Wall-crossing from Boltzmannian Black Hole Halos

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based on work with J. Manschot and A. Sen, to appear

Introduction I

- In SUSY field theories and string theory vacua with extended supersymmetry, the spectrum of BPS states can often be determined exactly at weak coupling.
- In following the BPS spectrum from weak to strong coupling, one must be wary of two issues:
  - short multiplets may pair up into a long multiplet,
  - single-particle states may decay into multi-particle states.
- The first issue can be avoided by considering a suitable index $\Omega(\gamma, t)$, designed such that contributions from long multiplets cancel. $\Omega(\gamma, t)$ is then a piecewise constant function of the charge vector γ and couplings/moduli t.

Introduction II

- To deal with the second issue, one must understand how $\Omega(\gamma, t)$ changes across a wall of marginal stability W, where a single-particle state with charge γ can decay into a multi-particle state with charges $\{\gamma_i\}$, such that $\gamma = \sum_i \gamma_i$, $M(\gamma) = \sum_i M(\gamma_i)$.
- Initial progress came from physics, by noting that single-particle states (in a certain limit) can be represented by multi-centered solitonic solutions. Those exist only on one side of the wall and decay into the continuum of multi-particle states on the other side.
- When $\gamma = \gamma_1 + \gamma_2$ is the sum of two primitive vectors, the index of the two-centered configuration is easily computed, leading to the primitive wall crossing formula for $D = 4, N = 2$ vacua:

$$\Delta\Omega(\gamma \to \gamma_1 + \gamma_2) = (-1)^{\gamma_{12}}\, \gamma_{12}\, \Omega(\gamma_1)\, \Omega(\gamma_2)\ , \qquad \gamma_{12} \equiv \langle \gamma_1, \gamma_2 \rangle$$

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Introduction III

- In the non-primitive case $\gamma = M\gamma_1 + N\gamma_2$ where $M, N > 1$ (γ_1, γ_2 being two primitive vectors), many multi-centered configurations in general contribute, and computing their index is much harder.
- The general answer to this problem came from the mathematical study of the wall-crossing properties of (generalized) Donaldson-Thomas invariants for Calabi-Yau three-folds. These are believed to be the mathematical translation of the BPS index $\Omega(\gamma)$ in type IIA CY vacua.
- Notably, Kontsevich & Soibelman (KS) and Joyce & Song (JS) have given implicit and explicit formulae for $\Delta\Omega(\gamma \to M\gamma_1 + N\gamma_2)$. Our main goal will be to interpret these formulae physically.

Physical interpretation of the KS/JS formulae I

- The KS formula was first interpreted physically in terms of the VM moduli space $\mathcal{M}_3$ of the $\mathcal{N}=2, D=4$ theory compactified on a circle S^1 of radius R. SUSY requires that $\mathcal{M}_3$ is hyperkähler (in field theory) / quaternion-Kähler (in SUGRA).
- The HK/QK metric on $\mathcal{M}_3$ is conveniently described in terms of the complex symplectic/contact structure on the twistor space $\mathcal{Z}$, a $\mathbb{P}^1$ bundle over $\mathcal{M}_3$.
- Above a fixed point $t \in \mathcal{M}_4$, the symplectic/contact structure is specified by a set of symplectomorphisms U_γ between Darboux coordinate patches. The KS formula guarantees the smoothness of the metric as t crosses a wall of marginal stability.

Gaiotto Moore Neitzke; Chen Dorey Petunin; Alexandrov BP Saueressig Vandoren

Physical interpretation of the KS/JS formulae II

- Recently, the (motivic/refined) KS formula was derived physically by using the notion of framed BPS states and supersymmetric galaxies. This reduces the general wall-crossing problem to a sequence of semi-primitive wall-crossings.

Andriyanash Denef Jafferis Moore

- The JS formula has not been interpreted physically yet. Its equivalence with KS is still conjectural.
- Here we shall interpret (and seek to derive) the KS/JS formulae in terms of the supersymmetric quantum mechanics of multi-centered solitonic/black hole configurations.

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Physical interpretation of the KS/JS formulae III

- In particular, we shall explain the physical relevance of the rational DT invariants

$$\bar{\Omega}(\gamma) \equiv \sum_{d|\gamma} \Omega(\gamma/d)/d^2 ,$$

  which feature prominently in the KS/JS formulae: replacing $\Omega(\gamma) \to \bar{\Omega}(\gamma)$ effectively reduces the Bose-Fermi statistics of the centers to Boltzmannian statistics !

- We shall also apply the KS/JS formulae to derive generalizations of the semi-primitive wall-crossing formula, and compute the index of D0-D6 bound states with $[D6] = 2, 3$.

Outline

Outline

Preliminaries I

- We consider $\mathcal{N} = 2$ supergravity in 4 dimensions (this includes field theories with rigid $\mathcal{N} = 2$ as a special case). Let $\Gamma = \Gamma_e \oplus \Gamma_m$ be the lattice of electric and magnetic charges, with symplectic pairing

$$\langle \gamma, \gamma' \rangle = \langle (p^\Lambda, q_\Lambda), \gamma' = (p'^\Lambda, q'_\Lambda) \rangle \equiv q_\Lambda p'^\Lambda - q'_\Lambda p_\Lambda \in \mathbb{Z}$$

- BPS states preserve 4 out of 8 supercharges, and saturate the bound $M(\gamma) \geq |Z(\gamma)|$ with $Z(\gamma) = e^{\mathcal{K}/2}(q_\Lambda X^\Lambda - p^\Lambda F_\Lambda)$.
- We are interested in the index $\Omega(\gamma; t^a) = \mathrm{Tr}_{\mathcal{H}'_\gamma(t^a)}(-1)^{2J_3}$ where $\mathcal{H}'_\gamma(t^a)$ is the Hilbert space of states with charge $\gamma \in \Gamma$ in the vacuum where the VM scalars asymptote to t^a at spatial infinity, after factoring out the bosonic and fermionic center of motion d.o.f.

