Wall-crossing, Black holes, and Quivers

Boris Pioline

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based on work with J. Manschot and A. Sen,

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Introduction I

- The main topic of these lectures:

$$\boxed{\text{BPS states in } D=4, N=2 \text{ gauge theories/string vacua}}$$

- A very active topic of research since seminal work of Seiberg and Witten.
- Despite much recent progress, determining the complete spectrum of BPS states in a given theory or string vacuum remains open.
- Offers a window into strongly coupled phenomena triggered by non-perturbative states:
  - monopole condensation in softly broken $N=2$ theories,
  - conifold transitions between CY vacua, etc

Introduction II

- Allow precision tests of non-perturbative dualities, which can then be extended beyond BPS sector:
  - Electric-magnetic duality
  - String dualities
  - D-brane/black hole duality, statistical origin of BH entropy
- Fascinating connections with mathematics:
  - Enumerative geometry, topological strings
  - Donaldson-Thomas theory
  - Quiver representations
  - Cluster algebras

Wall-crossing I

- The reason BPS states can be followed from weak to strong coupling reliably is that they lie in short multiplets. They may pair up, but the net number $\Omega(\gamma, t) = \mathrm{Tr}'(-1)^{2J_3}$ is preserved.
- Still, the BPS index $\Omega(\gamma, t)$ may jump on certain walls in moduli space, where the central charge of two charge vectors align:

$$W(\gamma_1, \gamma_2) = \{t \in \mathcal{M} \ : \ Z(\gamma_1, t)/Z(\gamma_2, t) \in \mathbb{R}^+\}$$

Since BPS states have mass $M(\gamma) = |Z(\gamma, t)|$, and Z is linear in γ, the decay $[\gamma_1 + \gamma_2] \mapsto [\gamma_1] + [\gamma_2]$ becomes energetically allowed, leading to a jump $\Delta\Omega(\gamma_1 + \gamma_2)$ across the wall.

Wall-crossing II

- The jump in $\Omega(\gamma_1+\gamma_2)$ can be computed from the indices $\Omega(\gamma_1)$ and $\Omega(\gamma_2)$ of the constituents, by studying the quantum mechanics of the two-particle bound state.
- In the vicinity of the wall, on the side where the bound state is stable, the internal degrees of freedom of $[\gamma_1]$ and $[\gamma_2]$ decouple from the relative degrees of freedom, leading to the primitive wall-crossing formula

$$\Delta\Omega(\gamma_1+\gamma_2)=(-1)^{\gamma_{12}+1}\,|\gamma_{12}|\,\Omega(\gamma_1)\,\Omega(\gamma_2)$$

Denef Moore

Wall-crossing III

- More generally, a bound state of charge $[M\gamma_1 + N\gamma_2]$ can decay into a multi-particle state of charges $[M_i\gamma_1 + N_i\gamma_2]$ as long as $\sum(M_i, N_i) = (M, N)$.
- The jump $\Delta\Omega(M\gamma_1 + N\gamma_2)$ can again be computed by by studying the quantum mechanics of n-particle bound state. This will be the topic of the first part of these lectures:

I. Wall-crossing from multi-soliton quantum mechanics

Elementary constitutents of BPS I

- In general the decay products $[M_i\gamma_1 + N_i\gamma_2]$ may themselves be bound states of more elementary constituents.
- An interesting question is to express, at any point in moduli space, the total index $\Omega(\gamma, t)$ in terms of the degeneracies $\Omega^{\rm S}(\gamma)$ of elementary, absolutely stable constituents.
- In gauge theories, there often exist a strong coupling chamber where only a finite number of BPS states exist (monopole and dyon in SW theory). In other chambers the BPS spectrum can be understood as bound states of these elementary constituents.

Elementary constitutents of BPS II

- In SUGRA, these elementary constituents are single centered BPS black holes. $\Omega^S(\gamma)$ should count the degrees of freedom associated to the $AdS_2 \times S^2$ near-horizon geometry, dual to some microscopic CFT.
- Importantly, since single-centered BPS black holes have zero angular momentum, $\Omega^S(\gamma)$ counts the actual number of states, not only the index !

