$, $k = 1 \dots l$, such that the total charge vectors $\delta_k = \beta_{b_{k-1}+1} + \cdots + \beta_{b_k}$, $k = 1 \dots l$ in each packets all have the same phase $\arg Z(\delta_k)$.

The Joyce-Song formula IV

- Define the U-factor as the sum

$$U(\alpha_1,\ldots,\alpha_n) \equiv \sum_l \frac{(-1)^{l-1}}{l} \cdot \prod_{k=1}^{l} \prod_{j=1}^{m} \frac{1}{(a_j - a_{j-1})!} \; S(\beta_{b_{k-1}+1}, \beta_{b_{k-1}+2}, \ldots, \beta_{b_k}) \; .$$

  over all partitions of α_i and β_j satisfying the conditions above.
- If none of the phases of the vectors α_i coincide, $S = U$. Contributions with $l > 1$ arise only when $\{\alpha_i\}$ can be split into two (or more) packets with the same total charge, e.g.

$$U[\gamma_1,\gamma_2,\gamma_1,\gamma_2] = S[\gamma_1,\gamma_2,\gamma_1,\gamma_2] - \frac{1}{2} S[\gamma_1,\gamma_2]^2 = 1 - \frac{1}{2}(-1)^2 = \frac{1}{2}$$

The Joyce-Song formula V

- Finally (departing slightly from JS), define the (Landau) $\mathcal{L}$ factor Landau factor $\mathcal{L}$ is a

$$\mathcal{L}(\alpha_1, \ldots, \alpha_n) = \sum_{\text{trees}} \prod_{\text{edges(i,j)}} \langle \alpha_i, \alpha_j \rangle$$

  where the sum runs over all labeled trees with n vertices labelled $\{1, \ldots, n\}$, with edges oriented from i to j if $i < j$.
- Each tree can be labelled by its Prüfer code, a sequence of $n - 2$ numbers in $\{1, \ldots n\}$.
- With these definitions,

$$g(\{\alpha_i\}) = \frac{1}{2^{n-1}} (-1)^{n-1+\sum_{i<j} \langle \alpha_i, \alpha_j \rangle} \sum_{\sigma \in \Sigma_n} \mathcal{L}\left(\alpha_{\sigma(1)}, \ldots \alpha_{\sigma(n)}\right) U\left(\alpha_{\sigma(1)}, \ldots \alpha_{\sigma(n)}\right)$$

The Joyce-Song formula VI

- To derive the primitive wcf, note that there is only one oriented tree with 2 nodes. Assuming $\gamma_{12} < 0$, the JS data is then

$\sigma(12)$	S	U	$\mathcal{L}$
12	a	-1	γ_{12}
21	b	1	$-\gamma_{12}$

leading again to

$$\Delta\Omega(\gamma \to \gamma_1 + \gamma_2) = (-1)^{\gamma_{12}}\, \gamma_{12}\, \Omega(\gamma_1)\, \Omega(\gamma_2)\ , \qquad \gamma_{12} \equiv \langle \gamma_1, \gamma_2 \rangle$$

The Joyce-Song formula VII

- For generic 3-body decay, assuming the same phase ordering as before and taking into account the 3 possible oriented trees, the JS data

$\sigma(123)$	S	U	$\mathcal{L}$
123	bb	1	$\alpha_{12}\alpha_{13}+\alpha_{13}\alpha_{23}+\alpha_{12}\alpha_{23}$
132	b-	0	$\alpha_{12}\alpha_{13}-\alpha_{13}\alpha_{23}-\alpha_{12}\alpha_{23}$
213	ab	-1	$-\alpha_{12}\alpha_{23}+\alpha_{13}\alpha_{23}-\alpha_{12}\alpha_{13}$
231	-a	0	$\alpha_{12}\alpha_{13}-\alpha_{13}\alpha_{23}-\alpha_{12}\alpha_{23}$
312	ab	-1	$\alpha_{13}\alpha_{23}-\alpha_{12}\alpha_{23}-\alpha_{13}\alpha_{12}$
321	aa	1	$\alpha_{13}\alpha_{23}+\alpha_{12}\alpha_{13}+\alpha_{12}\alpha_{23}$

leads to the same answer as KS,

$$g(\{\alpha_1,\alpha_2,\alpha_3\}) = (-1)^{\alpha_{12}+\alpha_{23}+\alpha_{13}}\,\alpha_{12}\,(\alpha_{13}+\alpha_{23})$$