Preliminaries II

- The BPS invariants $\Omega(\gamma; t^a)$ are locally constant functions of t^a, but may jump across codimension-one subspaces

$$W(\gamma_1, \gamma_2) = \{t^a \,/\, \arg[Z(\gamma_1)] = \arg[Z(\gamma_2)]\}$$

  where γ_1 and γ_2 are two primitive (non-zero) vectors such that $\gamma = M\gamma_1 + N\gamma_2$, $M, N \geq 1$.
- Let $c_\pm$ be the chamber in which $\arg(Z_{\gamma_1}) \gtrless \arg(Z_{\gamma_2})$. Our aim is to compute $\Delta\Omega(\gamma) \equiv \Omega^-(\gamma) - \Omega^+(\gamma)$ as a function of $\Omega^+(\gamma)$ (say).
- Assume that close to $W(\gamma_1, \gamma_2)$, $\Omega(M\gamma_1 + N\gamma_2) = 0$ whenever $MN < 0$ (root property). Let $\tilde{\Gamma}$ be the positive cone

$$\tilde{\Gamma} : \quad \{M\gamma_1 + N\gamma_2, \quad M, N \geq 0, \quad (M, N) \neq (0, 0)\} .$$

Wall-crossing from semi-classical solutions I

- Assume that $M(\gamma_1), M(\gamma_2)$ are much greater than the dynamical scale (Λ or m_P). In this limit, single-particle states (potentially unstable across W) can be described by classical configurations with $m_{r,s}$ centers of charge $r\gamma_1 + s\gamma_2 \in \tilde{\Gamma}$, satisfying $(M, N) = \sum (r, s) m_{r,s}$.
- *In addition, in either chamber, there may be multi-centered configurations whose charge vectors do not lie in $\tilde{\Gamma}$. However, they remain bound across W and do not contribute to $\Delta\Omega(\gamma)$.*
- Assume for definiteness that $\gamma_{12} < 0$. Then multi-centered solutions with charges in $\tilde{\Gamma}$ exist only in chamber c_-, not c_+. E.g. two-centered solutions can only exist when

$$r_{12} = \frac{1}{2} \frac{\langle \alpha_1, \alpha_2 \rangle \, |Z(\alpha_1) + Z(\alpha_2)|}{\mathrm{Im}[Z(\alpha_1) \bar{Z}(\alpha_2)]} > 0 \; .$$

Denef

Wall-crossing from semi-classical solutions II

- At the wall, $r_{ij} \to \infty$: the single-particle bound state decays into the continuum of multi-particle state. $\Delta\Omega(\gamma)$ is equal to the index of the SUSY quantum mechanics describing the internal d.o.f. of the multi-centered configurations which are gained/lost across the wall.
- Close to the wall, this reduces to the SUSY quantum mechanics of point-like particles, each carrying its own set of degrees of freedom with index $\Omega(\gamma_i)$, interacting via Newtonian and Coulomb forces. The statistics of each center is bosonic or fermionic, depending on the sign of $\Omega(\gamma_i)$.

Wall-crossing from semi-classical solutions III

- For primitive decay $\gamma \to \gamma_1 + \gamma_2$, one recovers the primitive WCF

$$\Delta\Omega(\gamma \to \gamma_1 + \gamma_2) = (-1)^{\gamma_{12}+1}\, |\gamma_{12}|\, \Omega^+(\gamma_1)\, \Omega^+(\gamma_2)\ ,$$

  where $(-1)^{\gamma_{12}+1}\, |\gamma_{12}|$ is the index of Landau states on a sphere of radius r_{12} threaded by a magnetic flux $\gamma_{1,2}$.

- This argument generalizes to semi-primitive wall-crossing $\gamma \to \gamma_1 + N\gamma_2$: one set of classical configurations consists of a “halo" of m_s particles of charge $s\gamma_2$, $\sum s m_s = N$, orbiting around one particle of charge γ_1.

$$\begin{aligned} Z_{\rm halo}(\gamma_1, q) \equiv & 1 + \sum\nolimits_{\{m_s\}} \Delta\Omega(\gamma \to \gamma_1 + \sum s\, m_s \gamma_2)\, q^{s\, m_s} \\ = & \prod_{k>0} \left(1 - (-1)^{k\gamma_{12}} q^k\right)^{k\, |\gamma_{12}|\ \Omega^+(k\gamma_2)} \ . \end{aligned}$$

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Wall-crossing from semi-classical solutions IV

- More generally however, there are configurations with a core of charge $\gamma_1 + l\gamma_2$ and halo of total charge $(N-l)\gamma_2$. Defining

$$Z^{\pm}(1,q) = \sum_{N\geq 0} \Omega^{\pm}(\gamma_1 + N\gamma_2)\, q^N \,,$$

  the final formula is $Z^{-}(1,q) = Z^{+}(1,q)\, Z_{\rm halo}(\gamma_1, q)$. E.g.

$$\Delta\Omega(1,2) = \Omega^+(1,0)\left[2\gamma_{12}\,\Omega^+(0,2) + \frac{1}{2}\gamma_{12}\,\Omega^+(0,1)\left(\gamma_{12}\Omega^+(0,1)+1\right)\right] + \Omega^+(1,1)\left[(-1)^{\gamma_{12}}\gamma_{12}\Omega^+(0,1)\right] \,.$$

- The term in red reflects the Bose/Fermi statistics of the particles with degeneracy $\gamma_{12}\Omega^+(0,1)$ and apparently “violates charge conservation" !