$\boxed{\text{II. Total index from single-centered degeneracies}}$

BPS states from quivers I

- For $\mathcal{N}=2$ string vacua it is not known how to compute $\Omega(\gamma,t)$ nor $\Omega^{\rm S}(\gamma)$ for general charge vectors. Mathematically, it amounts to computing Donaldson-Thomas invariants for the derived category of coherent sheaves, a hard problem !
- However, BPS states can sometimes be constructed as D-brane bound states, described by some quiver quantum mechanics, which can often be solved exactly.
- The analogue of single-centered black holes are then middle cohomology states (Lefschetz singlets). A prediction of the black hole picture is that these states should be robust under wall-crossing.

Bena Berkooz El Showk de Boer van den Bleeken;Lee Wang Yi

BPS states from quivers II

- This also relates back to BPS states in $N = 2$ gauge theories, which in many cases are also described by quivers.

Cecotti, Vafa, ...

$\boxed{\text{III. Applications to quivers}}$

A word on string vacua with higher SUSY

- For maximally supersymmetric $N = 8$ SUGRA, the spectrum of 1/8-BPS dyons is entirely determined by U-duality. No wall-crossing in this case, since marginal stability occurs only in higher codimension.
- For $N = 4$ string vacua, the spectrum of 1/4-BPS dyons is still largely determined by U-duality, but does involves wall-crossing. Only bound states of two 1/2-BPS states contribute due to fermionic zero-modes. Elementary degeneracies Ω^S are given by Fourier coefficients of Mock modular forms.

Dijkgraaf Verlinde^2; de Wit et al; David Jatkar Sen; Dabholkar Murthy Zagier

- A long term goal is to extend this results to $N = 2$ string vacua.

Outline

Outline

Preliminaries I

- Consider $\mathcal{N} = 2$ supergravity in 4 dimensions. Let $\Gamma = \Gamma_e \oplus \Gamma_m$ be the lattice of electric and magnetic charges, with antisymmetric (Dirac-Schwinger- Zwanziger) integer pairing
$$\langle \gamma, \gamma' \rangle = q_\Lambda p'^\Lambda - q'_\Lambda p_\Lambda \in \mathbb{Z}$$
- BPS states preserve 4 out of 8 supercharges, and saturate the bound $M \geq |Z(\gamma, t^a)|$ where $Z(\gamma, t^a) = \langle Y(t^a), \gamma \rangle$ is the central charge.
- The index $\Omega(\gamma; t^a) = \mathrm{Tr}_{\mathcal{H}'_\gamma(t^a)}(-1)^{2J_3}$ (where $\mathcal{H}'_\gamma(t^a)$ is the Hilbert space of one-particle states with charge $\gamma \in \Gamma$ in the vacuum with vector moduli t^a) receives contributions from short multiplets only.

Preliminaries II

- In $N = 2$ gauge theories (but not SUGRA/string vacua), there is an additional $SU(2)_R$ symmetry, and the spin character
$$\Omega(\gamma; t, y) = \mathrm{Tr}(-1)^{2J_3}\, y^{2(I_3+J_3)}$$
is protected. The 'non exotics' conjecture asserts that all states have $I_3 = 0$, and the PSC coincides with the refined index
$$\Omega(\gamma; t, y) = \mathrm{Tr}(-1)^{2J_3}\, y^{2J_3}$$
Gaiotto Neitzke Moore
- In $N = 2$ SUGRA/string vacua, one can still define $\Omega(\gamma; t, y)$ but it is no longer protected, hence could get contributions from non-BPS states and depend on HM moduli.

Preliminaries III

- $\Omega(\gamma; t)$ is locally constant, but may jump on codimension 1 loci in VM moduli space called 'walls of marginal stability', where the bound state spectrum mixes with the continuum of multi-particle states:

$$W(\gamma_1, \gamma_2) = \{t \in \mathcal{M} \ : \ Z(\gamma_1, t)/Z(\gamma_2, t) \in \mathbb{R}^+\}$$

- Basic mechanism: Some of the BPS states with charge $\gamma = M\gamma_1 + N\gamma_2$ are bound states of more elementary BPS states with charge $\alpha_i = M_i\gamma_1 + N_i\gamma_2$, and these bound states exist only on one side of the wall.

