Recounting Black Hole Counting

Boris Pioline, LPTHE

50th Anniversary of LPTHE, 28/11/2017

Foreword

- Special thanks to Karim for putting up such a nice meeting, and happy $51\frac{1}{2}$-th birtthday to LPTHE !

Foreword

- Special thanks to Karim for putting up such a nice meeting, and happy $51\frac{1}{2}$-th birtthday to LPTHE !
- It has been wonderful working at LPTHE since 1999. My only complaint is that trees in front of my window are not trimmed more often.

Foreword

- Special thanks to Karim for putting up such a nice meeting, and happy $51\frac{1}{2}$-th birtthday to LPTHE !
- It has been wonderful working at LPTHE since 1999. My only complaint is that trees in front of my window are not trimmed more often.

- In this talk I will recollect some work done over the years on supersymmetric black holes, to which I have participated. Apologies to my string theory colleagues whose work will not be represented.

In search of black hole microstates I

- One of the few clues about quantum gravity is the celebrated Bekenstein-Hawking entropy law,

$$S_{BH} = \frac{c^3}{4 G_N \hbar} \mathcal{A}(M, J, Q, P)$$

  - The total entropy of an isolated system can only increase
  - Semi-classically, a black hole emits a thermal radiation with temperature $1/T = \partial S/\partial M$.

- A quantum-mechanical description of black holes should produce a discrete set of micro-states, such that in the thermodynamical limit $M \gg m_P = \sqrt{c^3/G_N \hbar}$,

$$S_{BH} \sim \log \Omega(M, J, Q, P)$$

Since string theory claims to be a consistent theory of quantum gravity, it is important to check if it passes this check.

In search of black hole microstates II

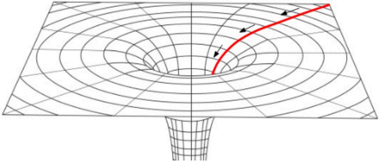
- Four-dimensional vacua of string theory with extended SUSY are described at low energy by Einstein's theory of GR, along with Maxwell fields, scalar fields, etc, as well as higher derivative extension. This effective theory admits black hole solutions, which obey the Bekenstein-Hawking-(Wald) entropy law.
- Among these black holes, those which saturate the BPS bound

$$M \geq |Z(Q, P; \varphi)|$$

are invariant under some fraction of SUSY and so are particularly robust. They automatically have vanishing Hawking temperature, i.e. are extremal.

In search of black hole microstates III

- BPS black holes with $J = 0$ are spherically symmetric, and interpolate between $\mathbb{R}^{3,1}$ at spatial infinity and $AdS_2 \times S^2$ near the horizon. The scalar fields $\varphi(r)$ flow from φ_∞ to $\varphi_*(Q, P)$ according to the "attractor mechanism".



- The Bekenstein-Hawking entropy of these BPS black holes

$$S_{BH} = \pi |Z(Q, P; \varphi_*)|$$

depends only on the electromagnetic charge $\gamma = (Q, P)$ and scales like $\|\gamma\|^2$. For $\mathcal{N} \geq 4$ SUGRA, $S_{BH} = \pi\sqrt{I_4(\gamma)}$ where I_4 is a specific quartic polynomial.

Small black holes and perturbative strings I

- A possible source of black hole microstates are fundamental strings: e.g. in Heterotic string on $\mathbb{R}^{3,1} \times T^6$, obtained by tensoring $[c_L = 26] \otimes [\hat{c}_R = 10]$, BPS states with charge $\gamma = (Q, 0) \in \mathbb{Z}^{28}$ arise by taking an arbitrary excited state on the bosonic side times the ground state on the fermionic side, subject to the matching condition $N = \frac{1}{2} Q^2$.

