Wall-crossing made easy and smooth

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Based on joint work with

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The man who could walk through walls

Marcel Aymé, *Le passe-muraille*, 1943

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“When he left [his mistress’ room], Dutilleul passed through the walls of the house and felt an unusual rubbing sensation against his hips and shoulders. He felt as though he were moving through some gel-like substance that was growing thicker (...) Dutilleul was immobilized inside the wall. He is there to this very day, imprisoned in the stone.”

Introduction

- Unlike the real world, gauge theories and string vacua with extended SUSY abound with massless scalar fields / moduli. How does the spectrum of bound states depend on them ?
- More often than not, bound states decay into multi-particle states across certain codimension-one walls in moduli space: a way to learn about their elementary constituents !
- Using semi-classical methods, one may sometimes determine the spectrum at weak coupling. Understanding these decays systematically is important to extrapolate to strong coupling.

BPS states and BPS index

- This can be achieved for BPS states, annihilated by a fraction of SUSY: their mass is computable exactly and possible decays are highly constrained.
- While the number of BPS states may change erratically, the BPS index is constant – at least away from the walls. In theories with $\mathcal{N} = 2$ SUSY,

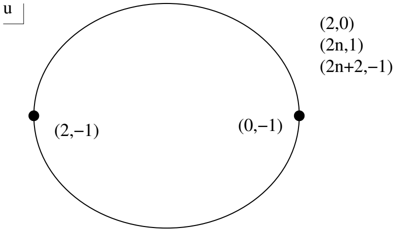
$$\Omega(\gamma, u) = -\frac{1}{2}\mathrm{Tr}_{\mathcal{H}_1(\gamma,u)}(-1)^{2J_3}\,(2J_3)^2$$

  where $\mathcal{H}_1(\gamma, u)$ is the Hilbert space of one-particle states.
- The jump $\Delta\Omega$ across the wall is determined by certain universal wall-crossing formulae, first discovered in the math literature.

Joyce Song 2008; Kontsevich Soibelman 2008

Wall-crossing in gauge theories

- E.g., in $D = 4, \mathcal{N} = 2$ SYM with $G = SU(2)$ (Seiberg-Witten) on the Coulomb branch,



All BPS states in the weak coupling region can be viewed as bound states of the magnetic monopole $(0, 1)$ and dyon $(2, -1)$. Those are absolutely stable, i.e. exist everywhere on the Coulomb branch.

Seiberg Witten 1994; Bilal Ferrari 1996

Bound states as multi-centered solutions

- In the low energy field theory, all these bound states are described semi-classically by multi-centered BPS solitons (or black holes).

  Denef 2000; Denef Moore 2007

- Near the wall, the centers become farther apart, and behave like point particles interacting by Coulomb, Lorentz, (Newton) and scalar forces.
- As I'll explain in part I, the degeneracy of the bound state (hence the jump in Ω) is determined by the index of the SUSY quantum mechanics of these point particles, which is computable by localization.

Denef 2002; Manschot BP Sen 2010

Witten index and multi-particle states I

- Another protected quantity is the Witten index

$$\varpi(R,\gamma,u) = -\tfrac{1}{2}\mathrm{Tr}_{\mathcal{H}(u,\gamma)}(-1)^{2J_3}(2J_3)^2\, e^{-2\pi R H} \quad \notin \mathbb{Z}!$$

  Here $\mathcal{H}(u)$ is the full Hilbert space of the four-dimensional theory on $\mathbb{R}^3$, including multi-particle states, H is the Hamiltonian.

- In center of mass frame, the Hamiltonian has a discrete spectrum starting at the BPS bound $E = |Z(\gamma,u)|$, and a continuum of (non-BPS) multi-particle states. They can still contribute to the Witten index, due to a possible spectral asymmetry between densities of bosonic and fermionic states.

Kaul Rajaraman 1983, Akhoury Comtet 1984, Troost 2010

- Since only multi-particle states made of BPS constituents can contribute to the Witten index, $\varpi(R,\gamma,u)$ should be a universal function of the BPS indices $\Omega(\alpha_i,u)$.

Witten index and multi-particle states II

- $\varpi(R, \gamma, u)$ is computed by a path integral on $\mathbb{R}^3 \times S^1(R)$ with periodic boundary conditions, and with an insertion of a 4-fermion vertex to soak up fermionic zero-modes.
- Because the path integral has no phase transition, the Witten index $\varpi(R, \gamma, u)$ is expected to be smooth across walls of marginal stability.

Alexandrov Moore Neitzke BP 2014

- $\varpi(R, \gamma, u)$ is an analogue of the 'new supersymmetric index' in $D = 2$ massive theories with $(2, 2)$ supersymmetry. It is closely related to the metric on the Coulomb branch in the theory compactified on S^1.

Cecotti Fendley Intriligator Vafa 1992; Gaiotto Moore Neitzke 2008

Outline

1. Wall-crossing made easy
2. The Coulomb branch formula for quiver moduli spaces
3. Wall-crossing made smooth

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3. Wall-crossing made smooth

Generalities

- Supersymmetric gauge theories or supergravity models in 4 dimensions typically include a large number of massless scalars $u \in \mathcal{M}$ and Abelian gauge fields A_μ^Λ.
- Bound states are labelled by their electric and magnetic charges q_Λ, p^Λ, by their mass M and spin J_3.
- The charge vector $\gamma = (p^\Lambda, q_\Lambda)$ takes values in a lattice equipped with an integer antisymmetric pairing, corresponding to the angular momentum carried by the electromagnetic field:

$$\langle \gamma, \gamma' \rangle \equiv q_\Lambda \, p'^\Lambda - q'_\Lambda \, p^\Lambda \in \mathbb{Z}$$

Dirac 1931; Schwinger 1966; Zwanziger 1968

States with $\langle \gamma, \gamma' \rangle \neq 0$ are ‘mutually non-local’.