Wall-crossing from semi-classical solutions V

- It is instructive to rewrite the semi-primitive wcf using the rational BPS invariants

$$\bar{\Omega}(\gamma) \equiv \sum_{d|\gamma} \Omega(\gamma/d)/d^2 \,, \qquad \Omega(\gamma) = \sum_{d|\gamma} \mu(d)\,\bar{\Omega}(\gamma/d)/d^2 \,,$$

  where $\mu(d)$ is the Möbius function (i.e. 1 if d is a product of an even number of distinct primes, -1 if d is a product of an odd number of primes, or 0 otherwise).

- Using the identity $\prod_{d=1}^{\infty}(1-q^d)^{\mu(d)/d} = e^{-q}$, or working backwards, we can rewrite

$$Z_{\rm halo}(\gamma_1, q) = \exp\left[\sum_{s=1}^{\infty} q^s (-1)^{\langle\gamma_1, s\gamma_2\rangle} \langle\gamma_1, s\gamma_2\rangle \bar{\Omega}^+(s\gamma_2)\right] \,.$$

Wall-crossing from semi-classical solutions VI

- Thus, the halo partition function can be equivalently obtained by treating the particles in the halo as distinguishable, each carrying an effective index $\bar{\Omega}(s\gamma_2)$, and applying Boltzmann statistics !
- In terms of the rational invariants, the WCF is simpler, and manifestly consistent with charge conservation. E.g.,

$$\Delta\bar{\Omega}(1,2) = \bar{\Omega}^{+}(1,0)\left[2\gamma_{12}\,\bar{\Omega}^{+}(0,2) + \frac{1}{2}\gamma_{12}\,\bar{\Omega}^{+}(0,1)^2\right] + \bar{\Omega}^{+}(1,1)\left[(-1)^{\gamma_{12}}\gamma_{12}\bar{\Omega}^{+}(0,1)\right] \; .$$

- The rational DT invariants $\bar{\Omega}(\gamma)$ are also useful in constructing modular invariant black hole partition functions, and in computing instanton corrections to hypermultiplet moduli spaces.

Manschot; Alexandrov BP Saueressig Vandoren

The main conjecture I

- In general, we expect that the WCF is given by a sum

$$\Delta\bar{\Omega}(\gamma) = \sum_{n\geq 2} \sum_{\substack{\{\alpha_1,\dots\alpha_n\}\in\tilde{\Gamma} \\ \gamma=\alpha_1+\dots+\alpha_n}} \frac{g(\{\alpha_i\})}{|\mathrm{Aut}(\{\alpha_i\})|} \prod_{i=1}^n \bar{\Omega}^+(\alpha_i) \,,$$

  over all unordered decompositions of the total charge vector γ into a sum of n vectors $\alpha_i \in \tilde{\Gamma}$. The symmetry factor $|\mathrm{Aut}(\{\alpha_i\})|$ is the one relevant for Boltzmannian statistics.
- We conjecture that the coefficient $g(\{\alpha_i\})$ is equal to the index of the SUSY quantum mechanics of n distinguishable particles with charge α_i.

The main conjecture II

- The KS/JS formulae give a mathematical (implicit/explicit) prediction for the coefficients $g(\{\alpha_i\})$. We shall show that this prediction is correct for $n = 2, 3$.
- The computation of the SUSY index for $n \geq 4$ is a difficult problem, which may be amenable to localization methods. Hopefully, this will lead to a new, elementary physical derivation of the JS/KS formula.

Outline

The Kontsevich-Soibelman formula I

- Consider the Lie algebra $\mathcal{A}$ spanned by abstract generators $\{e_\gamma, \gamma \in \Gamma\}$, satisfying the commutation rule

$$[e_{\gamma_1}, e_{\gamma_2}] = \kappa(\langle\gamma_1, \gamma_2\rangle)\, e_{\gamma_1+\gamma_2} \ , \qquad \kappa(x) = (-1)^x\, x \ .$$

- For a given charge vector γ and value of the VM moduli t^a, consider the operator $U_\gamma(t^a)$ in the Lie group $\exp(\mathcal{A})$

$$U_\gamma(t^a) \equiv \exp\left(\Omega(\gamma; t^a) \sum_{d=1}^{\infty} \frac{e_{d\gamma}}{d^2}\right)$$

- The operators e_γ / U_γ can be realized as Hamiltonian vector fields / symplectomorphisms of a twisted torus.