The Joyce-Song formula VIII

- We have checked that JS and KS also agree for generic 4-body decay (involving 16 trees), 5-body decay (125 trees) and for special cases (2,3), (2,4) (up to 1296 trees !).
- While I do not know of a combinatorial proof, it seems that the JS formula (derived for Abelian categories) is equivalent to the classical KS formula (stated for triangulated categories).
- Note that the JS formula is restricted to $y = 1$, and involves large denominators and cancellations. We shall find a more economic formula which also works at the motivic level.

Outline

Quantum mechanics of multi-centered solutions I

- The moduli space $\mathcal{M}_n$ of BPS configurations with n centers in $\mathcal{N}=2$ SUGRA is described by solutions to Denef's equations

$$\sum_{j=1\dots n, j\neq i}^{n} \frac{\alpha_{ij}}{|\vec{r}_i - \vec{r}_j|} = c_i \,, \qquad \left\{ \begin{array}{l} c_i = 2\,\mathrm{Im}\left[e^{-i\phi} Z(\alpha_i)\right] \\ \phi = \arg[Z(\alpha_1 + \cdots \alpha_n)] \end{array} \right. .$$

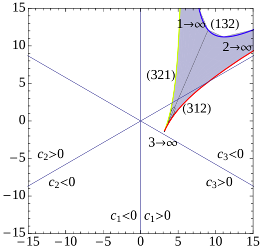
- $\mathcal{M}_n$ is a symplectic manifold of dimension $2n-2$, and carries an Hamiltonian action of $SU(2)$:

$$\omega = \frac{1}{2} \sum_{i<j} \alpha_{ij} \sin\theta_{ij}\, \mathrm{d}\theta \wedge \mathrm{d}\phi_{ij} \,, \qquad \vec{J} = \frac{1}{2} \sum_{i<j} \alpha_{ij} \frac{\vec{r}_{ij}}{|r_{ij}|}$$

de Boer El Showk Messamah Van den Bleeken

Quantum mechanics of multi-centered solutions II

- When the α_i's lie in the positive cone $\tilde{\Gamma}$ (more generally, whenever $\text{sign}(\alpha_{ij})$ defines an ordering of the α_i, $\mathcal{M}_n$ is compact, and the fixed points of J_3 are isolated.
- E.g for 3 centers with $\alpha_{12} > 0, \alpha_{23} > 0, \alpha_{13} > 0$, the domain of the plane $c_1 + c_2 + c_3 = 0$ allowed by Denef's equations is:



For fixed c_i, the range of r_{ij} is read off by intersecting the shaded area with a radial line which joins c_i to the origin. Here, 3-centered solutions only exist in the region $c_1 > 0, c_3 < 0$, and have r_{ij} bounded from below and from above. Fixed points of J_3 correspond to collinear solutions, and lie on the boundary of this domain.

Quantum mechanics of multi-centered solutions III

- The symplectic form $\omega/2\pi \in H^2(\mathcal{M}_n, \mathbb{Z})$ is the curvature of a complex line bundle $\mathcal{L}$ over $\mathcal{M}_n$, with connection

$$\lambda = \frac{1}{2} \sum_{i<j} \alpha_{ij} \left(1 - \cos\theta_{ij}\right) \mathrm{d}\phi_{ij} \ , \quad \mathrm{d}\lambda = \omega \ .$$

- Assuming that $\mathcal{M}_n$ is spin, let $S = S_+ \oplus S_-$ be the spin bundle. Let $D = D_+ \oplus D_-$ be the Dirac operator for the metric obtained by restricting the flat metric on $\mathbb{R}^{3n-3}$ to $\mathcal{M}_n$, with $D_\pm : S_\pm \mapsto S_\mp$. The action of $SO(3)$ on $\mathcal{M}_n$ lifts to an action of $SU(2)$ on $S_\pm$.
- We assume that BPS states correspond to harmonic spinors, i.e. sections of $S \otimes \mathcal{L}$ annihilated by the Dirac operator D.