BPS states and BPS index

- In models with $\mathcal{N}=2$ supersymmetries, the mass of any state is bounded from below by the BPS bound

$$M \geq |Z(\gamma,u)|\ , \qquad Z(\gamma,u) = e^{\mathcal{K}/2}(q_\Lambda X^\Lambda - p^\Lambda F_\Lambda)$$

- States saturating the BPS bound are called BPS states. They are annihilated by half of the supersymmetry, therefore form short SUSY multiplets.

Witten Olive 1978

- Two short multiplets might combine into a long multiplet and desaturate the BPS bound, but the (refined) index Ω stays constant under this process:

$$\Omega(\gamma; y, u) = \frac{1}{1/y - y} \mathrm{Tr}_{\mathcal{H}_1(\gamma,u)} (-1)^{2J_3} (2J_3) y^{2(I_3+J_3)}$$

Walls of marginal stability

- The index $\Omega(\gamma; u)$ may fail to be constant when the single-particle spectrum mixes with the continuum of multi-particle states, i.e. when the bound state decays.
- The decay of BPS bound states is constrained by the triangular inequality

$$M(\gamma_1+\gamma_2) = |Z(\gamma_1+\gamma_2)| = |Z(\gamma_1)+Z(\gamma_2)| \le M(\gamma_1)+M(\gamma_2)$$

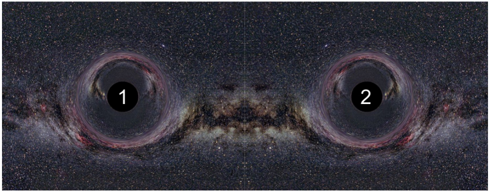
The decay is energetically possible only when the constituents are BPS, and central charges are aligned, i.e. on the wall

$$W(\gamma_1,\gamma_2) = \{u \,/\, \arg[Z(\gamma_1,u)] = \arg[Z(\gamma_2,u)]\} \subset \mathcal{M}$$

Cecotti Vafa 1992; Seiberg Witten 1994

Primitive wall-crossing from two-centered solutions I

- For $\langle \gamma_1, \gamma_2 \rangle \neq 0$, there exists a two-centered BPS solution of charge $\gamma = \gamma_1 + \gamma_2$:



$$\frac{\langle \gamma_1, \gamma_2 \rangle}{R} = \frac{2\,\mathrm{Im}[\bar{Z}(\gamma_1)\, Z(\gamma_2)]}{|Z(\gamma_1 + \gamma_2)|}$$

Denef 2002

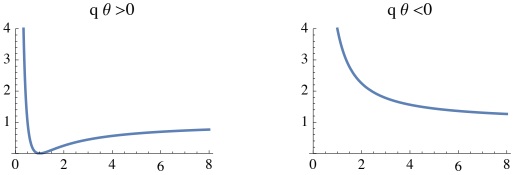
- The solution exists only on one side of the wall. As u approaches the wall, the distance r_{12} diverges and the bound state decays into its constituents γ_1 and γ_2.

Primitive wall-crossing from two-centered solutions I

- Near the wall, the two monopoles can be treated as pointlike particles with charge $\Omega(\gamma_i)$ internal degrees of freedom, interacting via $\mathcal{N}=4$ supersymmetric quantum mechanics,

$$H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 - \frac{q}{2m}\vec{B}\cdot\vec{\sigma}\otimes(1_2 - \sigma_3) + \frac{1}{2m}\left(\vartheta - \frac{q}{r}\right)^2$$

$$\vec{\nabla}\wedge\vec{A} = \vec{B} = \frac{\vec{r}}{r^3}\ ,\ q = \frac{1}{2}\langle\gamma_1,\gamma_2\rangle\ ,\ m = \frac{|Z_{\gamma_1}||Z_{\gamma_2}|}{|Z_{\gamma_1}|+|Z_{\gamma_2}|}\ ,\ \frac{\vartheta^2}{2m} = |Z_{\gamma 1}|+|Z_{\gamma_2}|-|Z_{\gamma_1+\gamma_2}|$$



Primitive wall-crossing from two-centered solutions II

- H describes two bosonic degrees of freedom with helicity $h = 0$, and one helicity $h = \pm 1/2$ fermionic doublet with gyromagnetic ratio $g = 4$.