Dabholkar Harvey 1989

- The number of such states is $p(N)$, the number of integer partitions of N with 24 colors:

$$\sum_{N \geq 0} p(N)\, q^{N-1} = \frac{1}{q \prod_{n \geq 1} (1 - q^n)^{24}} = \frac{1}{\Delta(\tau)}\,, \quad q = e^{2\pi i \tau}$$

Small black holes and perturbative strings II

- The asymptotic growth of $p(N)$ is very well known since Hardy-Ramanujan, thanks to the modular invariance of $\Delta(\tau)$,

$$\log p(N) \sim 4\pi\sqrt{N} \sim \mathcal{O}(|Q|) \ll |Q|^2!$$

  For such purely electric black hole (or more generally whenever Q and P are collinear), the BH entropy vanishes at leading order, but becomes $\mathcal{O}(|Q|)$ when higher derivative corrections are taken into account. This qualitative agreement can be made quantitative under certain assumptions.

  Sen 1995; Dabholkar 2004; Dabholkar Denef Moore BP, 2005

- Still, it would be desirable to show agreement for large black holes, where classical GR is valid.

Large black holes and D-brane bound states I

- Starting with Strominger and Vafa's 1996 paper, it was understood that micro-states of large BPS black holes can be constructed from D-brane bound states: e.g. in type IIA on $\mathbb{R}^{3,1} \times K3 \times T^2$, the following cocktail
  1. P_1 D4-branes wrapped on $T^2 \times C_1$ with $C_1 \subset K_3$
  2. P_2 D4-branes wrapped on $T^2 \times C_2$ with $C_2 \subset K_3$
  3. P_3 D6-branes wrapped on $T^2 \times K_3$
  4. Q_0 D0-branes

  behaves as a BPS black hole with area $\mathcal{A} \propto \sqrt{Q_0 P^1 P^2 P^3}$ in $\mathbb{R}^{3,1}$
- Upon lifting type IIA to 11D SUGRA, one finds that the black hole is really a black string wrapped on S_1, with computable central charges (c_L, c_R). Cardy's formula leads to $\log \Omega \sim \sqrt{Q_0 P^1 P^2 P^3}$, in quantitative agreement with the Bekenstein-Hawking entropy formula.

Large black holes and D-brane bound states II

- This agreement was (rightfully) hailed as a major success of string theory, and opened several avenues of research:
  a) relax the amount of SUSY: extremal non-SUSY black holes, near extremal, etc;
  b) relax the thermodynamic limit, and compare finite size effects on microscopic side with quantum corrections on macroscopic side.

  I shall mostly focus on b) and explain some milestones on the topic of "precision black hole counting".

Precision black hole counting I

- Returning to the topic of large BPS black holes in $\mathcal{N}=4$ string vacua (Het/T^6, II/$K3\times T_2$ and variants), it was boldly conjectured by Dijkgraaf Verlinde Verlinde (1996) that the exact number of black hole microstates (counted with sign) is given by a Fourier coefficient of a Siegel modular form of genus two:

$$\Omega(Q,P)=\int_C \mathrm{d}^3\tau\, \frac{e^{\mathrm{i}\pi(\rho Q^2+\sigma P^2+vP\cdot Q)}}{\Phi_{10}}\,, \quad \tau=\begin{pmatrix}\rho & v\\ v & \sigma\end{pmatrix}\in\mathcal{H}_2$$

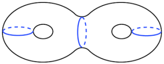
  where Φ_{10} is the Igusa cusp form of weight 10 under $Sp(4,\mathbb{Z})$. The main evidence at the time was

  a) $\Omega(Q,P)=\Omega(aQ+bP,cQ+dP)\in\mathbb{Z}$, $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{Z})$

  b) $\log\Omega(Q,P)\sim\pi\sqrt{I_4(Q,P)}=S_{BH}$

Precision black hole counting II

- This striking conjecture raised many questions, including
  a) Why on earth should BH degeneracies have anything to do with genus two curves ?



  b) Due to the vanishing of $\Phi_{10} \sim v^2 \Delta(\rho)\Delta(\sigma)$ near the separating degeneration locus $v = 0$, the answer depends on the choice of contour C. Which is the correct choice ?

