A smooth index for $D = 4, \mathcal{N} = 2$ theories on $\mathbb{R}^3$

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based on work with S. Alexandrov, G. Moore, A. Neitzke, arXiv:1406.2360

and work in progress with S. Alexandrov

Introduction I

- In recent years, much progress has been made on understanding the spectrum of BPS states in $D = 4$ theories with $\mathcal{N} = 2$ supersymmetry. Powerful methods (wall-crossing, quivers, spectral networks, ...) allow to compute the BPS index in a large class of models:

$$\Omega(\gamma, u) = -\frac{1}{2}\mathrm{Tr}_{\mathcal{H}_1(\gamma,u)}(-1)^{2J_3}(2J_3)^2 \quad \in \mathbb{Z}$$

  where $\mathcal{H}_1(\gamma, u)$ is the space of single particle states, and u labels the Coulomb branch.

- While $\Omega(\gamma, u)$ is locally constant, it can jump across walls of marginal stability, when some of the BPS states with total charge γ decay into their constituents. The jump is determined by a universal wall-crossing formula.

Kontsevich Soibelman 2008; Denef Moore 2008; ... ; Manschot BP Sen 2010

Introduction II

- Another protected quantity is the Witten index

$$\mathcal{I}(R,u,C) = -\tfrac{1}{2}\mathrm{Tr}_{\mathcal{H}(u)}(-1)^{2J_3}(2J_3)^2\sigma_\gamma\, e^{-2\pi R H - 2\pi i\langle\gamma,C\rangle} \quad \notin \mathbb{Z}!$$

  Here $\mathcal{H}(u)$ is the full Hilbert space of the four-dimensional theory on $\mathbb{R}^3$, including multi-particle states, H is the Hamiltonian, C are chemical potentials conjugate to the charges, σ_γ is a pesky sign.

- Equivalently, consider the path integral on $\mathbb{R}^3 \times S^1(R)$ with periodic boundary conditions, and with an insertion of a 4-fermion vertex to soak up fermionic zero-modes.
- In a sector with charge $\gamma \neq 0$, and in center of mass frame, the Hamiltonian has a discrete spectrum starting at $E = |Z(\gamma)|$ (assuming $\Omega(\gamma,u) \neq 0$), and a continuum starting at

$$E_c = \min\{\sum_i |Z_{\alpha_i}|; \gamma = \sum \alpha_i, \Omega(\alpha_i,u) \neq 0\}$$

Introduction III

- Although multi-particle states do not saturate the BPS bound, they can still contribute to the Witten index, due to a possible spectral asymmetry between densities of bosonic and fermionic states.

Kaul Rajaraman 1983, Akhoury Comtet 1984, Troost 2010

- Because the path integral has no phase transition, the Witten index $\mathcal{I}(R, u, C)$ is expected to be smooth across walls of marginal stability. *The single-particle contribution to $\mathcal{I}(R, u, C)$ may jump, but the discontinuity must be cancelled by multi-particle contributions.*
- The BPS indices $\Omega(\gamma, u)$ can be recovered from $\mathcal{I}(R, u, C)$ by Fourier transforming with respect to C and taking the limit $R \to \infty$.

Introduction IV

- A similar phenomenon occurs in $D=2$ massive theories with $(2,2)$ supersymmetry: the 'new supersymmetric index' $Z_{ab} = \mathrm{Tr}(-1)^F\, F e^{-2\pi R H}$ receives contributions from multi-kinks interpolating between vacua a and b, and is a smooth function of R, u determined by the single-particle degeneracies $\Omega_{ab}(u)$.

  Cecotti Fendley Intriligator Vafa 1992

- In the context of $D=4, \mathcal{N}=2$ theories, close cousins of $\mathcal{I}(R,u,C)$ are the hyperkähler metric on the Coulomb branch in $\mathbb{R}^3 \times S^1$, and the line defects vevs $\langle L_\zeta \rangle$: both are expressable in terms of the (framed) BPS indices $\Omega(\gamma, u)$, and are smooth across the walls.

  Gaiotto Moore Neitzke 2008, 2010

- Our aim is to propose a natural candidate for the Witten index $\mathcal{I}(R,u,C)$, which has the expected large radius limit and is smooth across the walls.