D'Hoker Vinet 1985; Denef 2002; Avery Michelson 2007;Lee Yi 2011

- H commutes with 4 supercharges,

$$Q_4 = \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 & -\mathrm{i}\left(\vartheta - \frac{q}{r}\right) + \vec{\sigma} \cdot (\vec{p} - q\vec{A}) \\ \mathrm{i}\left(\vartheta - \frac{q}{r}\right) + \vec{\sigma} \cdot (\vec{p} - q\vec{A}) & 0 \end{pmatrix}$$

$$\{Q_m, Q_n\} = 2H\,\delta_{mn}$$

Primitive wall-crossing from two-centered solutions III

- When $q\vartheta > 0$, H has a BPS ground state with degeneracy $2|q|$, transforming as a multiplet of spin $j = \frac{1}{2}(|\langle\gamma_1, \gamma_2\rangle| - 1)$ under rotations (plus a number of non-BPS bound states which cancel pairwise in the index).
- Equivalently, one may first truncate the dynamics to the BPS phase space, a two-sphere with symplectic form $\omega = \frac{1}{2}\langle\gamma_1, \gamma_2\rangle \sin\theta \, \mathrm{d}\theta \mathrm{d}\phi$. The geometric quantization of $\mathcal{M}_2$ produces the same multiplet of BPS states.
- In addition, there is a continuum of non-BPS states starting at $E = \vartheta^2/(2m)$, which will become important in part II.

Primitive wall-crossing from two-centered solutions IV

Primitive wall-crossing formula (Denef Moore 2007)

$$\Delta\Omega(\gamma_1+\gamma_2) = \pm \underbrace{|\langle\gamma_1,\gamma_2\rangle|}_{\text{angular momentum}} \times \underbrace{\Omega(\gamma_1)}_{\text{internal states of 1}} \times \underbrace{\Omega(\gamma_2)}_{\text{internal states of 2}}$$

Multi-centered solutions

- On the same wall, many other bound states will decay: those represented by multi-centered BPS solutions with charges $\alpha_i = M_i \gamma_1 + N_i \gamma_2$, with $M_i \geq 0, N_i \geq 0$ and $(M_i, N_i) \neq 0$.
- Stationary BPS solutions with n centers at $\vec{r} = \vec{r}_i$ exist whenever

Denef's equations (Denef 2000)

$$\forall i \; : \; \sum_{j \neq i} \frac{\alpha_{ij}}{|\vec{r}_i - \vec{r}_j|} = c_i(u)$$

Here $\alpha_{ij} \equiv \langle \alpha_i, \alpha_j \rangle$, $c_i = 2\,\mathrm{Im}\left[e^{-i\phi} Z(\alpha_i, u)\right]$, $\phi = \arg[Z(\sum_i \alpha_i, u)]$.

BPS phase space

- For fixed charges α_i and moduli u, the space of solutions modulo overall translations is a compact symplectic manifold $\mathcal{M}_n$ of dimension $2n-2$, invariant under $SO(3)$:

$$\omega = \frac{1}{2} \sum_{i<j} \alpha_{ij} \sin\theta_{ij}\, \mathrm{d}\theta_{ij} \wedge \mathrm{d}\phi_{ij} , \quad \vec{J} = \frac{1}{2} \sum_{i<j} \alpha_{ij} \frac{\vec{r}_{ij}}{|r_{ij}|}$$

de Boer El Showk Messamah Van den Bleeken 2008

- The solution exists only on one side of the wall. In the vicinity of the wall, the centers move away from each other, and can again be treated like point-like particles with
  1. $\Omega(\gamma_i)$ internal states at each center
  2. $g(\{\alpha_i\})$ external states obtained by geometric quantization of $\mathcal{M}_n$

Geometric quantization and localization

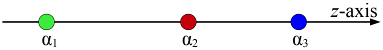
- Given a symplectic manifold $(\mathcal{M}, \omega)$, geometric quantization produces a graded Hilbert space $\mathcal{H}$, the space of harmonic spinors for the Dirac operator D coupled to ω. If $\mathcal{M}$ is compact, $\mathcal{H}$ is finite dimensional.
- Working assumption: the index $g(\{\alpha_i\}) = \mathrm{Tr}(-1)^{2J_3}$ of the SUSY quantum mechanics is the index of the Dirac operator D. More generally, the refined index $g(\{\alpha_i\}, y) \equiv \mathrm{Tr}(-y)^{2J_3}$ in the SUSY quantum mechanics is equal to the equivariant index of D.
- Since $\mathcal{M}_n$ admits a $U(1)$ action, the equivariant index can be computed by localization:

$$\mathrm{Ind}(D) = \lim_{y \to 1} \mathrm{Ind}(D, y) \ , \quad \mathrm{Ind}(D, y) = \sum_{\text{fixed pts}} \mathrm{Jac}(p)\, y^{2J_3(p)}$$

Atiyah Bott, Berline Vergne

The Coulomb branch formula

- For any n, the fixed points of the action of J_3 are collinear multi-centered configurations along the z-axis:



$$\forall i\,,\quad \sum_{j\neq i} \frac{\alpha_{ij}}{|z_i - z_j|} = c_i\,,\qquad J_3 = \frac{1}{2}\sum_{i<j} \alpha_{ij}\,\mathrm{sign}(z_j - z_i)\,.$$

- These fixed points are isolated, and labelled by permutations σ:

Coulomb branch wall-crossing formula

$$g(\{\alpha_i\}, y) = \frac{(-1)^{\sum_{i<j}\alpha_{ij}+n-1}}{(y-y^{-1})^{n-1}} \sum_{\sigma} s(\sigma)\, y^{\sum_{i<j}\alpha_{\sigma(i)\sigma(j)}}\;,\qquad s(\sigma) = 0, \pm 1$$

