Wall-crossing from Boltzmannian Black Hole Halos

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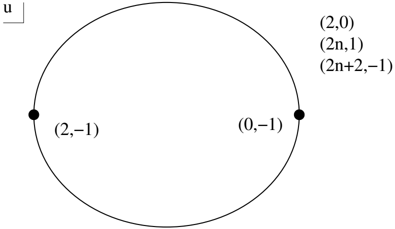
based on work with J. Manschot and A. Sen, to appear

Introduction I

- In $D = 4$, $N = 2$ supersymmetric field and string theories, the exact spectrum of BPS states can often be determined at weak coupling, and extrapolated to strong coupling.
- E.g., in pure $SU(2)$ Seiberg-Witten theory,



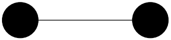
Seiberg Witten; Bilal Ferrari

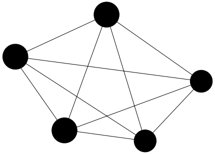
Introduction II

- In following the BPS spectrum from weak to strong coupling, one must be wary of two issues:
  - short multiplets may pair up into a long multiplet,
  - single-particle states may decay into multi-particle states.
- The first issue can be avoided by considering a suitable index $\Omega(\gamma, t)$, designed such that contributions from long multiplets cancel. $\Omega(\gamma, t)$ is then a piecewise constant function of the charge vector γ and couplings/moduli t.
- To deal with the second issue, one must understand how $\Omega(\gamma, t)$ changes across a wall of marginal stability W, where a single-particle state with charge γ can decay into a multi-particle state with charges $\{\gamma_i\}$, such that $\gamma = \sum_i \gamma_i$, $\arg Z(\gamma_i) = \arg Z(\gamma)$.

Introduction III

- Initial progress came from physics, by noting that single-particle states (in a certain limit) can be represented by multi-centered solitonic solutions. Those exist only on one side of the wall and decay into the continuum of multi-particle states on the other side.





- The simplest "primitive" decay $\gamma \to \gamma_1 + \gamma_2$ involves only two-centered configurations, whose index is easily computed.

Denef Moore

- In the non-primitive case $\gamma = M\gamma_1 + N\gamma_2$ where $M, N > 1$ (γ_1, γ_2 being two primitive vectors), many multi-centered configurations in general contribute, and computing their index is in general difficult.

Introduction IV

- A general answer to this problem has come from the mathematical study of the wall-crossing properties of (generalized) Donaldson-Thomas invariants for Calabi-Yau three-folds, or more generally CY-3 categories.
- These DT invariants are believed to be the mathematical translation of the BPS index $\Omega(\gamma)$ in type IIA CY vacua.
- Notably, Kontsevich & Soibelman (KS) and Joyce & Song (JS) gave two different-looking formulae for $\Delta\Omega(\gamma \to M\gamma_1 + N\gamma_2)$.
- Our goal will be to interpret/derive/generalize these formulae from physical reasoning.

Physical interpretation of the KS/JS formulae I

- The KS formula was first interpreted physically by considering Seiberg-Witten theory on $\mathbb{R}^3 \times S^1$: the resulting moduli space $\mathcal{M}_3$ is a torus bundle over $\mathcal{M}_4$, and is equipped with a hyperkähler metric. Instanton corrections to the metric on $\mathcal{M}_3$ come from BPS states in $D = 4$.

Seiberg Witten; Donagi

- The twistor space of $\mathcal{M}_3$ is a complex symplectic manifold. It can be specified by a set of symplectomorphisms U_γ between Darboux coordinate patches. The KS formula guarantees the consistency of this construction, and smoothness of the hyperkähler metric across the walls !

Gaiotto Moore Neitzke; Chen Dorey Petunin

Physical interpretation of the KS/JS formulae II

- The same story works formally in string theory, upon replacing (hyperkähler, symplectic) with (quaternion-Kähler, contact), and ignoring the exponential growth of DT invariants.

Alexandrov BP Saueressig Vandoren; BP Vandoren

- Recently, the (motivic/refined) KS formula was derived physically by using the notion of framed BPS states and supersymmetric galaxies. This reduces the general wall-crossing problem to a sequence of semi-primitive wall-crossings.

Andriyanash Denef Jafferis Moore

Our main results I

- We shall approach the wall-crossing problem by quantizing multi-centered solitonic/black hole configurations.