Coulomb branch on $\mathbb{R}^3 \times S^1$ I

- On $\mathbb{R}^{3,1}$, the Coulomb branch $\mathcal{M}_4$ is a special Kähler manifold determined by the central charge function $Z : \Gamma \to \mathbb{C}$.
- After compactification on a circle of radius R, and dualizing the vector fields in $D = 3$ into scalars C, the low energy dynamics can be formulated in terms of a non-linear sigma model $\mathbb{R}^3 \to \mathcal{M}_3(R)$.
- The target space $\mathcal{M}_3(R)$ is a torus bundle over $\mathcal{M}_4$, equipped with a hyperkähler metric.
- In the large radius limit, the metric is obtained by the 'rigid c-map' from $\mathcal{M}_4$, and has translational isometries along the torus fiber. At finite radius, instanton corrections from $D = 4$ BPS states winding around the circle and break the isometries.

Coulomb branch on $\mathbb{R}^3 \times S^1$ II

- The HK metric on $\mathcal{M}_3(R)$ is best described using twistorial methods: the twistor space $\mathcal{Z} = \mathbb{P}_t \times \mathcal{M}_3$ carries a natural complex structure and holomorphic 'symplectic' form

$$\omega = \mathrm{i}t^{-1}\omega_+ + \omega_3 + \mathrm{i}t\omega_- = \epsilon^{ab}\,\frac{\mathrm{d}\mathcal{X}_a}{\mathcal{X}_a} \wedge \frac{\mathrm{d}\mathcal{X}_b}{\mathcal{X}_b}$$

  The metric on $\mathcal{M}_3(R)$ can be read off from the holomorphic Darboux coordinates $\mathcal{X}_{\gamma_a}(t, u, C)$.
- In the infinite radius limit, a natural set of Darboux coordinates are exponentiated moment maps for the torus isometries,

$$\mathcal{X}_a = \mathcal{X}_{\gamma_a}\ , \quad \mathcal{X}^{\mathrm{sf}}_\gamma = \sigma_\gamma\, e^{-\pi \mathrm{i} R\left(t^{-1} Z_\gamma - t\bar{Z}_\gamma\right) - 2\pi\mathrm{i}\langle\gamma, C\rangle}.$$

Coulomb branch on $\mathbb{R}^3 \times S^1$ III

- Instanton corrections induce discontinuities across 'BPS rays' $\ell_{\gamma'} = \{t' \in \mathbb{C}^\times : Z_{\gamma'}/t' \in \mathrm{i}\mathbb{R}^-\}$,

$$\mathcal{X}_\gamma \to \mathcal{X}_\gamma (1 - \mathcal{X}_{\gamma'})^{\langle \gamma, \gamma' \rangle \, \Omega(\gamma')}$$

- The quantum corrected Darboux coordinates are solutions of the integral equations

$$\frac{\mathcal{X}_\gamma}{\mathcal{X}_\gamma^{\mathrm{sf}}} = \exp\left[\sum_{\gamma'} \frac{\Omega(\gamma', u)}{4\pi\mathrm{i}} \langle \gamma, \gamma' \rangle \int_{\ell_{\gamma'}} \frac{\mathrm{d}t'}{t'} \frac{t+t'}{t-t'} \log\left(1 - \mathcal{X}_{\gamma'}(t')\right) \right],$$

reminiscent of TBA equations in integrable systems.

Gaiotto Moore Neitzke 2008, 2010

- At large radius, a formal solution is obtained by iterating the system, leading to a 'multi-instanton sum'

$$\mathcal{X}_\gamma = \mathcal{X}_\gamma^{\rm sf} \exp\left[\sum_T \prod_{(i,j)\in T_1} \langle \alpha_i, \alpha_j\rangle \prod_{i\in T_0} \bar\Omega(\alpha_i)\, g_T\right]$$

where T runs over trees decorated by charges α_i such that $\gamma = \sum \alpha_i$, g_T are iterated contour integrals, and

$$\overline{\Omega}(\gamma) = \sum_{d|\gamma} \frac{1}{d^2} \Omega(\gamma/d)$$

are the 'rational BPS indices'.