Manschot, BP, Sen 2010

An example: 3-body decay

- E.g. for $n = 2$, $\mathcal{M}_2 = S^2$, $J_3 = \alpha_{12} \cos\theta$:

$$g(\{\alpha_1, \alpha_2\}, y) = \frac{(-1)^{\alpha_{12}}}{1/y - y} \left(\underbrace{y^{+\alpha_{12}}}_{\text{North pole}} - \underbrace{y^{-\alpha_{12}}}_{\text{South pole}} \right) \xrightarrow{y \to 1} \pm\alpha_{12}$$

- E.g. for $n = 3$ with $\alpha_{12} > \alpha_{23}$, there are 4 collinear configurations:

$$g(\{\alpha_i\}, y) = \frac{(-1)^{\alpha_{13}+\alpha_{23}+\alpha_{12}}}{(y-1/y)^2} \times$$
$$\left[\underbrace{y^{\alpha_{13}+\alpha_{23}+\alpha_{12}}}_{(123)} - \underbrace{y^{-\alpha_{13}-\alpha_{23}+\alpha_{12}}}_{(312)} - \underbrace{y^{\alpha_{13}+\alpha_{23}-\alpha_{12}}}_{(213)} + \underbrace{y^{-\alpha_{13}-\alpha_{23}-\alpha_{12}}}_{(321)} \right]$$
$$\xrightarrow{y \to 1} \pm\langle\alpha_1, \alpha_2\rangle \, \langle\alpha_1 + \alpha_2, \alpha_3\rangle$$

Non-primitive wall-crossing (naive)

- For fixed total charge $\gamma = M\gamma_1 + N\gamma_2$, the index $\Omega(\gamma)$ includes contributions from all n-centered solutions with charges $\alpha_i = M_i\gamma_1 + N_i\gamma_2$ such that $(M, N) = \sum_i (M_i, N_i)$. All these solutions disappear at once across the wall.
- Naively, the jump of the index across the wall should be

$$\Delta\Omega(\gamma, y) = \sum_{n\geq 2} \sum_{\gamma=\alpha_1+\dots+\alpha_n} g(\{\alpha_i\}, y) \prod_{i=1}^{n} \Omega(\alpha_i, y)$$

  where $g(\{\alpha_i\}, y)$ is the index of the SUSY quantum mechanics, and $\Omega(\alpha_i, y)$ is the refined index carried by the constituents (the same on both sides of the wall).
- This however ignores the issue of statistics.

Non-primitive wall-crossing (correct)

- Taking Bose-Fermi statistics into account, the formula for $\Delta\Omega(\gamma)$ is cumbersome (e.g. it involves products of $\Omega(\alpha_i)$ with $\gamma \neq \sum \alpha_i$).
- The correct formula is obtained by replacing $\Omega \to \bar{\Omega}$ where

$$\bar{\Omega}(\gamma, y) \equiv \sum_{d|\gamma} \frac{1}{d} \frac{y - 1/y}{y^d - y^{-d}} \Omega(\gamma/d, y^d)$$

Joyce Song

and introducing a Boltzmann symmetry factor:

Non-primitive wall-crossing formula

$$\Delta\bar{\Omega}(\gamma) = \sum_{n \geq 2} \sum_{\gamma = \alpha_1 + \dots + \alpha_n} \frac{g(\{\alpha_i\})}{|\mathrm{Aut}(\{\alpha_i\})|} \prod_{i=1}^{n} \bar{\Omega}(\alpha_i)$$

Manschot BP Sen 2010

An example: 3-body decay

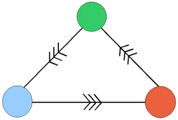
- E.g. for $\gamma = \gamma_1 + 2\gamma_2$, three types of bound states contribute:

$$\begin{aligned}
\Delta\Omega(\gamma) =& (-1)^{\gamma_{12}}\, \gamma_{12}\, \Omega(\gamma_2)\, \Omega(\gamma_1+\gamma_2) + 2\gamma_{12}\, \Omega(2\gamma_2)\, \Omega(\gamma_1) \\
&+ \tfrac{1}{2}\gamma_{12}\, \Omega(\gamma_2)\, (\gamma_{12}\Omega(\gamma_2)+1)\, \Omega(\gamma_1) \\
=& (-1)^{\gamma_{12}}\, \gamma_{12}\, \bar\Omega(\gamma_2)\, \bar\Omega(\gamma_1+\gamma_2) + 2\gamma_{12}\, \bar\Omega(2\gamma_2)\, \bar\Omega(\gamma_1) \\
&+ \tfrac{1}{2}(\gamma_{12})^2\, \bar\Omega(\gamma_2)\, \bar\Omega(\gamma_2)\, \bar\Omega(\gamma_1)
\end{aligned}$$

Outline

Quiver Matrix Mechanics

- In the weak coupling limit, the centers can be realized as D-branes interacting via open strings. At low energy, this is described by a Matrix Quantum Mechanics, with field content specified by a quiver with n nodes $\{1 \dots n\}$ of dimension 1 and $\langle \alpha_i, \alpha_j \rangle$ arrows from i to j.



- The Matrix Quantum Mechanics admits a Coulomb branch where the D-branes are well-separated, described by Denef's equations above. It also has a Higgs branch where all D-branes coincide.
- If the quiver has no oriented closed loop (e.g. if all α_i lie on a 2-dimensional cone spanned by γ_1, γ_2), one expects a 1-1 map between states on the Higgs branch and on the Coulomb branch.