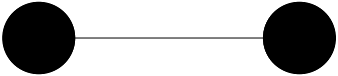
- The answer to the first question was sketched by Gaiotto and Dabholkar (2006), for details see C. Cosnier Horeau PhD thesis.
- To answer the second question, Dabhokar Gaiotto Nampuri (2007) noticed that the ambiguity due to the pole at $v = 0$ is of the form

$$\delta\Omega(Q,P) \sim (Q \cdot P)\, p(Q^2/2)\, p(P^2/2) ,$$

indicative of contributions from bound states of small black holes of charge $(Q, 0)$ and $(0, P)$.

Wall-crossing I

- A few years earlier, Denef (2001) had shown that two dyons with charge γ, γ' and Dirac-Schwinger-Zwanziger product $\kappa = \langle \gamma, \gamma' \rangle$ $= P \cdot Q' - P' \cdot Q$ can form $|\kappa|$ bound states for suitable values of the moduli.

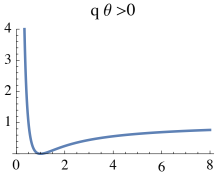


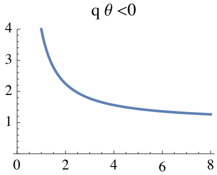
- In essence, the dynamics of two mutually non-local dyons is captured by a SUSY version of the Hamiltonian

$$H = \frac{1}{2m} (\vec{p} - \kappa \vec{A})^2 + \frac{1}{2m} \left(\varphi - \frac{\kappa}{|\vec{r}|} \right)^2$$

where $\vec{r}$ is the relative distance, m is the reduced mass, $\vec{A}$ is a unit charge magnetic monopole at $\vec{r} = 0$.

Wall-crossing II





- For $\kappa\varphi > 0$, the system has $|\kappa|$ BPS bound states, an infinite number of non-BPS bound states, and scattering states above $E_c = \varphi^2/(2m)$
- For $\kappa\varphi < 0$, the system has no bound states (but still scattering states above E_c)

The jump of the number of BPS bound states across the wall of marginal stability $\varphi = 0$ is known as wall-crossing phenomenon.

Wall-crossing III

- Soon, Cheng and Verlinde (2007) figured out a choice of contour $C(Q, P; \varphi)$ such that C crosses a zero of Φ_{10} whenever φ crosses a wall of marginal stability, and such that the discontinuity in $\Omega(Q, P; \varphi)$ agrees with the index of the corresponding bound state.
- In summary, the DVV formula counts both single centered (large) BPS black holes as well as bound states of two (small) BPS black holes:

$$\Omega(\gamma, \varphi) = \Omega_S(\gamma) + \sum_{\substack{\gamma=\gamma_1+\gamma_2 \\ P_1 \| Q_1, P_2 \| Q_2}} \langle \gamma_1, \gamma_2 \rangle \, \Theta\left(\langle \gamma_1, \gamma_2 \rangle \, \varphi_{12}\right) \Omega_S(\gamma_1) \, \Omega_S(\gamma_2)$$

where the single centered indices $\Omega_S(\gamma)$ are independent of the moduli.

Precision counting of single-centered black holes I

- Atish, Sameer Murthy and Don Zagier (2012) went on to construct partition functions for single centered indices, and show that they are given by Fourier coefficients of mock modular forms (i.e. either holomorphic, or modular, but not both). The breakdown of holomorphy can be understood as a consequence of the spectral asymmetry in the continuum of scattering states.

BP (2015); Murthy and BP (to appear)

- Rather than telling this story, I will change topic slighly and discuss modifications to this picture when the amount of SUSY is reduced to $\mathcal{N} = 2$ in four dimensions. This is joint work with Jan Manschot and Ashoke Sen, which started in 2010 right at the end of Ashoke's Blaise Pascal Chair at LPTHE.