Conjecture (Alexandrov Moore Neitzke BP 2014)

- Conjecture: the Witten index in $\mathcal{N}=2, D=4$ theories on the Coulomb branch is given by

$$\mathcal{I}(R,u,C) = \frac{R}{16\mathrm{i}\pi^2} \sum_\gamma \Omega(\gamma) \int_{\ell_\gamma} \frac{\mathrm{d}t}{t} \left(t^{-1} Z_\gamma - t \bar{Z}_\gamma \right) \log\left(1 - \mathcal{X}_\gamma(t)\right).$$

- This function first appeared as the contact potential on the hypermultiplet moduli space in type II string vacua. In the analogy between the integral equations of GMN and TBA, $\mathcal{I}(R,u,C)$ is the free energy. Similarly, in the TBA approach to null Wilson loops in AdS, $\mathcal{I}(R,u,C)$ is the regularized area.

Alexandrov Saueressig BP Vandoren 08; Alexandrov Roche; Alday Gaiotto Maldacena 09

- $\mathcal{I}(R,u,C)$ is closely related to the Kähler potential for the HK metric on $\mathcal{M}_3(R)$ (upon adding a classical term, plus the Kähler potential for a canonical hyperholomorphic line bundle)

Smoothness across the walls

- To support this claim, note that for any smooth function $F_\gamma(t, u, C)$ on $\Gamma \times \mathcal{Z}$, linear in γ,

$$\Phi(R, u, C) = \sum_\gamma \Omega(\gamma) \int_{\ell_\gamma} \frac{\mathrm{d}t}{t} F_\gamma \log\left(1 - \mathcal{X}_\gamma\right).$$

is smooth across the walls, as a result of the dilogarithm identities

$$\sum_\gamma \Omega^+(\gamma) L_{\sigma_\gamma}(\mathcal{X}_\gamma^+) = \sum_\gamma \Omega^-(\gamma) L_{\sigma_\gamma}(\mathcal{X}_\gamma^-)$$

implied by the KS motivic wall-crossing formula. Here $L_\sigma(z) = \mathrm{Li}_2(z) + \frac{1}{2}\log(g/\sigma)\log(1-z)$ is a variant of Rogers' dilogarithm.

Alexandrov Persson BP 2011

- The proposed index arises for $F_\gamma \propto t^{-1} Z_\gamma - t\bar{Z}_\gamma$.

One-particle contributions

- In the large radius limit, approximating $\mathcal{X}_\gamma$ by $\mathcal{X}_\gamma^{\rm sf}$,

$$\mathcal{I}(R,u,C) = \sum_\gamma \frac{R}{8\pi^2}\,\sigma_\gamma\,\overline{\Omega}(\gamma)\,|Z_\gamma| K_1(2\pi R|Z_\gamma|)\,e^{-2\pi \mathrm{i}\langle\gamma,C\rangle} + \dots$$

- For γ primitive, this is the expected contribution of a single-particle, relativistic BPS state of charge γ:

$$\mathrm{Tr} e^{-2\pi R\sqrt{-\Delta+M^2}+\mathrm{i}\theta J_3} = \frac{L}{2\pi}\,\frac{\chi_{\rm spin}(\theta)}{4\sin^2(\theta/2)}\,2M\,K_1(2\pi MR)$$

provided we define the Witten index as follows:

$$\mathcal{I}(R,u,C) = R \lim_{\substack{\theta\to 2\pi\\ L\to\infty}} \partial_\theta^2 \left[\frac{\sin^2(\theta/2)}{\pi L}\,\mathrm{Tr}\left(\sigma\, e^{-2\pi RH+\mathrm{i}\theta J_3-2\pi\mathrm{i}\langle\gamma,C\rangle}\right)\right].$$

Multi-particle contributions I

- If correct, this predicts the contribution of an arbitrary multi-particle state. E.g. for two particles,

$$\mathcal{I}_{\gamma,\gamma'} = -\frac{R}{64\pi^3}\,\overline{\Omega}(\gamma)\,\overline{\Omega}(\gamma')\,\langle\gamma,\gamma'\rangle\,J_{\gamma,\gamma'}$$

where

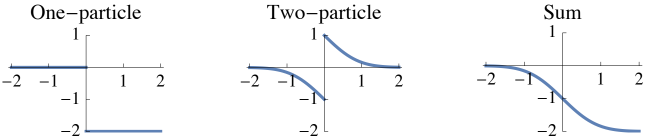
$$J_{\gamma,\gamma'} = \int_{\ell_\gamma}\frac{\mathrm{d}t}{t}\int_{\ell'_\gamma}\frac{\mathrm{d}t'}{t'}\,\frac{t+t'}{t-t'}\left(t^{-1}Z_\gamma - t\bar{Z}_\gamma\right)\mathcal{X}^{\rm sf}_\gamma(t)\mathcal{X}^{\rm sf}_{\gamma'}(t')\,,$$