Higgs branch formula

- The classical moduli space $\mathcal{M}_H$ on the Higgs branch is the moduli space of quiver representations with potential, i.e. the space of stable solutions of the F-term equations, modulo the complexified gauge group $\prod_\ell GL(N_\ell, \mathbb{C})$. Here 'stable' means that $\mu(\gamma') < \mu(\gamma)$ for any proper subrepresentation of γ, where $\mu(\gamma) = \frac{\sum c_\ell N_\ell}{\sum N_\ell}$

  King; Reineke

- BPS states on the Higgs branch correspond to Dolbeault cohomology classes in $H^{p,q}(\mathcal{M}_H, \mathbb{Z})$. The refined BPS index is the Hirzebruch polynomial, or χ_{y^2}-genus,

$$\Omega(\gamma; c_i; y) = \sum_{p,q=0}^{2d} h_{p,q}(\mathcal{M}_H)\,(-y)^{2p-d}$$

For $y = 1$, it reduces to the Euler number.

Quiver Matrix Mechanics I

- The wall-crossing formula suggests that it should be possible to express the BPS index $\Omega(\gamma; c_i; y)$ as a sum of bound states of a set of elementary (i.e. absolutely stable, or 'single-centered') constituents, carrying fixed internal index $\Omega_S(\alpha_i)$. Naively,

$$\bar{\Omega}(\gamma; c_i, y) = \sum_{\gamma = \sum \alpha_i} \frac{g(\{\alpha_i\}, \{c_i\}; y)}{|\mathrm{Aut}(\{\alpha_i\})|} \prod_i \bar{\Omega}_S(\alpha_i; y)$$

where $\Omega_S(\alpha_i) = 1$ is α_i is a basis vector and zero otherwise.

Manschot BP Sen 2011

- Things are not quite so simple, because when the quiver has loops, the Coulomb branch moduli space $\mathcal{M}_n$ is non-compact due to scaling solutions, and $g(\{\alpha_i\}, \{c_i\}; y)$ is not necessarily a symmetric Laurent polynomial.

Quiver Matrix Mechanics II

- This can be repaired by replacing $\Omega_S(\alpha_i; y)$ on the r.h.s. by

$$\Omega_{\rm tot}(\alpha; y) = \Omega_S(\alpha; y) + \sum_{\substack{\{\beta_i \in \Gamma\}, \{m_i \in \mathbb{Z}\} \\ m_i \geq 1,\, \sum_i m_i \beta_i = \alpha}} H(\{\beta_i\}; \{m_i\}; y) \prod_i \Omega_{\rm S}(\beta_i; y^{m_i})$$

- $H(\{\beta_i\}; \{m_i\}; y)$ is determined recursively by the conditions
  - H is symmetric under $y \to 1/y$,
  - H vanishes at $y \to 0$,
  - the coefficient of $\prod_i \Omega_{\rm S}(\beta_i; y^{m_i})$ in the expression for $\Omega(\sum_i m_i \beta_i; y)$ is a Laurent polynomial in y.

The formula is implemented in mathematica: `CoulombHiggs.m`

Manschot BP Sen 1302.5498; 1404.7154

Example I

- E.g., take a 3-node quiver with $\alpha_{12} = a$, $\alpha_{23} = b$, $\alpha_{31} = c$ satisfying triangular inequalities $0 < a < b + c$, etc. There exist scaling solutions of Denef's equations

$$\frac{a}{r_{12}} - \frac{c}{r_{13}} = c_1 , \quad \frac{b}{r_{23}} - \frac{a}{r_{12}} = c_2 , \quad \frac{c}{r_{31}} - \frac{b}{r_{23}} = c_3 ,$$

with $r_{12} \sim a\,\epsilon, r_{23} \sim b\,\epsilon, r_{13} \sim c\,\epsilon, \vec{J}^2 \sim \epsilon^2$ as $\epsilon \to 0$.

- For $c_1, c_2 > 0$, the only collinear configurations are (123) and (321), leading to a rational function rather than a Laurent polynomial,

$$g(\{\alpha_i\}) = \frac{(-1)^{a+b+c}(y^{a+b-c} + y^{-a-b+c})}{(y - 1/y)^2}$$

Example II

- The prescription gives

$$H(\{\gamma_1,\gamma_2,\gamma_3\};\{1,1,1\};y) = \begin{cases} -2/(y-y^{-1})^2 & , a+b+c \text{ even} \\ (y+y^{-1})/(y-y^{-1})^2 & , a+b+c \text{ odd} \end{cases}$$

so that the index of the Abelian 3-node quiver decomposes into

$$\Omega(\gamma,y,\{c_i\}) = g(\{\gamma_1,\gamma_2,\gamma_3\},\{c_i\},y) + H(\{\gamma_1,\gamma_2,\gamma_3\};\{1,1,1\};y) + \Omega_S(\gamma_1+\gamma_2+\gamma_3;y) \; .$$

- The single-centered invariant $\Omega_S(\gamma_1+\gamma_2+\gamma_3;y)$ is independent of c_i and y and grows exponentially with (a,b,c), while the first term grows polynomially.

Bena Berkooz El Showk de Boer van den Bleeken 2012

Outline

A new supersymmetric index in $D = 4$

- In this last part, we propose a universal formula for the Witten indices

$$\varpi(R,\gamma,u) = -\tfrac{1}{2}\mathrm{Tr}_{\mathcal{H}(u,\gamma)}(-1)^{2J_3}(2J_3)^2\, e^{-2\pi R\, H}$$

in terms of the BPS indices $\Omega(\gamma, u)$, which is smooth across the walls, thanks to an interplay between single-particle and multi-particle contributions.