Quantum mechanics of multi-centered solutions IV

- The 'refined index' is then given by

$$g_{\rm ref}(\{\alpha_i\}; y) = {\rm Tr}_{{\rm Ker}D_+}(-y)^{2J_3} + {\rm Tr}_{{\rm Ker}D_-}(-y)^{2J_3} \ .$$

- We further assume that ${\rm Ker}D_- = 0$, so that the refined index $g_{\rm ref}(\{\alpha_i\}; y)$ reduces to the equivariant index

$$g_{\rm ref}(\{\alpha_i\}; y) = {\rm Tr}_{{\rm Ker}D_+}(-y)^{2J_3} - {\rm Tr}_{{\rm Ker}D_-}(-y)^{2J_3} \ .$$

- The vanishing of ${\rm Ker}D_-$ can be shown to hold in special cases where $\mathcal{M}_n$ is Kähler. In gauge theories, the protected spin character presumably reduces to the equivariant index without further assumption.

Quantum mechanics of multi-centered solutions V

- The refined/equivariant index can be computed by the Atiyah-Bott Lefschetz fixed point formula:

$$g_{\rm ref}(\{\alpha_i\}, y) = \sum_{\rm fixed\ pts} \frac{y^{2J_3}}{\det((-y)^L - (-y)^{-L})}$$

where L is the matrix of the action of J_3 on the holomorphic tangent space around the fixed point.

- In the large charge limit, $\mathcal{L} \to k\mathcal{L}$ with $k \to \infty$, this reduces to the Duistermaat-Heckmann formula for the equivariant volume,

$$\frac{\int_{\mathcal{M}_n} \omega^{n-1}\, y^{2J_3}}{(2\pi)^{n-1}(n-1)!} = \sum_{\rm fixed\,pts} \frac{y^{2J_3}}{\det(L \log(-y))}$$

Quantum mechanics of multi-centered solutions VI

- The fixed points of the action of J_3 are collinear multi-centered configurations along the z-axis, such that

$$\sum_{j=1\dots n, j\neq i}^{n} \frac{\alpha_{ij}}{|z_i - z_j|} = c_i\,, \quad J_3 = \frac{1}{2}\sum_{i<j} \alpha_{ij}\,\mathrm{sign}(z_j - z_i)\,.$$

- Equivalently, fixed points are critical points of the 'superpotential'

$$W(\lambda, \{z_i\}) = -\sum_{i<j} \mathrm{sign}[z_j - z_i]\,\alpha_{ij} \ln|z_j - z_i| - \sum_i (c_i - \frac{\lambda}{n}) z_i$$

  These are isolated, and classified by permutations describing the order of z_i along the axis.

Quantum mechanics of multi-centered solutions VII

- In the vicinity of a fixed point p,

$$J_3 = \frac{1}{2}\sum_{i<j} \alpha_{ij}\,\mathrm{sign}[z_j - z_i] - \frac{1}{4} W_{ij}(x_i x_j + y_i y_j) + \cdots \,, \omega = \frac{1}{2} W_{ij} \mathrm{d}x_i \wedge \mathrm{d}y_j + \cdots$$

  where W_{ij} is the Hessian matrix of $W(\lambda, \{z_i\})$ wrt $z_1, \dots, z_n$, and (x_i, y_i) are coordinates in the plane transverse to the z-axis at the center i ($\sum_i x_i = \sum y_i = 0$).