- In the limit $R\to\infty$, $\psi_{\gamma,\gamma'}\to 0$, a saddle point approximation gives

$$J_{\gamma,\gamma'} \sim \mathrm{sgn}(\psi_{\gamma,\gamma'})\,\mathrm{Erfc}\left(|\psi_{\gamma\gamma'}|\sqrt{\frac{\pi R|Z_\gamma|\,|Z_{\gamma'}|}{|Z_\gamma|+|Z_{\gamma'}|}}\right)\,e^{-2\pi R|Z_\gamma+Z_{\gamma'}|-2\pi\mathrm{i}\langle\gamma+\gamma',C\rangle}$$

Multi-particle contributions II

- Using $\mathrm{Erf}(x) = \mathrm{sgn}(x)\,(1 - \mathrm{Erfc}(|x|)$, one checks that $\mathcal{I}_{\gamma+\gamma'} + \mathcal{I}_{\gamma,\gamma'}$ is smooth across the wall:



- In the remainder, we shall check this prediction by studying the quantum mechanics of a system of two mutually non-local non-relativistic particles.
- Note: the constituents can be treated as distinguishable and subject to Boltzmannian statistics, provided one uses the rational BPS index $\bar{\Omega}(\gamma)$ for the internal degrees of freedom.

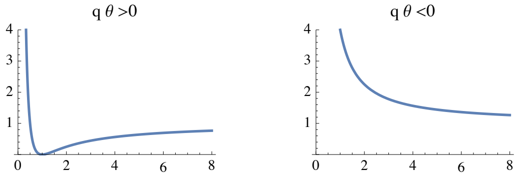
Manschot BP Sen 2010

Non-relativistic electron-monopole problem I

- Consider a non-relativistic particle of electric charge $q = \frac{1}{2}\langle\gamma, \gamma'\rangle$ in the field of a Dirac monopole of unit magnetic charge:

$$\boxed{H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 - \frac{q}{2m}\vec{B}\cdot\vec{\sigma}\otimes(1_2 - \sigma_3) + \frac{1}{2m}\left(\vartheta - \frac{q}{r}\right)^2}$$

$$\vec{\nabla}\wedge\vec{A} = \vec{B} = \frac{\vec{r}}{r^3}\,, \quad m = \frac{|Z_\gamma||Z_{\gamma'}|}{|Z_\gamma|+|Z_{\gamma'}|}\,, \quad \frac{\vartheta^2}{2m} = |Z_\gamma| + |Z_{\gamma'}| - |Z_{\gamma+\gamma'}|$$



Non-relativistic electron-monopole problem II

- H describes two bosonic degrees of freedom with helicity $h = 0$, and one helicity $h = \pm 1/2$ fermionic doublet with gyromagnetic ratio $g = 4$.

D'Hoker Vinet 1985; Denef 2002; Avery Michelson 2007;Lee Yi 2011

- H commutes with 4 supercharges,

$$Q_4 = \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 & -\mathrm{i}\left(\vartheta - \frac{q}{r}\right) + \vec{\sigma}\cdot(\vec{p} - q\vec{A}) \\ \mathrm{i}\left(\vartheta - \frac{q}{r}\right) + \vec{\sigma}\cdot(\vec{p} - q\vec{A}) & 0 \end{pmatrix}$$

$$Q_a = \dots$$

$$\{Q_m, Q_n\} = 2H\,\delta_{mn}$$

Non-relativistic electron-monopole problem III

- Going to a basis of monopole spherical harmonics, the Schrödinger equation with energy $E = k^2/(2m)$ becomes

$$\left[-\frac{1}{r}\partial_r^2 \, r + \frac{\nu^2 - q^2 - \frac{1}{4}}{r^2} + \left(\vartheta - \frac{q}{r} \right)^2 \right] \Psi(r) = k^2 \Psi \,,$$

  where

$$\nu = j + \frac{1}{2} + h \,, \quad j = |q| + h + \ell \,, \ell \in \mathbb{N} \,.$$

- Supersymmetric bound states exist for $q\vartheta > 0$, $h = -1/2$, $\ell = 0$, and form a multiplet of spin $j = |q| - \frac{1}{2}$, with $2j + 1 = |\langle \gamma_1, \gamma_2 \rangle|$.