Gaiotto Moore Neitzke

The Kontsevich-Soibelman formula II

- The KS wall-crossing formula states that the product

$$A_{\gamma_1,\gamma_2} = \prod_{\substack{\gamma=M\gamma_1+N\gamma_2,\\ M\geq 0, N\geq 0}} U_\gamma \ ,$$

ordered so that $\arg(Z_\gamma)$ decreases from left to right, stays constant across the wall. As t^a crosses W, $\Omega(\gamma;t^a)$ jumps and the order of the factors is reversed, but the operator A_{γ_1,γ_2} stays constant. Equivalently,

$$\prod_{\substack{M\geq 0, N\geq 0,\\ M/N\downarrow}} U^+_{M\gamma_1+N\gamma_2} = \prod_{\substack{M\geq 0, N\geq 0,\\ M/N\uparrow}} U^-_{M\gamma_1+N\gamma_2} \ ,$$

The Kontsevich-Soibelman formula III

- Noting that the operators $U_{k\gamma}$ for different $k \geq 1$ commute, one may combine them into a single factor

$$V_\gamma \equiv \prod_{k=1}^{\infty} U_{k\gamma} = \exp\left(\sum_{\ell=1}^{\infty} \bar{\Omega}(\ell\gamma)\, e_{\ell\gamma}\right), \qquad \bar{\Omega}(\gamma) = \sum_{m|\gamma} m^{-2} \Omega(\gamma/m)\,.$$

  and rewrite the KS formula as a product over primitive charge vectors only,

$$\prod_{\substack{M\geq 0, N\geq 0,\\ \gcd(M,N)=1, M/N\downarrow}} V^{+}_{M\gamma_1+N\gamma_2} = \prod_{\substack{M\geq 0, N\geq 0,\\ \gcd(M,N)=1, M/N\uparrow}} V^{-}_{M\gamma_1+N\gamma_2}\,,$$

The Kontsevich-Soibelman formula IV

- The algebra $\mathcal{A}$ is infinite dimensional, but the KS formula may be projected to any finite-dimensional algebra

$$\mathcal{A}_{M,N} = \mathcal{A}/\{ \sum_{m>M \text{ or } n>N} \mathbb{R} \cdot e_{m\gamma_1+n\gamma_2} \} \, .$$

  This truncation is sufficient to infer $\Delta\Omega(m\gamma_1 + n\gamma_2)$ for any $m \leq M, n \leq N$, e.g. using the Baker-Campbell-Hausdorff formula.

- For example, the primitive wcf follows in $\mathcal{A}_{1,1}$ from

$$\begin{aligned} &\exp(\bar\Omega^+(\gamma_1)e_{\gamma_1}) \, \exp(\bar\Omega^+(\gamma_1+\gamma_2)e_{\gamma_1+\gamma_2}) \, \exp(\bar\Omega^+(\gamma_2)e_{\gamma_2}) \\ = &\exp(\bar\Omega^-(\gamma_2)e_{\gamma_2}) \, \exp(\bar\Omega^-(\gamma_1+\gamma_2)e_{\gamma_1+\gamma_2}) \, \exp(\bar\Omega^-(\gamma_1)e_{\gamma_1}) \end{aligned}$$

  and the order 2 truncation of the BCH formula

$$e^X \, e^Y = e^{X+Y+\frac{1}{2}[X,Y]} \, .$$

The Kontsevich-Soibelman formula V

- In some cases, once can work directly with the full algebra $\mathcal{A}$, upon using the identity

$$U_{\gamma_2} \, U_{\gamma_1} = U_{\gamma_1} \, U_{\gamma_1+\gamma_2} \, U_{\gamma_2} \, , \qquad \gamma_{12} = -1$$

  which follows from the pentagonal identity for the di-logarithm.

- Using this identity repeatedly, one can e.g. establish

$$U_{2,-1} \cdot U_{0,1} = U_{0,1} \cdot U_{2,1} \cdot U_{4,1} \dots U_{2,0} \dots U_{3,-1} \cdot U_{2,-1} U_{1,-1}$$

  where $\Omega(\gamma) = 1$ in each factor except $\Omega(2,0) = -2$. This reproduces the wall-crossing from the strong to weak coupling region in $\mathcal{N} = 2$ SYM with $G = SU(2)$ and no flavor.

Seiberg Witten; Bilal Ferrari; Denef

The Kontsevich-Soibelman formula VI

- The semi-primitive formula can be derived similarly by projecting the KS formula to $\mathcal{A}_{1,\infty}$,

$$V^+_{\gamma_1}\, V^+_{\gamma_1+\gamma_2}\, V^+_{\gamma_1+2\gamma_2} \cdots V^+_{\gamma_2} = V^-_{\gamma_2} \cdots V^-_{\gamma_1+2\gamma_2}\, V^-_{\gamma_1+\gamma_2}\, V^-_{\gamma_1}$$

  and combining on either side the factors $V^+_{\gamma_1+N\gamma_2}$ in a single exponential using the order-2 BCH formula:

$$e^{X_1^+}\, V^+_{\gamma_2} = V^-_{\gamma_2} e^{X_1^-}$$

- Finally, the Hadamard lemma for $e^Y = V^+_{\gamma_2} = V^-_{\gamma_2}, X = e^{X_1^+}$

$$e^Y\, X\, e^{-Y} = X + [Y, X] + \frac{1}{2!}[Y, [Y, X]] + + \frac{1}{3!}[Y, [Y, [Y, X]]] + \dots$$

  leads directly to $Z^-(1, q) = Z^+(1, q)\, Z_{\rm halo}(\gamma_1, q)$.