- It will be convenient to consider the 'grand canonical index'

$$\mathcal{I}(R,u,C) = \mathcal{I}_0(R,u) + \sum_{\gamma} \sigma_\gamma\, \varpi(R,\gamma,u)\, e^{-2\pi\mathrm{i}\langle\gamma,C\rangle}$$

where $\mathcal{I}_0(R,u)$ is the perturbative contribution with zero charge, and σ_γ is a pesky sign.

Alexandrov Neitzke Moore BP, 2014

Coulomb branch on $\mathbb{R}^3 \times S^1$ I

- On $\mathbb{R}^{3,1}$, the Coulomb branch $\mathcal{M}_4$ is a special Kähler manifold determined by the central charge function $Z : \Gamma \to \mathbb{C}$.
- After compactification on a circle of radius R, and dualizing the vector fields in $D = 3$ into scalars C, the low energy dynamics can be formulated in terms of a non-linear sigma model $\mathbb{R}^3 \to \mathcal{M}_3(R)$.
- The target space $\mathcal{M}_3(R)$ is a torus bundle over $\mathcal{M}_4$, equipped with a hyperkähler metric.
- In the large radius limit, the metric is obtained by the 'rigid c-map' from $\mathcal{M}_4$, and has translational isometries along the torus fiber. At finite radius, instanton corrections from $D = 4$ BPS states winding around the circle and break the isometries.

Coulomb branch on $\mathbb{R}^3 \times S^1$ II

- The HK metric on $\mathcal{M}_3(R)$ is best described using twistorial methods: the twistor space $\mathcal{Z} = \mathbb{P}_t \times \mathcal{M}_3$ carries a natural complex structure and holomorphic 'symplectic' form

$$\omega = \mathrm{i} t^{-1} \omega_+ + \omega_3 + \mathrm{i} t \omega_- = \epsilon^{ab} \frac{\mathrm{d}\mathcal{X}_a}{\mathcal{X}_a} \wedge \frac{\mathrm{d}\mathcal{X}_b}{\mathcal{X}_b}$$

  The metric on $\mathcal{M}_3(R)$ can be read off from the holomorphic Darboux coordinates $\mathcal{X}_{\gamma_a}(t, u, C)$.

- In the infinite radius limit, a natural set of Darboux coordinates are exponentiated moment maps for the torus isometries,

$$\mathcal{X}_a = \mathcal{X}_{\gamma_a}\,, \quad \mathcal{X}^{\mathrm{sf}}_\gamma = \sigma_\gamma\, e^{-\pi \mathrm{i} R \left(t^{-1} Z_\gamma - t \bar{Z}_\gamma\right) - 2\pi \mathrm{i} \langle \gamma, C \rangle}.$$

Coulomb branch on $\mathbb{R}^3 \times S^1$ III

- Instanton corrections induce discontinuities across 'BPS rays' $\ell_{\gamma'} = \{t' \in \mathbb{C}^\times : Z_{\gamma'}/t' \in \mathrm{i}\mathbb{R}^-\}$, thru 'KS- symplectomorphisms',

$$\mathcal{X}_\gamma \to \mathcal{X}_\gamma (1 - \mathcal{X}_{\gamma'})^{\langle \gamma, \gamma' \rangle \, \Omega(\gamma')}$$

- The quantum corrected Darboux coordinates are solutions of the integral equations

$$\frac{\mathcal{X}_\gamma}{\mathcal{X}_\gamma^{\rm sf}} = \exp\left[\sum_{\gamma'} \frac{\Omega(\gamma', u)}{4\pi \mathrm{i}} \langle \gamma, \gamma' \rangle \int_{\ell_{\gamma'}} \frac{\mathrm{d}t'}{t'} \frac{t + t'}{t - t'} \log\left(1 - \mathcal{X}_{\gamma'}(t')\right) \right],$$

reminiscent of TBA equations in integrable systems.

Gaiotto Moore Neitzke 2008, 2010

Coulomb branch on $\mathbb{R}^3 \times S^1$ IV

- At large radius, a formal solution is obtained by iterating the system, leading to a ‘multi-instanton sum’

$$\mathcal{X}_\gamma = \mathcal{X}_\gamma^{\rm sf} \exp\left[\sum_T \prod_{(i,j)\in T_1} \langle \alpha_i, \alpha_j \rangle \prod_{i \in T_0} \bar\Omega(\alpha_i)\, g_T \right]$$

where T runs over trees decorated by charges α_i such that $\gamma = \sum \alpha_i$, g_T are iterated contour integrals.

A new supersymmetric index in $D = 4$

- Conjecture: the Witten index in $\mathcal{N} = 2, D = 4$ theories is

$$\mathcal{I}(R, u, C) = \frac{R}{16\mathrm{i}\pi^2} \sum_{\gamma} \Omega(\gamma) \int_{\ell_\gamma} \frac{\mathrm{d}t}{t} \left(t^{-1} Z_\gamma - t \bar{Z}_\gamma \right) \log\left(1 - \mathcal{X}_\gamma(t)\right)$$

- This function first appeared as the contact potential on the hypermultiplet moduli space in type II string vacua. In the analogy between the integral equations of GMN and TBA, $\mathcal{I}(R, u, C)$ is the free energy. Similarly, in the TBA approach to null Wilson loops in AdS, $\mathcal{I}(R, u, C)$ is the regularized area.