Denef; de Boer El Showk Messamah Van den Bleeken

- Our main new insight is a physical explanation of the relevance of the rational DT invariants

$$\bar{\Omega}(\gamma) \equiv \sum_{d|\gamma} \Omega(\gamma/d)/d^2 ,$$

which feature prominently in the KS/JS formulae: replacing $\Omega(\gamma) \to \bar{\Omega}(\gamma)$ effectively reduces the Bose-Fermi statistics of the centers to Boltzmannian statistics !

- We shall propose two new wall-crossing "Coulomb branch" and "Higgs branch" formulae, which appear to agree with KS/JS.

Outline

1. Introduction
2. A Boltzmannian view of wall-crossing
3. The Kontsevich-Soibelman formula
4. The Joyce-Song formula
5. Physical derivation of non-primitive wall-crossing

Outline

Preliminaries I

- We consider $\mathcal{N}=2$ supergravity in 4 dimensions (this includes field theories with rigid $\mathcal{N}=2$ as a special case). Let $\Gamma=\Gamma_e\oplus\Gamma_m$ be the lattice of electric and magnetic charges, with symplectic pairing

$$\langle\gamma,\gamma'\rangle=\langle(p^\Lambda,q_\Lambda),\gamma'=(p'^\Lambda,q'_\Lambda)\rangle\equiv q_\Lambda p'^\Lambda-q'_\Lambda p_\Lambda\in\mathbb{Z}$$

- BPS states preserve 4 out of 8 supercharges, and saturate the bound $M\geq|Z(\gamma,t^a)|$ where $Z(\gamma,t^a)=e^{\mathcal{K}/2}(q_\Lambda X^\Lambda-p^\Lambda F_\Lambda)$ is the central charge/stability data.
- We are interested in the index $\Omega(\gamma;t^a)=\mathrm{Tr}_{\mathcal{H}'_\gamma(t^a)}(-1)^{2J_3}$ where $\mathcal{H}'_\gamma(t^a)$ is the Hilbert space of stable states with charge $\gamma\in\Gamma$.

Preliminaries II

- The BPS invariants $\Omega(\gamma; t^a)$ are locally constant functions of t^a, but may jump across codimension-one subspaces

$$W(\gamma_1, \gamma_2) = \{ t^a \; / \; \arg[Z(\gamma_1)] = \arg[Z(\gamma_2)] \}$$

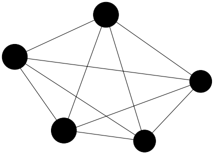
  where γ_1 and γ_2 are two primitive (non-zero) vectors such that $\gamma = M\gamma_1 + N\gamma_2$, $M, N \geq 1$.
- We choose γ_1, γ_2 such that $\Omega(\gamma; t^a)$ has support only on the positive cone (root basis property)

$$\tilde{\Gamma} : \quad \{ M\gamma_1 + N\gamma_2, \quad M, N \geq 0, \quad (M, N) \neq (0, 0) \} \; .$$

- Let $c_\pm$ be the chamber in which $\arg(Z_{\gamma_1}) \gtrless \arg(Z_{\gamma_2})$. Our aim is to compute $\Delta\Omega(\gamma) \equiv \Omega^-(\gamma) - \Omega^+(\gamma)$ as a function of $\Omega^+(\gamma)$ (say).

Wall-crossing from semi-classical solutions I

- Assume that $M(\gamma_1), M(\gamma_2)$ are much greater than the dynamical scale (Λ or m_P). In this limit, those single-particle states which are potentially unstable across W) can be described by classical configurations with n centers of charge $M_i\gamma_1 + N_i\gamma_2 \in \tilde{\Gamma}$, satisfying $(M, N) = \sum_i (M_i, N_i)$.



- *In addition, in either chamber, there may be multi-centered configurations whose charge vectors do not lie in $\tilde{\Gamma}$. However, they remain bound across W and do not contribute to $\Delta\Omega(\gamma)$.*

Wall-crossing from semi-classical solutions II

- Assume for definiteness that $\gamma_{12} < 0$. Then multi-centered solutions with charges in $\tilde{\Gamma}$ exist only in chamber c_-, not c_+. E.g. two-centered solutions can only exist when

$$r_{12} = \frac{1}{2} \frac{\langle \alpha_1, \alpha_2 \rangle \, |Z(\alpha_1) + Z(\alpha_2)|}{\mathrm{Im}[Z(\alpha_1) \bar{Z}(\alpha_2)]} > 0 \, .$$

Denef

- At the wall, r_{ij} diverges : the single-particle bound state decays into the continuum of multi-particle states.
- $\Delta\Omega(\gamma)$ is equal to the index of the SUSY quantum mechanics of n point-like particles, each carrying its own set of degrees of freedom with index $\Omega(\gamma_i)$, interacting via Newtonian and Coulomb forces.