- In particular, $U(1)$ acts as $L = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in each two-plane, leading to

$$\det(y^L - y^{-L}) = (y - 1/y)^{n-1} s(p) \,, \quad s(p) = -\mathrm{sign}(\det W_{IJ})$$

  where W_{IJ} is the Hessian of W with respect to $z_I = (\lambda, z_i)$. $s(p)$ is (minus) the Morse index of the critical point p.

Quantum mechanics of multi-centered solutions VIII

- This leads to the Coulomb branch formula

$$g_{\rm ref}(\{\alpha_i\},y) = \frac{(-1)^{\sum_{i<j}\alpha_{ij}+n-1}}{(y-y^{-1})^{n-1}} \sum_{p:\partial_I W(p)=0} s(p)\, y^{\sum_{i<j}\alpha_{ij}\,{\rm sign}(z_j-z_i)}$$

where $s(p) = -{\rm sign}(\det W_{IJ})$

- For $n \le 5$, we find perfect agreement with JS/KS !

$$g_{\rm ref}(\alpha_1,\alpha_2;y) = (-1)^{\alpha_{12}} \frac{\sinh(\nu\alpha_{12})}{\sinh\nu} \qquad (y=e^\nu)$$

$$g_{\rm ref}(\alpha_1,\alpha_2,\alpha_3;y) = (-1)^{\alpha_{13}+\alpha_{23}+\alpha_{12}} \frac{\sinh(\nu(\alpha_{13}+\alpha_{23}))\,\sinh(\nu\alpha_{12})}{\sinh^2\nu}$$

Higgs branch picture I

- An alternative formula can be given using the Higgs branch description of the multi-centered configuration, namely the quiver with n nodes $\{1\dots n\}$ of dimension 1 and α_{ij} arrows from i to j.
- Since α_i lie on a 2-dimensional sublattice $\tilde{\Gamma}$, the quiver has no oriented closed loop. Reineke's formula gives

$$g_{\rm ref} = \frac{(-y)^{-\sum_{i<j}\alpha_{ij}}}{(y-1/y)^{n-1}} \sum_{\rm partitions} (-1)^{s-1} y^{2\sum_{a\le b}\sum_{j<i}\alpha_{ji}\, m_i^{(a)}\, m_j^{(b)}} ,$$

where $\sum$ runs over all ordered partitions of $\gamma = \alpha_1 + \cdots + \alpha_n$ into s vectors $\beta^{(a)}$ $(1 \le a \le s,\ 1 \le s \le n)$ such that

1. $\beta^{(a)} = \sum_i m_i^{(a)} \alpha_i$ with $m_i^{(a)} \in \{0,1\}$, $\sum_a \beta^{(a)} = \gamma$
2. $\langle \sum_{a=1}^{b} \beta^{(a)}, \gamma \rangle > 0 \quad \forall \quad b \quad \text{with} \quad 1 \le b \le s-1$

Higgs branch picture II

- The Higgs branch formula agrees with KS/JS/Coulomb for $n = 2, 3, 4, 5$!
- The formula gives a prescription for what permutations are allowed in the Coulomb problem, and for their Morse index.
- It is perhaps not surprising that the Higgs branch formula agrees with KS/JS, since quiver categories are an example of Abelian categories. Unlike the JS formula, the Higgs branch formula works at $y \neq 1$.

Outline

Away from the wall I

- Having understood the jump $\Delta\Omega(\gamma; y)$ in terms of the index of multi-centered solutions, one would like to compute the BPS index $\Omega(\gamma; y, t^a)$ on either side of the wall, from the index $\Omega_S(\alpha_i)$ of single-centered black holes. Since spherically symmetric SUSY black holes cannot decay and carry zero angular momentum, $\Omega_S(\alpha_i)$ must be independent of t^a and y.