Alexandrov Saueressig BP Vandoren 08; Alexandrov Roche; Alday Gaiotto Maldacena 09

- $\mathcal{I}(R, u, C)$ is closely related to the Kähler potential for the HK metric on $\mathcal{M}_3(R)$ (upon adding a classical term, plus the Kähler potential for a canonical hyperholomorphic line bundle)

Smoothness across the walls

- To support this claim, note that for any smooth function $F_\gamma(t, u, C)$ on $\Gamma \times \mathcal{Z}$, linear in γ,

$$\Phi(R, u, C) = \sum_\gamma \Omega(\gamma) \int_{\ell_\gamma} \frac{\mathrm{d}t}{t} F_\gamma \log\left(1 - \mathcal{X}_\gamma\right).$$

is smooth across the walls, as a result of the dilogarithm identities

$$\sum_\gamma \Omega^+(\gamma) L_{\sigma_\gamma}(\mathcal{X}_\gamma^+) = \sum_\gamma \Omega^-(\gamma) L_{\sigma_\gamma}(\mathcal{X}_\gamma^-)$$

implied by the KS motivic wall-crossing formula. Here $L_\sigma(z) = \mathrm{Li}_2(z) + \frac{1}{2}\log(g/\sigma)\log(1-z)$ is a variant of Rogers' dilogarithm.

Alexandrov Persson BP 2011

- The proposed index arises for $F_\gamma \propto t^{-1} Z_\gamma - t\bar{Z}_\gamma$.

One-particle contributions

- In the large radius limit, approximating $\mathcal{X}_\gamma$ by $\mathcal{X}_\gamma^{\rm sf}$,

$$\mathcal{I}(R,u,C) = \sum_\gamma \frac{R}{8\pi^2}\,\sigma_\gamma\,\overline{\Omega}(\gamma)\,|Z_\gamma| K_1(2\pi R|Z_\gamma|)\,e^{-2\pi\mathrm{i}\langle\gamma,C\rangle} + \dots$$

- For γ primitive, this is the expected contribution of a single-particle, relativistic BPS state of charge γ:

$$\mathrm{Tr} e^{-2\pi R\sqrt{-\Delta+M^2}+\mathrm{i}\theta J_3} = \frac{L}{2\pi}\,\frac{\chi_{\rm spin}(\theta)}{4\sin^2(\theta/2)}\,2M\,K_1(2\pi MR)$$

provided we define the Witten index as follows:

$$\mathcal{I}(R,u,C) = R \lim_{\substack{\theta\to 2\pi \\ L\to\infty}} \partial_\theta^2 \left[\frac{\sin^2(\theta/2)}{\pi L}\,\mathrm{Tr}\Big(\sigma\, e^{-2\pi RH+\mathrm{i}\theta J_3-2\pi\mathrm{i}\langle\gamma,C\rangle}\Big)\right].$$

Multi-particle contributions I

- If correct, this predicts the contribution of an arbitrary multi-particle state. E.g. for two particles,

$$\mathcal{I}_{\gamma,\gamma'} = -\frac{R}{64\pi^3}\,\overline{\Omega}(\gamma)\,\overline{\Omega}(\gamma')\,\langle\gamma,\gamma'\rangle\,J_{\gamma,\gamma'}$$

where

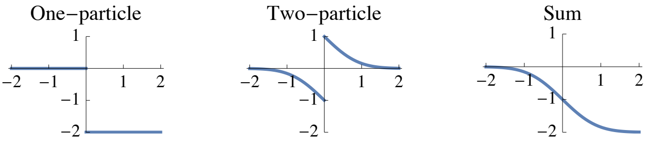
$$J_{\gamma,\gamma'} = \int_{\ell_\gamma}\frac{dt}{t}\int_{\ell'_\gamma}\frac{dt'}{t'}\,\frac{t+t'}{t-t'}\left(t^{-1}Z_\gamma - t\bar{Z}_\gamma\right)\mathcal{X}^{\rm sf}_\gamma(t)\mathcal{X}^{\rm sf}_{\gamma'}(t')\,,$$

- In the limit $R\to\infty$, $\psi_{\gamma,\gamma'}\to 0$, a saddle point approximation gives

$$J_{\gamma,\gamma'} \sim \mathrm{sgn}(\psi_{\gamma,\gamma'})\,\mathrm{Erfc}\left(|\psi_{\gamma\gamma'}|\sqrt{\frac{\pi R|Z_\gamma|\,|Z_{\gamma'}|}{|Z_\gamma|+|Z_{\gamma'}|}}\right)\,e^{-2\pi R|Z_{\gamma+\gamma'}|-2\pi i\langle\gamma+\gamma',C\rangle}$$

Multi-particle contributions II

- Using $\mathrm{Erf}(x) = \mathrm{sgn}(x)\,(1 - \mathrm{Erfc}(|x|)$, one checks that $\mathcal{I}_{\gamma+\gamma'} + \mathcal{I}_{\gamma,\gamma'}$ is smooth across the wall:



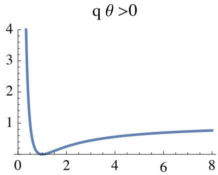
- In the remainder, we shall check this prediction by studying the quantum mechanics of a system of two mutually non-local non-relativistic particles.