The Kontsevich-Soibelman formula VII

- By projecting the KS formula to $\mathcal{A}_{M,\infty}$, one can obtain "order M" generalizations of the semi-primitive WCF, e.g. for $M = 2$

$$\widetilde{Z}_2^-(q) = \widetilde{Z}_2^+(q)\, Z_{\rm halo}(2\gamma_1, q)$$

where

$$\widetilde{Z}_2^\pm(q) \equiv \sum_{N\geq 0}^{\infty} \bar\Omega^\pm(2\gamma_1 + N\gamma_2)\, q^N$$
$$\pm \frac{1}{4} \sum_{N_1,N_2\geq 0} \kappa(|N_1 - N_2|\gamma_{12})\, \bar\Omega^\pm(\gamma_1 + N_1\gamma_2)\, \bar\Omega^+(\gamma_1 + N_2\gamma_2)\, q^{N_1+N_2} .$$

and $Z_{\rm halo}(2\gamma_1, q)$ is the same factor which appeared in the semi-primitive wcf, after replacing $\gamma_1 \mapsto 2\gamma_1$.

Toda; Stoppa; Cheung Diaconescu Pan

D6-D0 bound states I

- E.g for D6-D0 bound states (i.e. dimension zero sheaves on $\mathcal{X}$): at large volume, zero B-field,

$D6 \backslash D0$	0	1	2	3	4
0	$\cdot$	$-\chi$	$-\chi$	$-\chi$	$-\chi$
1	1	0	0	0	$\dots$
2	0	0	0	0	$\dots$
3	0	0	0	0	$\dots$

$$\Omega^+(1,0) = 1 \ , \qquad \Omega^+(0,n) = -\chi \quad (n > 0) \ .$$

D6-D0 bound states II

- As the B-field is increased, one enters the DT chamber, wherein

$D6\backslash D0$	0	1	2	3	4
0	$\cdot$	$-\chi$	$-\chi$	$-\chi$	$-\chi$
1	1	$-\chi$	$\frac{1}{2}(\chi^2+5\chi)$	$-\frac{1}{6}(\chi^3+15\chi^2+20\chi)$	$\dots$
2	0	0	$-\chi$	$-\frac{1}{6}(\chi^3+15\chi^2+20\chi)$	$\dots$
3	0	0	0	$-\chi$	$\dots$

- The partition function of rank 1 DT invariants is

$$Z^{-}(1,q) = [M(-q)]^{\chi}\ , \qquad M(q) = \prod_{n\geq 1} 1-q^n)^n$$

D6-D0 bound states III

- The partition function of rank 2 DT invariants is

$$Z^{-}(2,q) = \frac{1}{4}\left([M(q)]^{2\chi} - [M(-q^2)]^{\chi}\right) - \frac{1}{4}\sum_{n_1,n_2} \kappa(|n_1 - n_2|)\Omega^{-}(1,n_1)\,\Omega^{-}(1,n_2)\,q^{n_1+n_2}$$

Toda; Stoppa; Nagao

Generic decay I

- When α_i have generic phases, $g(\{\alpha_i\})$ can be computed by projecting the KS formula to the subalgebra spanned by $e_{\sum \alpha_j}$ where $\{\alpha_j\}$ runs over all subsets of $\{\alpha_i\}$.
- E.g., for $n = 3$, assuming that the phase of the charges are ordered according to

$$\alpha_1, \ (\alpha_1 + \alpha_2, \quad \alpha_1 + \alpha_3), \ \alpha_1 + \alpha_2 + \alpha_3, \ \alpha_2, \ \alpha_2 + \alpha_3, \ \alpha_3 ,$$

we find

$$g(\{\alpha_1, \alpha_2, \alpha_3\}) = (-1)^{\alpha_{12}+\alpha_{23}+\alpha_{13}} \, \alpha_{12} \, (\alpha_{13} + \alpha_{23})$$

As we shall see later, this fits the macroscopic index !

Generic decay II

- Similarly, for $n = 4$, assuming the clockwise ordering

$$\alpha_1\,,(\alpha_1+\alpha_2\,,\,\alpha_1+\alpha_3\,,\,\alpha_1+\alpha_2+\alpha_3)\,,\,\alpha_2\,,$$
$$(\alpha_2+\alpha_3\,,\,\alpha_1+\alpha_2+\alpha_4)\,,\alpha_1+\alpha_2+\alpha_3+\alpha_4\,,\,\alpha_1+\alpha_3+\alpha_4\,,$$
$$\alpha_3\,,\,(\alpha_1+\alpha_4\,,\,\alpha_2+\alpha_4\,,\,\alpha_2+\alpha_3+\alpha_4\,,\,\alpha_3+\alpha_4)\,,\,\alpha_4\ ,$$

we find

$$\begin{aligned} g(\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}) = & (-1)^{1+\sum_{i<j}\alpha_{ij}} \times \\ & [\langle\alpha_1,\alpha_2\rangle\,\langle\alpha_1+\alpha_2,\alpha_3\rangle\,\langle\alpha_1+\alpha_2+\alpha_3,\alpha_4\rangle \\ & +\langle\alpha_1,\alpha_3\rangle\,\langle\alpha_1+\alpha_3,\alpha_4\rangle\,\langle\alpha_2,\alpha_1+\alpha_3,\alpha_4\rangle \\ & +\langle\alpha_2,\alpha_3\rangle\,\langle\alpha_1,\alpha_4\rangle\,\langle\alpha_2+\alpha_3,\alpha_1+\alpha_4\rangle] \end{aligned}$$

which is a prediction for the index of the 4-body SUSY quantum mechanics.