Manchot BP Sen II

- Naively, one may expect

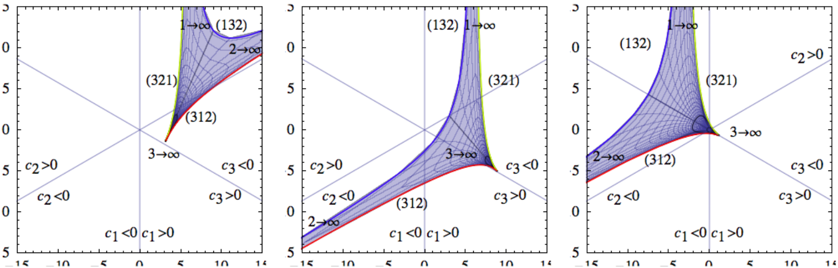
$$\bar{\Omega}(\gamma; y, t^a) = \sum_{n \geq 2} \sum_{\substack{\{\alpha_1, \dots \alpha_n\} \in \Gamma \\ \gamma = \alpha_1 + \dots + \alpha_n}} \frac{g(\{\alpha_i\}; y, c_i)}{|\mathrm{Aut}(\{\alpha_i\})|} \prod_{i=1}^{n} \bar{\Omega}_S(\alpha_i) \, ,$$

where $g(\{\alpha_i\}; y, c_i)$ is the refined index of the SUSY quantum mechanics on $\mathcal{M}_n(\alpha_i, c_i)$. This is similar to the formula for $\Delta\bar{\Omega}(\gamma)$, but with some important differences.

Away from the wall II

- Unlike the formula for $\Delta\bar{\Omega}(\gamma)$, the charges α_i of the constituents are no longer restricted to a two-dimensional subspace of the charge lattice, and there are a priori an infinite number of possible splittings $\gamma = \sum \alpha_i$. It is plausible that requiring that the multi-centered solution be regular may leave only a finite number of splittings. In addition, for a given splitting, the regularity constraint may rule out certain components of $\mathcal{M}_n(\alpha_i, c_i)$.
- The space $\mathcal{M}_n(\alpha_i, c_i)$ is in general no longer compact. E.g, in the 3-body case with $\alpha_{12} > 0, \alpha_{23} > 0, \alpha_{13} < 0$, the allowed values of c_i (and therefore r_{ij}) are plotted below:

Away from the wall III



- In particular, there can be scaling regions in $\mathcal{M}_n$, when some or all of the n centers approach each other at arbitrary small distances. Classically, these scaling solutions carry zero angular momentum and are invariant under $SO(3)$.
- Some of the distances r_{ij} can also diverge on walls of marginal stability, but the formula for $\Omega(\gamma, t^a)$ is by construction consistent with wall-crossing.

Away from the wall IV

- In the presence of scaling solutions, it appears that $\mathcal{M}_n$ admits a compactification $\overline{\mathcal{M}}_n$ with finite volume. However, this introduces new (non-collinear) fixed points of the action of J_3 which are no longer isolated, leading to additional contributions to the equivariant index.
- Rather than trying to compute these new contributions directly, we propose to determine them by requiring 1) that the resulting $\Omega(\gamma; y, t^a)$ is a finite Laurent polynomial in y and 2) that they carry the minimal angular momentum J_3 compatible with condition 1). This minimal modification hypothesis fixes $\Omega(\gamma; y, t^a)$ uniquely.
- We have checked that the minimal modification hypothesis works for an infinite class of 'dipole halo' configurations, where $\mathcal{M}_n$ is a toric manifold and can be quantized directly.

Conclusion I

- Multi-centered solitonic configurations provide a simple picture to derive and understand wall-crossing formulae for the BPS (refined) index.
- We have not proven the equivalence between the Coulomb branch, Higgs branch, JS and KS wall-crossing formula, but there is overwhelming evidence that they all agree.
- Our derivation was made in the context of $\mathcal{N}=2$ supergravity, it would be interesting to develop our understanding of multi-centered dyonic solutions in $\mathcal{N}=2$ gauge theories.

Lee Yi

- In principle, our formulae can be used to extract the degeneracies $\Omega_S(\gamma)$ of single-centered black holes from the moduli-dependent BPS index $\Omega(\gamma)$. The former is the one that should be compared with Sen's quantum entropy function.

THANK YOU !