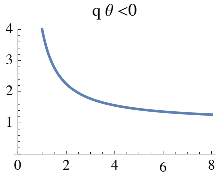
Non-relativistic electron-monopole problem I

- Let us return to the $\mathcal{N}=4$ supersymmetric quantum mechanics of the non-relativistic electron-monopole problem:

$$\boxed{H=\frac{1}{2m}(\vec{p}-q\vec{A})^2-\frac{q}{2m}\vec{B}\cdot\vec{\sigma}\otimes(1_2-\sigma_3)+\frac{1}{2m}\left(\vartheta-\frac{q}{r}\right)^2}$$

$$\vec{\nabla}\wedge\vec{A}=\vec{B}=\frac{\vec{r}}{r^3}\,,\quad m=\frac{|Z_\gamma||Z_{\gamma'}|}{|Z_\gamma|+|Z_{\gamma'}|}\,,\quad \frac{\vartheta^2}{2m}=|Z_\gamma|+|Z_{\gamma'}|-|Z_{\gamma+\gamma'}|$$





Non-relativistic electron-monopole problem II

- Going to a basis of monopole spherical harmonics, the Schrödinger equation with energy $E = k^2/(2m)$ becomes

$$\left[-\frac{1}{r}\partial_r^2\, r + \frac{\nu^2 - q^2 - \frac{1}{4}}{r^2} + \left(\vartheta - \frac{q}{r}\right)^2 \right] \Psi(r) = k^2 \Psi\,,$$

where

$$\nu = j + \frac{1}{2} + h\,, \quad j = |q| + h + \ell\,, \ell \in \mathbb{N}\,.$$

- Supersymmetric bound states exist for $q\vartheta > 0$, $h = -1/2$, $\ell = 0$, and form a multiplet of spin $j = |q| - \frac{1}{2}$, with $2j + 1 = |\langle \gamma_1, \gamma_2 \rangle|$.

Denef 2002

Non-relativistic electron-monopole problem III

- The S-matrix for partial waves is similar to that of H-atom,

$$S_\nu(k) = \frac{\Gamma\left(\frac{1}{2} + \nu + \mathrm{i}\frac{q\vartheta}{\sqrt{k^2-\vartheta^2}}\right)}{\Gamma\left(\frac{1}{2} + \nu - \mathrm{i}\frac{q\vartheta}{\sqrt{k^2-\vartheta^2}}\right)} = e^{2\mathrm{i}\delta_\nu(k)}.$$

BP 2015

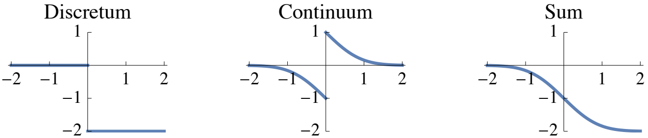
- The contribution of the continuum to $\mathrm{Tr}(-1)^F\, e^{-2\pi R H}$ is thus

$$\sum_{h=0^2,\pm\frac{1}{2}} (-1)^{2h} \sum_{\ell=0}^{\infty} \int_{k=\vartheta}^{\infty} \frac{\mathrm{d}k\,\partial_k}{2\pi\mathrm{i}} \log \frac{\Gamma\left(|q| + \ell + 2h + 1 + \mathrm{i}\frac{q\vartheta}{\sqrt{k^2-\vartheta^2}}\right)}{\Gamma\left(|q| + \ell + 2h + 1 - \mathrm{i}\frac{q\vartheta}{\sqrt{k^2-\vartheta^2}}\right)} e^{-\frac{\pi R k^2}{m}}$$

Non-relativistic electron-monopole problem IV

- Terms with $\ell > 0$ cancel, leaving the contribution from $\ell = 0$ only:

$$\begin{aligned}
\mathrm{Tr}(-1)^F\, e^{-2\pi R H} &= -\,|2q|\,\Theta(q\vartheta) - \frac{2q\vartheta}{\pi}\int_{k=|\vartheta|}^{\infty} \frac{\mathrm{d}k}{k\sqrt{k^2-\vartheta^2}}\, e^{-\frac{\pi R k^2}{m}} \\
&= -\,2|q|\,\Theta(q\vartheta) + |q|\,\mathrm{sgn}(q\vartheta)\,\mathrm{Erfc}\left(|\vartheta|\sqrt{\frac{\pi R}{m}}\right) \\
&= -\,|q| - q\,\mathrm{Erf}\left(\vartheta\sqrt{\frac{\pi R}{m}}\right) .
\end{aligned}$$



Conclusion

- Wall-crossing phenomena in four-dimensional SUSY gauge theories and string vacua can be interpreted as the (dis)appearance of multi-centered solitons.
- The wall-crossing formula is universal, and follows from the SUSY quantum mechanics of point-like particles interacting by Coulomb, Lorentz, (Newton) and scalar exchange forces.
- This suggests that the complete BPS spectrum can be constructed from bound states of a set of absolutely stable constituents, with fixed internal degeneracy $\Omega_S(\alpha)$. What is their mathematical meaning, e.g. in context of quiver moduli spaces ?
- The BPS indices $\Omega(\gamma, u)$ can be combined into a smooth function $\mathcal{I}(R, u, C)$, which should provide a new supersymmetric index for $\mathcal{N} = 2, D = 4$ theories. Can one check the spectral density for multi-particle states ? Can one compute $\mathcal{I}(R, u, C)$ exactly ?

Thank you for your attention !