The motivic Kontsevich-Soibelman formula I

- KS have proposed a quantum deformation of their formula, which governs wall-crossing properties of motivic DT invariants $\Omega_{\rm ref}(\gamma; y, t)$. Physically, these correspond to the "refined index"

$$\Omega_{\rm ref}(\gamma, y) = \mathrm{Tr}'_{\mathcal{H}(\gamma)} (-y)^{2J_3} \equiv \sum_{n\in\mathbb{Z}} (-y)^n \, \Omega_{{\rm ref},n}(\gamma) \,,$$

  where J_3 is the angular momentum in 3 dimensions along the z axis (more accurately, a combination of angular momentum and $SU(2)_R$ quantum numbers). As $y \to 1$, $\Omega_{\rm ref}(\gamma; y, t) \to \Omega(\gamma; t)$.

  Dimofte Gukov Soibelman

- While this index is protected in $\mathcal{N} = 2, D = 4$ field theories, this is not so in in supergravity/string theory, where $SU(2)_R$ is generically broken. Still, one may consider the wall-crossing properties of $\Omega_{\rm ref}(\gamma; y, t)$ at fixed coupling.

The motivic Kontsevich-Soibelman formula II

- To state the formula, consider the Lie algebra $\mathcal{A}(y)$ spanned by generators $\{\tilde{e}_\gamma, \gamma \in \Gamma\}$, satisfying the commutation rule

$$[\tilde{e}_{\gamma_1}, \tilde{e}_{\gamma_2}] = \kappa(\langle\gamma_1, \gamma_2\rangle)\, \tilde{e}_{\gamma_1+\gamma_2} , \qquad \kappa(x) = \frac{(-y)^x - (-y)^{-x}}{y - 1/y} .$$

- To any charge vector γ, attach the operator

$$\hat{U}_\gamma = \prod_{n\in\mathbb{Z}} \mathbf{E}\left(\frac{y^n\, \tilde{e}_\gamma}{y-1/y}\right)^{-(-1)^n \Omega_{\mathrm{ref},n}(\gamma)} , \quad \mathbf{E}(x) \equiv \exp\left[\sum_{k=1}^{\infty} \frac{(xy)^k}{k(1-y^{2k})}\right]$$

where **E** is the quantum dilogarithm function.

The motivic Kontsevich-Soibelman formula III

- The motivic version of the KS wall-crossing formula again states that the product

$$\hat{A}_{\gamma_1,\gamma_2} = \prod_{\substack{\gamma=M\gamma_1+N\gamma_2,\\ M\geq 0, N\geq 0}} \hat{U}_\gamma \ ,$$

ordered such that $\arg Z_\gamma$ decreases from left to right, is constant across the wall.

- As before, one may combine the $\hat{U}_{k\gamma}$ into a single factor

$$\hat{V}_\gamma = \prod_{\ell\geq 1} \hat{U}_{\ell\gamma} = \exp\left[\sum_{N=1}^{\infty} \bar{\Omega}_{\rm ref}(N\gamma, y)\, \tilde{e}_{N\gamma}\right]$$

where $\bar{\Omega}_{\rm ref}(N\gamma, y)$ are the “rational motivic invariants", defined by

$$\bar{\Omega}^{+}_{\rm ref}(\gamma, y) \equiv \sum_{m|\gamma} \frac{(y-y^{-1})}{m(y^m - y^{-m})} \Omega^{+}_{\rm ref}(\gamma/m, y^m)\,.$$

The motivic Kontsevich-Soibelman formula IV

Manschot

- The motivic KS formula becomes

$$\prod_{\substack{M\geq 0, N\geq 0>0,\\ \gcd(M,N)=1, M/N\downarrow}} \hat{V}^{+}_{M\gamma_1+N\gamma_2} = \prod_{\substack{M\geq 0, N\geq 0>0,\\ \gcd(M,N)=1, M/N\uparrow}} \hat{V}^{-}_{M\gamma_1+N\gamma_2} ,$$

- $\Delta\bar{\Omega}_{\rm ref}(\gamma, y)$ can be computed using the same techniques as before, e.g. the primitive wcf read

$$\Delta\Omega_{\rm ref}(\gamma_1+\gamma_2, y) = \frac{(-y)^{\langle\gamma_1,\gamma_2\rangle} - (-y)^{-\langle\gamma_1,\gamma_2\rangle}}{y-1/y}\, \Omega_{\rm ref}(\gamma_1, y)\, \Omega_{\rm ref}(\gamma_2, y)$$

The motivic Kontsevich-Soibelman formula V

- The refined semi-primitive wall-crossing formula is given by

$$Z^-(1,q,y) = Z^+(1,q,y)\, Z_{\rm halo}(\gamma_1,q,y)$$

where

$$Z_{\rm halo}(\gamma_1,q,y) \equiv \exp\left(\sum_{\ell=1}^{\infty} \frac{(-y)^{\langle\gamma_1,\ell\gamma_2\rangle} - (-y)^{-\langle\gamma_1,\ell\gamma_2\rangle}}{y-y^{-1}}\, \bar\Omega_{\rm ref}(\ell\gamma_2,y)\, q^\ell\right)$$

or in terms of the integer motivic invariants,

$$Z_{\rm halo}(\gamma_1,q,y) = \prod_{\substack{k\geq 1, n\in\mathbb{Z}\\ 1\leq j\leq k|\gamma_{12}|}} \left(1-(-1)^{k|\gamma_{12}|} q^k\, y^{n+2j-1-k|\gamma_{12}|}\right)^{(-1)^n\, \Omega_{{\rm ref},n}(k\gamma_2)}$$