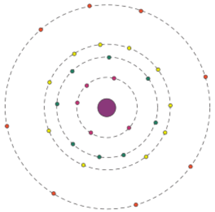
Wall-crossing from semi-classical solutions III

- For primitive decay $\gamma \to \gamma_1 + \gamma_2$, the quantization of the phase space of two-centered configuration reproduces the primitive WCF

$$\Delta\Omega(\gamma \to \gamma_1 + \gamma_2) = (-1)^{\gamma_{12}+1} \, |\gamma_{12}| \, \Omega^+(\gamma_1) \, \Omega^+(\gamma_2) \, ,$$

where $(-1)^{\gamma_{12}+1} \, |\gamma_{12}|$ is the index of Landau states on a sphere of radius r_{12} threaded by a magnetic flux $\gamma_{1,2}$.

- This generalizes to semi-primitive wall-crossing $\gamma \to \gamma_1 + N\gamma_2$: the potentially unstable configurations consist of of a “halo" of m_s particles of charge $s\gamma_2$, $\sum s m_s = N$, orbiting around a “core" of charge γ_1.



Denef Moore

Wall-crossing from semi-classical solutions IV

- This leads to a Mac-Mahon type partition function,

$$\frac{\sum_{N\geq 0} \Omega^-(1,N)\, q^N}{\sum_{N\geq 0} \Omega^+(1,N)\, q^N} = \prod_{k>0} \left(1-(-1)^{k\gamma_{12}} q^k\right)^{k\,|\gamma_{12}|\;\Omega^+(k\gamma_2)} \quad .$$

- E.g. for $\gamma \mapsto \gamma_1 + 2\gamma_2$,

$$\Delta\Omega(1,2) = \Omega^+(1,0)\left[2\gamma_{12}\,\Omega^+(0,2) + \frac{1}{2}\gamma_{12}\,\Omega^+(0,1)\left(\gamma_{12}\Omega^+(0,1)+1\right)\right] + \Omega^+(1,1)\left[(-1)^{\gamma_{12}}\gamma_{12}\Omega^+(0,1)\right] \; .$$

- The term $\frac{1}{2}d(d+1)$ with $d = \gamma_{12}\Omega^+(0,1)$, reflects the Bose/Fermi statistics of identical particles, i.e. the projection on (anti)symmetric wave functions.

Wall-crossing from semi-classical solutions V

- It is instructive to rewrite the semi-primitive wcf using the rational BPS invariants

$$\bar{\Omega}(\gamma) \equiv \sum_{d|\gamma} \Omega(\gamma/d)/d^2 \ ,$$

- By the Möbius inversion formula,

$$\Omega(\gamma) = \sum_{d|\gamma} \mu(d)\, \bar{\Omega}(\gamma/d)/d^2$$

  where $\mu(d)$ is the Möbius function (i.e. 1 if d is a product of an even number of distinct primes, -1 if d is a product of an odd number of primes, or 0 otherwise).

- The rational DT invariants $\bar{\Omega}(\gamma)$ appear in the JS formula, in constructions of modular invariant black hole partition functions, and in instanton corrections to hypermultiplet moduli spaces.

Manschot; Alexandrov BP Saueressig Vandoren

Wall-crossing from semi-classical solutions VI

- In the (1,2) example,

$$\Delta\bar{\Omega}(1,2) = \bar{\Omega}^{+}(1,0)\left[2\gamma_{12}\,\bar{\Omega}^{+}(0,2) + \frac{1}{2}\gamma_{12}\,\bar{\Omega}^{+}(0,1)^{2}\right] + \bar{\Omega}^{+}(1,1)\left[(-1)^{\gamma_{12}}\gamma_{12}\bar{\Omega}^{+}(0,1)\right] .$$

  is simpler, and manifestly consistent with charge conservation.
- More generally, using the identity $\prod_{d=1}^{\infty}(1-q^{d})^{\mu(d)/d} = e^{-q}$, or working backwards, the semi-primitive wcf can be rewritten as

$$\frac{\sum_{N\geq 0}\bar{\Omega}^{-}(1,N)\,q^{N}}{\sum_{N\geq 0}\bar{\Omega}^{+}(1,N)\,q^{N}} = \exp\left[\sum_{s=1}^{\infty} q^{s}(-1)^{\langle\gamma_1,s\gamma_2\rangle}\langle\gamma_1,s\gamma_2\rangle\bar{\Omega}^{+}(s\gamma_2)\right] .$$