Preliminaries IV

- Rk1: for a given wall, we can always choose the basis γ_1, γ_2 such that $\Omega(M\gamma_1 + N\gamma_2)$ has support on the cone $MN \geq 0$. The constituents have $M_i \geq 0, N_i \geq 0$, so only a finite number of bound states can occur.
- Rk2: The index of states with $\gamma \notin \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$ is constant across the wall. So are $\Omega(\gamma_1)$ and $\Omega(\gamma_2)$.
- Rk3: The index $\Omega(M\gamma_1 + N\gamma_2)$ may contain contributions from bound states of constituents with charges lying outside the lattice $\mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$. But those are insensitive to the wall.
- Rk4: There is no BPS bound state on the wrong side of the wall, but there could exist some non-BPS bound states. Such states do not contribute to $\Delta\Omega(\gamma)$, but in string/SUGRA they could contribute to $\Delta\Omega(\gamma, y)$.

Primitive wall-crossing I

- For $\langle \gamma_1, \gamma_2 \rangle \neq 0$, there exists a two-centered BPS solution of charge $\gamma = \gamma_1 + \gamma_2$, angular momentum $\vec{J} = \frac{1}{2} \langle \gamma_1, \gamma_2 \rangle \, \vec{u}$:

$$|x_1 - x_2| = \sqrt{G_4} \frac{\langle \Gamma_1, \Gamma_2 \rangle}{2} \frac{|Z(\Gamma_1 + \Gamma_2, t)|}{\mathrm{Im}(Z(\Gamma_1, t) \bar{Z}(\Gamma_2, t))}$$

Denef 2002

- The solution exists only on one side of the wall. As t approaches the wall, the distance r_{12} diverges and the bound state decays into its constituents γ_1 and γ_2.

Primitive wall-crossing II

- Near the wall, the two centers can be treated as pointlike particles with $\Omega(\gamma_i)$ internal degrees of freedom, interacting via Newton, Coulomb, Lorentz, scalar exchange forces.
- The classical BPS phase space $\mathcal{M}_2$ for the two-particle system is the two-sphere, with symplectic form

$$\omega = \tfrac{1}{2}\gamma_{12} \sin\theta \, \mathrm{d}\theta \mathrm{d}\phi \; , \quad \gamma_{12} \equiv \langle \gamma_1, \gamma_2 \rangle$$

  such that rotations ∂_ϕ are generated by $J_3 = \frac{1}{2}\gamma_{12} \cos\theta$.
- Quantum mechanically, one obtains $|\gamma_{12}|$ states transforming as a spin $j = \frac{1}{2}(|\gamma_{12}| - 1)$ multiplet under rotations.

Primitive wall-crossing III

- Near the wall, the $\Omega(\gamma_i)$ internal degrees of freedom decouple from the configurational degrees of freedom. The index of the two-particle bound state is then

$$\Omega_{\text{bound}} = \underbrace{(-1)^{\gamma_{12}+1}\gamma_{12}}_{\substack{\text{angular}\\ \text{momentum}}} \times \underbrace{\Omega(\gamma_1)}_{\substack{\text{internal}\\ \text{states of 1}}} \times \underbrace{\Omega(\gamma_2)}_{\substack{\text{internal}\\ \text{states of 2}}}$$

- These are the only bound states of charge $\gamma = \gamma_1 + \gamma_2$ which (dis)appear across the wall. Thus

$$\Delta\Omega(\gamma_1+\gamma_2) = (-1)^{\gamma_{12}+1}\,\gamma_{12}\,\Omega(\gamma_1)\,\Omega(\gamma_2)\;,$$

Denef Moore

Primitive wall-crossing IV

- Similarly, the variation of the refined index is

$$\Delta\Omega(\gamma_1+\gamma_2;y) = \frac{(-y)^{\gamma_{12}} - (-y)^{-\gamma_{12}}}{y-1/y}\,\Omega(\gamma_1;y)\,\Omega(\gamma_2;y)$$

Diaconescu Moore; Dimofte Gukov

- Let us try to extend this reasoning to compute $\Delta\Omega(M\gamma_1+N\gamma_2)$ for general (M,N).