Denef 2002

Non-relativistic electron-monopole problem IV

- The S-matrix for partial waves is similar to that of H-atom,

$$S_\nu(k) = \frac{\Gamma\left(\frac{1}{2}+\nu+\mathrm{i}\frac{q\vartheta}{\sqrt{k^2-\vartheta^2}}\right)}{\Gamma\left(\frac{1}{2}+\nu-\mathrm{i}\frac{q\vartheta}{\sqrt{k^2-\vartheta^2}}\right)} = e^{2\mathrm{i}\delta_\nu(k)}.$$

Alexandrov BP, to appear

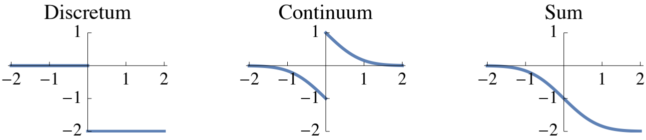
- The contribution of the continuum to $\mathrm{Tr}(-1)^F\, e^{-2\pi RH}$ is thus

$$\sum_{h=0^2,\pm\frac{1}{2}}(-1)^{2h}\sum_{\ell=0}^{\infty}\int_{k=\vartheta}^{\infty}\frac{\mathrm{d}k\,\partial_k}{2\pi\mathrm{i}}\log\frac{\Gamma\left(|q|+\ell+2h+1+\mathrm{i}\frac{q\vartheta}{\sqrt{k^2-\vartheta^2}}\right)}{\Gamma\left(|q|+\ell+2h+1-\mathrm{i}\frac{q\vartheta}{\sqrt{k^2-\vartheta^2}}\right)}\, e^{-\frac{\pi Rk^2}{m}}$$

Non-relativistic electron-monopole problem V

- Terms with $\ell > 0$ cancel, leaving the contribution from $\ell = 0$ only:

$$\begin{aligned}
\mathrm{Tr}(-1)^F\, e^{-2\pi R H} &= -\,|2q|\,\Theta(q\vartheta) - \frac{2q\vartheta}{\pi}\int_{k=|\vartheta|}^{\infty} \frac{\mathrm{d}k}{k\sqrt{k^2-\vartheta^2}}\, e^{-\frac{\pi R k^2}{m}} \\
&= -\,2|q|\,\Theta(q\vartheta) + |q|\,\mathrm{sgn}(q\vartheta)\,\mathrm{Erfc}\left(|\vartheta|\sqrt{\frac{\pi R}{m}}\right) \\
&= -\,|q| - q\,\mathrm{Erf}\left(\vartheta\sqrt{\frac{\pi R}{m}}\right) .
\end{aligned}$$



Non-relativistic electron-monopole problem VI

- The result is robust under perturbations of the potential and metric at short distance, and depends only on the ratio $S_{\nu+1}(k)/S_{\nu}(k)$, which is fixed by supersymmetry. In fact, the spectral asymmetry

$$\frac{\mathrm{d}k}{k\sqrt{k^2-\vartheta^2}}$$

is as predicted by Callias' index theorem.

Summary and open problems I

- The Witten index in $\mathcal{N} = 2, D = 4$ theories on $\mathbb{R}^3$ is smooth across walls of marginal stability, due to interplay between bound states and multiparticle states.
- Our proposal for $\mathcal{I}(R, u, C)$ is consistent with the quantum mechanics of the relativistic one-particle and non-relativitistic two-particle systems.
- The replacement $\mathrm{sgn}(x) \to \mathrm{Erf}(x)$ is a well-known trick to ensure modularity of indefinite theta series. It was also postulated in early studies of D4-D2-D0 black hole partition functions to ensure S-duality. Its physical justification is now transparent !

Zwegers 2002; Manschot 2009; Cardoso Cirafici Nampuri 2013

- Can one compute the index for the quantum mechanics of n mutually non-local particles using localization ?
- How about the refined index $\mathrm{Tr}(-1)^{2I_3} y^{2(J_3+I_3)} e^{-2\pi R H}$?

Cecotti Neitzke Vafa 2014