- Physically, this follows by treating the particles in the halo as distinguishable, each carrying an effective index $\bar{\Omega}(s\gamma_2)$, and applying Boltzmann statistics !

- In general, we expect that the WCF is given by a sum

$$\Delta\bar{\Omega}(\gamma) = \sum_{n\geq 2} \sum_{\substack{\{\alpha_1,\dots\alpha_n\}\in\tilde{\Gamma} \\ \gamma=\alpha_1+\dots+\alpha_n}} \frac{g(\{\alpha_i\})}{|\mathrm{Aut}(\{\alpha_i\})|} \prod_{i=1}^{n} \bar{\Omega}^{+}(\alpha_i)\ ,$$

  over all unordered decompositions of the total charge vector γ into a sum of n vectors $\alpha_i \in \tilde{\Gamma}$. The symmetry factor $|\mathrm{Aut}(\{\alpha_i\})|$ is conventional, but natural in Boltzmannian statistics.

- The KS and JS formulae give a mathematical (implicit/explicit) prediction for the coefficients $g(\{\alpha_i\})$. After reviewing these formulae, we shall check them against a physical derivation based on black hole halo picture.

Outline

The Kontsevich-Soibelman formula I

- Consider the Lie algebra $\mathcal{A}$ spanned by abstract generators $\{e_\gamma, \gamma \in \Gamma\}$, satisfying the commutation rule

$$[e_{\gamma_1}, e_{\gamma_2}] = \kappa(\langle\gamma_1, \gamma_2\rangle)\, e_{\gamma_1+\gamma_2} \ , \qquad \kappa(x) = (-1)^x\, x \ .$$

- For a given charge vector γ and value of the VM moduli t^a, consider the operator $U_\gamma(t^a)$ in the Lie group $\exp(\mathcal{A})$

$$U_\gamma(t^a) \equiv \exp\left(\Omega(\gamma; t^a) \sum_{d=1}^{\infty} \frac{e_{d\gamma}}{d^2}\right)$$

- The operators e_γ / U_γ can be realized as Hamiltonian vector fields / symplectomorphisms of a twisted torus.

Gaiotto Moore Neitzke

The Kontsevich-Soibelman formula II

- The KS wall-crossing formula states that the product

$$A_{\gamma_1,\gamma_2} = \prod_{\substack{\gamma=M\gamma_1+N\gamma_2,\\ M\geq 0, N\geq 0}} U_\gamma \ ,$$

ordered so that $\arg(Z_\gamma)$ decreases from left to right, stays constant across the wall. As t^a crosses W, $\Omega(\gamma; t^a)$ jumps and the order of the factors is reversed, but the operator A_{γ_1,γ_2} stays constant. Equivalently,

$$\prod_{\substack{M\geq 0, N\geq 0,\\ M/N\downarrow}} U^+_{M\gamma_1+N\gamma_2} = \prod_{\substack{M\geq 0, N\geq 0,\\ M/N\uparrow}} U^-_{M\gamma_1+N\gamma_2} \ ,$$

The Kontsevich-Soibelman formula III

- The algebra $\mathcal{A}$ is infinite dimensional but filtered. The KS formula may be projected to any finite-dimensional algebra

$$\mathcal{A}_{M,N} = \mathcal{A}/\{ \sum_{m>M \text{ or } n>N} \mathbb{R} \cdot e_{m\gamma_1+n\gamma_2} \} \; .$$

This projection is sufficient to infer $\Delta\Omega(m\gamma_1 + n\gamma_2)$ for any $m \leq M, n \leq N$, e.g. using the Baker-Campbell-Hausdorff formula.