Multi-centered black holes I

- Precision black hole counting in $\mathcal{N}=2$ is vastly more complicated than in $\mathcal{N}=4$:
  - the moduli space receives quantum corrections,
  - the constraints from duality are much weaker, and most importantly,
  - bound states involving an arbitrary number of BPS constituents can form, leading to an intricate pattern of walls of marginal stability !
- In the context of $\mathcal{N}=2$ (ungauged) supergravity, Denef (2000) constructed multi-centered BPS solutions from charges $\gamma_1,\dots\gamma_n$ such that

$$\forall i \quad : \quad \sum_{j\neq i} \frac{\langle \gamma_i,\gamma_j\rangle}{|\vec r_i-\vec r_j|} = c_i(\varphi)$$

$$\vec J = \sum_{i<j} \langle \gamma_i,\gamma_j\rangle \frac{\vec r_{ij}}{|r_{ij}|}$$

Multi-centered black holes II

- Accordingly, one may expect that the total index decomposes as

$$\Omega(\gamma,\varphi) = \Omega_S(\gamma) + \sum_{n\geq 2} \sum_{\gamma=\gamma_1+\dots\gamma_n} g(\{\gamma_i\},\varphi) \prod_{i=1}^{n} \Omega_S(\gamma_n)$$

  where $g(\{\gamma_i\},\varphi)$ is the Witten index of the relativistic quantum mechanics of n non-local dyons, while $\Omega_S(\gamma_i)$ counts the degrees of freedom of the individual constituents, independent of φ.
- While solving the quantum many-body problem is hopeless, computing the Witten index turns out to be tractable using localization.

Solving the supersymmetric n-body problem I

- For this, note that the space of solutions to Denef's equations (after factoring out center of motion) is a $(2n-2)$-dimensional symplectic space $\mathcal{M}_n(\{\gamma_i\},\varphi)$, with an Hamiltonian action of $SO(3)$ generated by $\vec{J}$.

de Boer El Showk Messamah van den Bleeken 2008

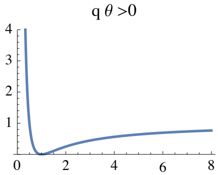
- The BPS Hilbert space is obtained by geometric quantization of $\mathcal{M}_n$. Provided $\mathcal{M}_n$ is compact, the equivariant Dirac index can be computed using the Atiyah-Bott Lefschetz fixed point theorem:

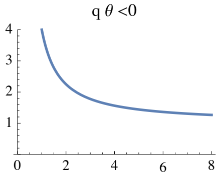
$$
\begin{aligned}
g(\{\gamma_i\},\varphi) &= \lim_{y\to 1} g(\{\gamma_i\},\varphi,y)\ , \\
g(\{\gamma_i\},\varphi,y) &= \frac{1}{(y-1/y)^{n-1}} \sum_{\text{fixed points of } J_3} y^{2J_3}
\end{aligned}
$$

Solving the supersymmetric n-body problem II

- For two-centers, $\mathcal{M}_2 = (S_2, \kappa \sin\theta \mathrm{d}\theta \mathrm{d}\phi)$ when $\kappa\varphi > 0$, or $\emptyset$ if $\kappa\varphi < 0$. Localization produces

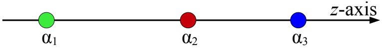
$$g(\{\gamma_1, \gamma_2\}, y) = \frac{(-1)^\kappa}{y - 1/y} \left(\underbrace{y^{+\kappa}}_{\text{North pole}} - \underbrace{y^{-\kappa}}_{\text{South pole}} \right) \xrightarrow{y\to 1} \pm\kappa$$





Solving the supersymmetric n-body problem III

- For $n > 2$, the fixed points of the action of J_3 are collinear multi-centered configurations along the z-axis:



$$\forall i\ , \quad \sum_{j\neq i} \frac{\kappa_{ij}}{|z_i - z_j|} = c_i\ , \qquad J_3 = \frac{1}{2} \sum_{i<j} \kappa_{ij}\, \mathrm{sign}(z_j - z_i)\,.$$

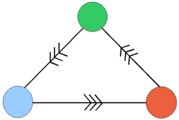
- The fixed points are isolated, and labelled by permutations σ:

$$g(\{\gamma_i\}, y) = \frac{(-1)^{\sum_{i<j} \kappa_{ij} + n - 1}}{(y - y^{-1})^{n-1}} \sum_{\sigma} s(\sigma)\, y^{\sum_{i<j} \kappa_{\sigma(i)\sigma(j)}}\ , \qquad s(\sigma) = 0, \pm 1$$

Manschot, BP, Sen 2010

Solving the supersymmetric n-body problem IV

- When $\mathcal{M}_n$ is compact, $g(\{\gamma_i\}, y)$ is a symmetric Laurent polynomial in y, and the limit $y \to 1$ exists. This breaks down however the quiver obtained by drawing κ_{ij} oriented arrows between the nodes labelled by γ_i and γ_j admits closed loops.