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Outline

The Joyce-Song formula I

- Joyce & Song have derived an explicit wall-crossing formula in the context of the Abelian category of coherent sheaves on a Calabi-Yau three-fold.
- KS instead considered the derived category of coherent sheaves, which is not an Abelian but rather a triangulated category. In spite of these subtleties, we shall find evidence that the two formulae agree.
- To formulate the JS formula, we need to introduce S, U and $\mathcal{L}$ factors, which are functions of an ordered list of charge vectors $\alpha_i \in \tilde{\Gamma}, i = 1 \ldots n$.

The Joyce-Song formula II

- We define $S(\alpha_1,\dots,\alpha_n) \in \{0,\pm 1\}$ as follows. If $n=1$, set $S(\alpha_1)=1$. If $n>1$ and, for every $i=1\dots n-1$, either

$$
\begin{array}{lllll}
(a) & \langle \alpha_i, \alpha_{i+1} \rangle \le 0 & \text{and} & \langle \alpha_1+\dots+\alpha_i, \alpha_{i+1}+\dots+\alpha_n \rangle < 0, \\
(b) & \langle \alpha_i, \alpha_{i+1} \rangle > 0 & \text{and} & \langle \alpha_1+\dots+\alpha_i, \alpha_{i+1}+\dots+\alpha_n \rangle \ge 0 \,,
\end{array}
$$

  let $S(\alpha_1,\dots,\alpha_n)=(-1)^r$, where r is the number of times option (a) is realized; otherwise, $S(\alpha_1,\dots,\alpha_n)=0$.

The Joyce-Song formula III

- To define the U factor, consider all ordered partitions of the n vectors α_i into $1 \le m \le n$ packets $\{\alpha_{a_{j-1}+1}, \cdots, \alpha_{a_j}\}$, $j = 1 \dots m$, with $0 = a_0 < a_1 < \cdots < a_m = n$, such that all vectors in each packet have the same phase $\arg Z(\alpha_i)$. Let

$$\beta_j = \alpha_{a_{j-1}+1} + \cdots + \alpha_{a_j}, \qquad j = 1 \dots m$$

  be the sum of the charge vectors in each packet.
- Next, consider all ordered partitions of the m vectors β_j into $1 \le l \le m$ packets $\{\beta_{b_{k-1}+1}, \cdots, \beta_{b_k}\}$, with $0 = b_0 < b_1 < \cdots < b_l = m$, $k = 1 \dots l$, such that the total charge vectors $\delta_k = \beta_{b_{k-1}+1} + \cdots + \beta_{b_k}$, $k = 1 \dots l$ in each packets all have the same phase $\arg Z(\delta_k)$.

The Joyce-Song formula IV

- Define the U-factor as the sum

$$U(\alpha_1,\ldots,\alpha_n) \equiv \sum_l \frac{(-1)^{l-1}}{l} \cdot \prod_{k=1}^{l} \prod_{j=1}^{m} \frac{1}{(a_j - a_{j-1})!}\; S(\beta_{b_{k-1}+1}, \beta_{b_{k-1}+2}, \ldots, \beta_{b_k})\ .$$

  over all partitions of α_i and β_j satisfying the conditions above.
- If none of the phases of the vectors α_i coincide, $S = U$. Contributions with $l > 1$ arise only when $\{\alpha_i\}$ can be split into two (or more) packets with the same total charge, e.g.

$$U[\gamma_1,\gamma_2,\gamma_1,\gamma_2] = S[\gamma_1,\gamma_2,\gamma_1,\gamma_2] - \frac{1}{2} S[\gamma_1,\gamma_2]^2 = 1 - \frac{1}{2}(-1)^2 = \frac{1}{2}$$

The Joyce-Song formula V

- Finally (departing slightly from JS), define the (Landau) $\mathcal{L}$ factor Landau factor $\mathcal{L}$ is a

$$\mathcal{L}(\alpha_1, \ldots, \alpha_n) = \sum_{\text{trees}} \prod_{\text{edges(i,j)}} \langle \alpha_i, \alpha_j \rangle$$

  where the sum runs over all labeled trees with n vertices labelled $\{1, \ldots, n\}$, with edges oriented from i to j if $i < j$.
- Each tree can be labelled by its Prüfer code, a sequence of $n - 2$ numbers in $\{1, \ldots n\}$.

The Joyce-Song formula VI

- The JS formula then says that the coefficient $g(\{\alpha_i\})$ in

$$\Delta\bar{\Omega}(\gamma) = \sum_{n\geq 2} \sum_{\substack{\{\alpha_1,\dots\alpha_n\}\in\tilde{\Gamma} \\ \gamma=\alpha_1+\dots+\alpha_n}} \frac{g(\{\alpha_i\})}{|\mathrm{Aut}(\{\alpha_i\})|} \prod_{i=1}^{n} \bar{\Omega}^+(\alpha_i)\ .$$

  is given by a sum over permutations

$$g(\{\alpha_i\}) = \frac{1}{2^{n-1}} (-1)^{n-1+\sum_{i<j}\langle\alpha_i,\alpha_j\rangle} \sum_{\sigma\in\Sigma_n} \mathcal{L}\left(\alpha_{\sigma(1)},\dots\alpha_{\sigma(n)}\right)\ U\left(\alpha_{\sigma(1)},\dots\alpha_{\sigma(n)}\right)$$