- For example, the primitive wcf follows in $\mathcal{A}_{1,1}$ from

$$\begin{aligned} & \exp(\bar{\Omega}^+(\gamma_1) e_{\gamma_1}) \exp(\bar{\Omega}^+(\gamma_1+\gamma_2) e_{\gamma_1+\gamma_2}) \exp(\bar{\Omega}^+(\gamma_2) e_{\gamma_2}) \\ = & \exp(\bar{\Omega}^-(\gamma_2) e_{\gamma_2}) \exp(\bar{\Omega}^-(\gamma_1+\gamma_2) e_{\gamma_1+\gamma_2}) \exp(\bar{\Omega}^-(\gamma_1) e_{\gamma_1}) \end{aligned}$$

and the order 2 truncation of the BCH formula

$$e^X \, e^Y = e^{X+Y+\frac{1}{2}[X,Y]} \; .$$

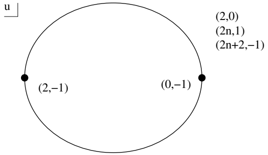
The Kontsevich-Soibelman formula IV

- In some simple cases, one may work in the full algebra $\mathcal{A}$, and use the “pentagonal identity"

$$U_{\gamma_2} U_{\gamma_1} = U_{\gamma_1} U_{\gamma_1+\gamma_2} U_{\gamma_2} \,, \qquad \gamma_{12} = -1$$

- Using this identity repeatedly, one can e.g. establish the wall-crossing identity in pure Seiberg-Witten $SU(2)$ theory,

$$U_{2,-1} \cdot U_{0,1} = U_{0,1} \cdot U_{2,1} \cdot U_{4,1} \dots U_{2,0}^{(-2)} \dots U_{3,-1} \cdot U_{2,-1} U_{1,-1}$$



Denef Moore; Dimofte Gukov Soibelman

The Kontsevich-Soibelman formula V

- Noting that the operators $U_{k\gamma}$ for different $k \geq 1$ commute, one may combine them into a single factor

$$V_\gamma \equiv \prod_{k=1}^{\infty} U_{k\gamma} = \exp\left(\sum_{\ell=1}^{\infty} \bar{\Omega}(\ell\gamma)\, e_{\ell\gamma}\right), \qquad \bar{\Omega}(\gamma) = \sum_{m|\gamma} m^{-2}\Omega(\gamma/m)\,.$$

  and rewrite the KS formula as a product over primitive charge vectors only,

$$\prod_{\substack{M\geq 0, N\geq 0,\\ \gcd(M,N)=1, M/N\downarrow}} V^{+}_{M\gamma_1+N\gamma_2} = \prod_{\substack{M\geq 0, N\geq 0,\\ \gcd(M,N)=1, M/N\uparrow}} V^{-}_{M\gamma_1+N\gamma_2}\,,$$

- Using the BCH formula, one easily derives the semi-primitive wcf formula, and generalizations to $\gamma \to 2\gamma_1 + N\gamma_2, \ldots$.

Generic decay I

- When α_i have generic phases, $g(\{\alpha_i\})$ can be computed by projecting the KS formula to the subalgebra spanned by $e_{\sum \alpha_j}$ where $\{\alpha_j\}$ runs over all subsets of $\{\alpha_i\}$.
- E.g., for $n = 3$, assuming that the phase of the charges are ordered according to

$$\alpha_1,\ \alpha_1 + \alpha_2,\ \alpha_1 + \alpha_3,\ \alpha_1 + \alpha_2 + \alpha_3,\ \alpha_2,\ \alpha_2 + \alpha_3,\ \alpha_3\ ,$$

we find

$$g(\{\alpha_1, \alpha_2, \alpha_3\}) = (-1)^{\alpha_{12}+\alpha_{23}+\alpha_{13}}\, \alpha_{12}\, (\alpha_{13} + \alpha_{23})$$

As we shall see later, this fits the macroscopic index of 3-centered configurations !

The motivic Kontsevich-Soibelman formula I

- KS have proposed a quantum deformation of their formula, which governs wall-crossing properties of motivic DT invariants $\Omega_{\rm ref}(\gamma; y, t)$. Physically, these correspond to the "refined index"

$$\Omega_{\rm ref}(\gamma, y) = {\rm Tr}'_{\mathcal{H}(\gamma)}(-y)^{2J_3} \equiv \sum_{n\in\mathbb{Z}} (-y)^n \, \Omega_{{\rm ref},n}(\gamma) \,,$$

  where J_3 is the angular momentum in 3 dimensions along the z axis (more accurately, a combination of angular momentum and $SU(2)_R$ quantum numbers). As $y \to 1$, $\Omega_{\rm ref}(\gamma; y, t) \to \Omega(\gamma; t)$.