- In this case, $\mathcal{M}_n$ is not longer compact, but has singular regions where a subset of points becomes arbitrarily closed to each other: the scaling solutions, which classically have $\vec{J} = 0$.

Bena Berkooz El Showk de Boer van den Bleeken 2012

The Coulomb branch formula I

- Ashoke, Jan and I (2012) devised a somewhat ad-hoc prescription to determine the contributions from scaling regions recursively, and express the total index $\Omega(\gamma, \varphi)$ in terms of single-centered indices $\Omega_S(\gamma_i)$:

$$\bar{\Omega}(\gamma; \varphi, y) = \sum_{\gamma = \sum \gamma_i} \frac{g(\{\gamma_i\}, \{c_i\}; y)}{|\mathrm{Aut}(\{\gamma_i\})|} \prod_i \bar{\Omega}_{\rm tot}(\gamma_i, y)$$

where $\bar{\Omega}(\gamma, y)$ is the rational (or Boltzmannian) invariant

$$\bar{\Omega}(\gamma, y) \equiv \sum_{d|\gamma} \frac{1}{d} \frac{y - 1/y}{y^d - y^{-d}} \Omega(\gamma/d, y^d)$$

The Coulomb branch formula II

$$\begin{aligned}\Omega_{\rm tot}(\gamma;y) =&\Omega_S(\gamma;y) \\ &+ \sum_{\substack{\{\beta_i\in\Gamma\},\{m_i\in\mathbb{Z}\}\\ m_i\geq 1,\ \sum_i m_i\beta_i=\gamma}} H(\{\beta_i\};\{m_i\};y)\prod_i \Omega_S(\beta_i;y^{m_i})\end{aligned}$$

- $H(\{\beta_i\};\{m_i\};y)$ is determined recursively by the conditions
  - H is symmetric under $y \to 1/y$,
  - H vanishes at $y \to 0$,
  - the coefficient of $\prod_i \Omega_S(\beta_i;y^{m_i})$ in the expression for $\Omega(\sum_i m_i\beta_i;y)$ is a Laurent polynomial in y.

The formula is implemented in mathematica: `CoulombHiggs.m`

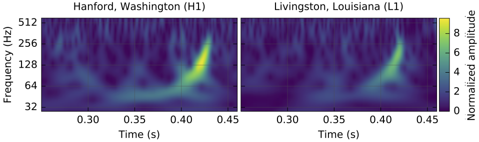
Manschot BP Sen 1302.5498; 1404.7154

The Coulomb branch formula III

- This formula is referred to as the "Coulomb branch formula", since it arises on the Coulomb branch of a $D = 0 + 1$ dimensional gauge theory known as $\mathcal{N} = 4$ quiver quantum mechanics. On the Higgs branch of the same theory, BPS ground states correspond to harmonic forms on the moduli space of quiver representations.
- Our conjectural formula can be used to compute all the Hodge numbers of quiver moduli spaces in terms of new BPS invariants, which in turn can be determined from the χ_y genus. Needless to say, it would be very interesting to prove our conjecture !

By way of a conclusion I

- Precision counting of BPS black holes has had very fruitful implications for mathematics: mock modular forms, Donaldson-Thomas invariants, quiver moduli...
- These results support the general consistency of string theory as a theory of quantum gravity. Whether it is the one relevant for Nature remains to be seen.
- Still, it is comforting to think than in a parallel Universe, a super-LIGO/VIRGO detector may hear the celestial dance of multi-BPS black holes, as opposed to the squeaking and squashing of Schwarschild black holes.



Thank you for your attention !