The Joyce-Song formula VII

- To derive the primitive wcf, note that there is only one oriented tree with 2 nodes. Assuming $\gamma_{12} < 0$, the JS data is then

$\sigma(12)$	S	U	$\mathcal{L}$
12	a	-1	γ_{12}
21	b	1	$-\gamma_{12}$

leading again to

$$\Delta\Omega(\gamma \to \gamma_1 + \gamma_2) = (-1)^{\gamma_{12}}\, \gamma_{12}\, \Omega(\gamma_1)\, \Omega(\gamma_2)\ , \qquad \gamma_{12} \equiv \langle \gamma_1 , \gamma_2 \rangle$$

The Joyce-Song formula VIII

- For generic 3-body decay, assuming the same phase ordering as before and taking into account the 3 possible oriented trees, the JS data

$\sigma(123)$	S	U	$\mathcal{L}$
123	bb	1	$\alpha_{12}\alpha_{13}+\alpha_{13}\alpha_{23}+\alpha_{12}\alpha_{23}$
132	b-	0	$\alpha_{12}\alpha_{13}-\alpha_{13}\alpha_{23}-\alpha_{12}\alpha_{23}$
213	ab	-1	$-\alpha_{12}\alpha_{23}+\alpha_{13}\alpha_{23}-\alpha_{12}\alpha_{13}$
231	-a	0	$\alpha_{12}\alpha_{13}-\alpha_{13}\alpha_{23}-\alpha_{12}\alpha_{23}$
312	ab	-1	$\alpha_{13}\alpha_{23}-\alpha_{12}\alpha_{23}-\alpha_{13}\alpha_{12}$
321	aa	1	$\alpha_{13}\alpha_{23}+\alpha_{12}\alpha_{13}+\alpha_{12}\alpha_{23}$

leads to the same answer as KS,

$$g(\{\alpha_1,\alpha_2,\alpha_3\}) = (-1)^{\alpha_{12}+\alpha_{23}+\alpha_{13}}\,\alpha_{12}\,(\alpha_{13}+\alpha_{23})$$

The Joyce-Song formula IX

- We have checked that JS and KS also agree for generic 4-body decay (involving 16 graphs), and for special cases (2,3), (2,4) (involving up to 1296 graphs !).
- While there is no general proof yet, it seems that the JS formula is equivalent to the classical KS formula. Finding a motivic generalization of JS seems an interesting problem.
- Note that the JS formula involves large denominators and leads to many cancellations. There may be a more economic way to state the solution to KS.

Outline

The quantum mechanics of multi-centered solutions I

- The space $\mathcal{M}_n$ of BPS configurations with n centers in $\mathcal{N}=2$ SUGRA is described by solutions to Denef's equations

$$\sum_{j=1\dots n, j\neq i}^{n} \frac{\alpha_{ij}}{|\vec{r}_{ij}|} = 2\,\mathrm{Im}\left[e^{-i\alpha} Z(\alpha_i)\right]\,, \qquad \alpha = \arg[Z(\alpha_1 + \cdots \alpha_n)]\,.$$

Denef

- $\mathcal{M}_n$ has real dimension $2n-2$, and carries a symplectic form

$$\omega = \frac{1}{2}\sum_{i<j} \alpha_{ij}\, \frac{\mathrm{d}\vec{r}_{ij} \wedge \mathrm{d}\vec{r}_{ij} \cdot \mathrm{d}\vec{r}_{ij}}{|r_{ij}|^3}$$

with an Hamiltonian action of spatial rotations $SU(2)$. In the case of interest to us, $\mathcal{M}_n$ is compact (no "scaling" solutions).

de Boer El Showk Messamah Van den Bleeken

The quantum mechanics of multi-centered solutions II

- Quantizing the internal degrees of freedom of the multi-centered configurations amounts to quantizing the symplectic space $\mathcal{M}_n$. The index is given, at least in the classical limit where all α_{ij} are large, by

$$g(\{\alpha_i\}, y) = \int_{\mathcal{M}_n} \omega^{n-1} \, (-y)^{j_3}$$

  where j_3 is the moment map associated to $U(1)$ rotations.

- For $n = 2, 3$, $\mathcal{M}_n$ is toric. The integral localizes to the fixed points of the torus action:

$$g(\alpha_1, \alpha_2; y) = \frac{(-y)^{\langle\gamma_1,\gamma_2\rangle} - (-y)^{-\langle\gamma_1,\gamma_2\rangle}}{y - 1/y}$$

$$g(\alpha_1, \alpha_2, \alpha_3; y) = \frac{(-1)^{\alpha_{13}+\alpha_{23}+\alpha_{12}}}{\sinh^2 \nu} \sinh(\nu(\alpha_{13} + \alpha_{23})) \sinh(\nu\alpha_{12}) \, ,$$

  where $\nu = \ln y$. This is in precise agreement with KS/JS !

The quantum mechanics of multi-centered solutions III

- For $n > 3$, we expect that $\mathcal{M}_n$ is still toric. This is manifest on the dual, Higgs branch description of the same problem, given by a quiver with n nodes. Hopefully, the JS formula can be derived by computing the index via localization.
- If so, one may also get a motivic version of the JS formula.

THANK YOU !