  Dimofte Gukov Soibelman

- Caution: this index is protected in $\mathcal{N} = 2, D = 4$ field theories, but not in supergravity/string theory, where $SU(2)_R$ is generically broken.

The motivic Kontsevich-Soibelman formula II

- To state the formula, consider the Lie algebra $\mathcal{A}(y)$ spanned by generators $\{\tilde{e}_\gamma, \gamma \in \Gamma\}$, satisfying the commutation rule

$$[\tilde{e}_{\gamma_1}, \tilde{e}_{\gamma_2}] = \kappa(\langle\gamma_1,\gamma_2\rangle)\, \tilde{e}_{\gamma_1+\gamma_2} \ , \qquad \kappa(x) = \frac{(-y)^x - (-y)^{-x}}{y - 1/y} \ .$$

- To any charge vector γ, attach the operator

$$\hat{U}_\gamma = \prod_{n\in\mathbb{Z}} \mathbf{E}\left(\frac{y^n\, \tilde{e}_\gamma}{y-1/y}\right)^{-(-1)^n \Omega_{\mathrm{ref},n}(\gamma)} \ , \quad \mathbf{E}(x) \equiv \exp\left[\sum_{k=1}^{\infty} \frac{(xy)^k}{k(1-y^{2k})}\right]$$

where **E** is the quantum dilogarithm function.

The motivic Kontsevich-Soibelman formula III

- The motivic version of the KS wall-crossing formula again states that the ordered product

$$\hat{A}_{\gamma_1,\gamma_2} = \prod_{\substack{\gamma=M\gamma_1+N\gamma_2,\\ M\geq 0, N\geq 0}} \hat{U}_\gamma \ ,$$

  is constant across the wall.

- As before, one may combine the $\hat{U}_{k\gamma}$ into a single factor

$$\hat{V}_\gamma = \prod_{\ell\geq 1} \hat{U}_{\ell\gamma} = \exp\left[\sum_{N=1}^{\infty} \bar{\Omega}_{\rm ref}(N\gamma, y)\, \tilde{e}_{N\gamma}\right]$$

  where $\bar{\Omega}_{\rm ref}(N\gamma, y)$ are the “rational motivic invariants", defined by

$$\bar{\Omega}^+_{\rm ref}(\gamma, y) \equiv \sum_{m|\gamma} \frac{(y-y^{-1})}{m(y^m - y^{-m})} \Omega^+_{\rm ref}(\gamma/m, y^m)\,.$$

The motivic Kontsevich-Soibelman formula IV

- The motivic KS formula becomes

$$\prod_{\substack{M\geq 0, N\geq 0>0,\\ \gcd(M,N)=1, M/N\downarrow}} \hat{V}^{+}_{M\gamma_1+N\gamma_2} = \prod_{\substack{M\geq 0, N\geq 0>0,\\ \gcd(M,N)=1, M/N\uparrow}} \hat{V}^{-}_{M\gamma_1+N\gamma_2} ,$$

- $\Delta\bar{\Omega}_{\rm ref}(\gamma, y)$ can be computed using the same techniques as before, e.g. the primitive wcf read

$$\Delta\Omega_{\rm ref}(\gamma_1+\gamma_2, y) = \frac{(-y)^{\langle\gamma_1,\gamma_2\rangle} - (-y)^{-\langle\gamma_1,\gamma_2\rangle}}{y - 1/y}\, \Omega_{\rm ref}(\gamma_1, y)\, \Omega_{\rm ref}(\gamma_2, y)$$

Outline

The Joyce-Song formula I

- In the context of the Abelian category of coherent sheaves on a Calabi-Yau three-fold, Joyce & Song have shown that the jump of (generalized, rational) DT invariants across the wall is given by

$$\Delta\bar{\Omega}(\gamma) = \sum_{n\geq 2} \sum_{\substack{\{\alpha_1,\dots\alpha_n\}\in\tilde{\Gamma} \\ \gamma=\alpha_1+\dots+\alpha_n}} \frac{g(\{\alpha_i\})}{|\mathrm{Aut}(\{\alpha_i\})|} \prod_{i=1}^n \bar{\Omega}^+(\alpha_i) \, .$$

where the coefficient g is given by

$$g(\{\alpha_i\}) = \frac{1}{2^{n-1}} (-1)^{n-1+\sum_{i<j}\langle\alpha_i,\alpha_j\rangle} \sum_{\sigma\in\Sigma_n} \mathcal{L}\left(\alpha_{\sigma(1)}, \dots \alpha_{\sigma(n)}\right) \, U\left(\alpha_{\sigma(1)}, \dots \alpha_{\sigma(n)}\right)$$

The Joyce-Song formula II

- The definition of the $\mathcal{L}$ and $\mathcal{U}$ factors is technical and will be omitted. Let us simply say that they depend on the initial and final stability data and involve a sum over connected graphs.
- To derive the primitive wcf, note that there is only one oriented tree with 2 nodes. Assuming $\gamma_{12} < 0$, the JS data is then

$\sigma(12)$	S	U	$\mathcal{L}$
12	a	-1	γ_{12}
21	b	1	$-\gamma_{12}$

leading again to

$$\Delta\Omega(\gamma \to \gamma_1 + \gamma_2) = (-1)^{\gamma_{12}}\, \gamma_{12}\, \Omega(\gamma_1)\, \Omega(\gamma_2)\ , \qquad \gamma_{12} \equiv \langle \gamma_1, \gamma_2 \rangle$$

The Joyce-Song formula III

- For generic 3-body decay, assuming the same phase ordering as before and taking into account the 3 possible oriented trees, the JS data

$\sigma(123)$	S	U	$\mathcal{L}$
123	bb	1	$\alpha_{12}\alpha_{13} + \alpha_{13}\alpha_{23} + \alpha_{12}\alpha_{23}$
132	b-	0	$\alpha_{12}\alpha_{13} - \alpha_{13}\alpha_{23} - \alpha_{12}\alpha_{23}$
213	ab	-1	$-\alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{23} - \alpha_{12}\alpha_{13}$
231	-a	0	$\alpha_{12}\alpha_{13} - \alpha_{13}\alpha_{23} - \alpha_{12}\alpha_{23}$
312	ab	-1	$\alpha_{13}\alpha_{23} - \alpha_{12}\alpha_{23} - \alpha_{13}\alpha_{12}$
321	aa	1	$\alpha_{13}\alpha_{23} + \alpha_{12}\alpha_{13} + \alpha_{12}\alpha_{23}$

leads to the same answer as KS,

$$g(\{\alpha_1, \alpha_2, \alpha_3\}) = (-1)^{\alpha_{12}+\alpha_{23}+\alpha_{13}}\, \alpha_{12}\,(\alpha_{13} + \alpha_{23})$$

The Joyce-Song formula IV

- We have checked that JS and KS also agree for generic 4-body decay (involving 16 trees), 5-body decay (125 trees) and for special cases (2,3), (2,4) (up to 1296 graphs !).
- While there is no general proof yet, it seems that the JS formula (derived for Abelian categories) is equivalent to the classical KS formula (stated for triangulated categories).
- Note that the JS formula involves large denominators and leads to many cancellations. We shall find a more economic formula which also works at the motivic level.

Outline

Quantum mechanics of multi-centered solutions I

- The moduli space $\mathcal{M}_n$ of BPS configurations with n centers in $\mathcal{N}=2$ SUGRA is described by solutions to Denef's equations

$$\sum_{j=1\dots n, j\neq i}^{n} \frac{\alpha_{ij}}{|\vec{r}_i - \vec{r}_j|} = c_i \,, \qquad \begin{cases} c_i = 2\,\mathrm{Im}\left[e^{-i\alpha} Z(\alpha_i)\right] \\ \alpha = \arg[Z(\alpha_1 + \cdots \alpha_n)] \end{cases} .$$

- $\mathcal{M}_n$ is a compact symplectic manifold of dimension $2n-2$, and carries an Hamiltonian action of $SU(2)$:

$$\omega = \frac{1}{2}\sum_{i<j} \alpha_{ij} \frac{\mathrm{d}\vec{r}_{ij} \wedge \mathrm{d}\vec{r}_{ij} \cdot \vec{r}_{ij}}{|r_{ij}|^3} \,, \qquad \vec{J} = \frac{1}{2}\sum_{i<j} \alpha_{ij} \frac{\vec{r}_{ij}}{|r_{ij}|}$$

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Quantum mechanics of multi-centered solutions II

- Quantizing the internal degrees of freedom of the multi-centered configurations amounts to quantizing the symplectic space $\mathcal{M}_n$. The index is given, at least when $|\alpha_{ij}| \gg 1, y \to 1$, by

$$g(\{\alpha_i\}, y) = \frac{(-1)^{\sum_{i<j} \alpha_{ij} - n+1}}{(2\pi \sinh \nu/\nu)^{n-1}} \int_{\mathcal{M}_n} \omega^{n-1}\, e^{2\nu J_3} \,, \qquad \nu \equiv \log y$$

  We conjecture that this is exact for all α_{ij}, y.

- By the Duistermaat-Heckmann theorem, the integral localizes to the fixed points of the action of J_3, i.e. collinear multi-centered configurations along the z-axis, such that

$$\sum_{j=1\dots n, j\neq i}^{n} \frac{\alpha_{ij}}{|z_i - z_j|} = c_i \,, \quad J_3 = \frac{1}{2} \sum_{i<j} \alpha_{ij}\, \mathrm{sign}(z_i - z_j) \,.$$

Quantum mechanics of multi-centered solutions III

- These are classified by permutations σ describing the order of z_i along the axis. Let $\mathcal{S}(t)$ be the set of permutations allowed by Denef's equations. Localization leads to the Coulomb branch formula

$$g_{\rm ref}(\{\alpha_i\}, y) = \frac{(-1)^{\sum_{i<j}\alpha_{ij}+n-1}}{(y-y^{-1})^{n-1}} \sum_{\sigma\in\mathcal{S}(t)} s(\sigma)\, y^{\sum_{i<j}\alpha_{ij}\,{\rm sign}(\sigma(j)-\sigma(i))} .$$

  where $s(\sigma) = (-1)^{\#\{i;\sigma(i+1)<\sigma(i)\}}$ originates from Hessian(J_3).
- For $n \le 5$, we find perfect agreement with JS/KS !

$$g(\alpha_1, \alpha_2; y) = (-1)^{\alpha_{12}} \frac{\sinh(\nu\alpha_{12})}{\sinh\nu}$$

$$g(\alpha_1, \alpha_2, \alpha_3; y) = (-1)^{\alpha_{13}+\alpha_{23}+\alpha_{12}} \frac{\sinh(\nu(\alpha_{13}+\alpha_{23}))\,\sinh(\nu\alpha_{12})}{\sinh^2\nu} ,$$

Higgs branch picture I

- An alternative formula can be given using the Higgs branch description of the multi-centered configuration, namely the quiver with n nodes $\{1 \dots n\}$ of dimension 1 and α_{ij} arrows from i to j.
- Since α_i lie on a 2-dimensional sublattice $\tilde{\Gamma}$, the quiver has no oriented closed loop. Reineke's formula gives

$$g_{\rm ref} = \frac{(-y)^{-\sum_{i<j}\alpha_{ij}}}{(y-1/y)^{n-1}} \sum_{\rm partitions} (-1)^{s-1} y^{2\sum_{a\leq b}\sum_{j<i}\alpha_{ji}\, m_i^{(a)}\, m_j^{(b)}} ,$$

  where $\sum$ runs over all ordered partitions of $\gamma = \alpha_1 + \dots + \alpha_n$ into s vectors $\beta^{(a)}$ ($1 \leq a \leq s$, $1 \leq s \leq n$) such that
  1. $\beta^{(a)} = \sum_i m_i^{(a)} \alpha_i$ with $m_i^{(a)} \in \{0,1\}$, $\sum_a \beta^{(a)} = \gamma$
  2. $\langle \sum_{a=1}^{b} \beta^{(a)}, \gamma \rangle > 0 \quad \forall \quad b \quad$ with $\quad 1 \leq b \leq s-1$
- The formula agrees with KS/JS/Coulomb for $n = 2,3,4,5$!

Conclusion I

- Multi-centered black hole configurations provide a simple way to derive wall-crossing formulae for DT invariants.
- We have not proven the equivalence between our formulae, JS and KS, but there is overwhelming evidence that they all agree.
- The Coulomb branch formula seems the most economic: no denominators, no cancellations. Sadly, we do not know how to characterize $\mathcal{S}(t)$ (yet).
- The derivation of JS/KS relies on Ringel-Hall algebras. What does this mean physically ? is this the long-sought Algebra of BPS states ?

THANK YOU !