Multi-centered solutions I

- The most general stationary, BPS solution of $\mathcal{N}=2$ SUGRA is

$$\mathrm{d}s^2 = -e^{2U}\left(\mathrm{d}t + \mathcal{A}\right)^2 + e^{-2U}\mathrm{d}\vec{r}^2$$

$$2\, e^{-U(\vec{r})}\mathrm{Im}\left[e^{-\mathrm{i}\phi}Y\left(t^a(\vec{r})\right)\right] = \beta + \sum_{i=1}^{n}\frac{\alpha_i}{|\vec{r}-\vec{r}_i|},$$

$$\phi = \arg Z_\gamma\ , \quad \gamma = \alpha_1 + \dots + \alpha_n\ , \quad \beta = 2\,\mathrm{Im}\left[e^{-\mathrm{i}\phi}\,Y(t_\infty)\right]$$

- The integrability condition for $\mathcal{A}$ requires

$$[*] \qquad \sum_{\substack{j=1\\ j\neq i}}^{n}\frac{\langle\alpha_i,\alpha_j\rangle}{|\vec{r}_i-\vec{r}_j|} = c_i\ , \qquad c_i \equiv 2\,\mathrm{Im}\left(e^{-\mathrm{i}\phi}Z_{\alpha_i}\right)$$

Denef; Bates Denef

Multi-centered solutions II

- This provides $n-1$ conditions on $3n$ locations $\vec{r}_i$. Modding out by translations in $\mathbb{R}^3$, we define $\mathcal{M}_n(\{\alpha_i\},\{c_i\})$ to be the $2n-2$ dimensional space of solutions to [$*$].
- For the solution to be regular, one must also ensure

$$e^{-2U(\vec{r})} = \frac{1}{\pi} S \left(\beta + \sum_{i=1}^{n} \frac{\alpha_i}{|\vec{r}-\vec{r}_i|} > 0 \,, \qquad \forall\, \vec{r} \in \mathbb{R}^3 \right) \,.$$

  This may remove some connected components in $\mathcal{M}_n$.
- When all charges α_i lie in a two-dimensional lattice $\mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$ and satisfy the cone condition $MN \geq 0$, this condition appears to be automatically satisfied. Moreover $c_i = \Lambda \sum_{j\neq i} \alpha_{ij}$ with $\Lambda \to 0$ at the wall.

Q. mech. of multi-centered black holes I

- At least when the centers are well-separated, the dynamics of the bound state is described by $\mathcal{N}=4$ quantum mechanics with $3n$ bosonic coordinates, $4n$ fermionic coordinates with Lagrangian

$$\mathcal{L} = \sum_i [W_i(\vec{r}_i)]^2 + \sum_i \vec{A}_i \dot{\vec{r}}_i + \sum_{i,j} \gamma_{ij} \dot{\vec{r}}_i \dot{\vec{r}}_j + \dots$$

$$W_i = \sum_{j \neq i} \frac{\langle \alpha_i, \alpha_j \rangle}{|\vec{r}_i - \vec{r}_j|} - c_i \ , \qquad \vec{A}_i = \sum_{j \neq i} \langle \alpha_i, \alpha_j \rangle \, \vec{A}_{\rm Dirac}(\vec{r}_i - \vec{r}_j)$$

Denef;Kim Park Wang Yi

Q. mech. of multi-centered black holes II

- The classical ground state dynamics is then first order quantum mechanics on the BPS phase space $\mathcal{M}_n = \{W_i = 0\}$ equipped with the symplectic form

$$\omega = \frac{1}{2} \sum_{i<j} \alpha_{ij} \sin\theta_{ij}\, \mathrm{d}\theta_{ij} \wedge \mathrm{d}\phi_{ij} \ , \qquad \vec{J} = \frac{1}{2} \sum_{i<j} \alpha_{ij} \frac{\vec{r}_{ij}}{r_{ij}}$$

de Boer El Showk Messamah Van den Bleeken

- Semiclassically, the number of states is equal to the (equivariant) symplectic volume

$$g_{\mathrm{class}}(\{\alpha_i\}, \{c_i\}; y) = \frac{(-1)^{\sum_{i<j} \alpha_{ij} - n + 1}}{(2\pi)^{n-1}(n-1)!} \int_{\mathcal{M}_n} \omega^{n-1}\, y^{2J_3}$$

Q. mech. of multi-centered black holes III

- Quantum mechanically, the refined index is equal to the equivariant Dirac index of the symplectic manifold $(\mathcal{M}_n, \omega)$,

$$g_{\rm ref}(\{\alpha_i\},\{c_i\};y) = \mathrm{Tr}_{\mathrm{Ker}D_+}(-y)^{2J_3} - \mathrm{Tr}_{\mathrm{Ker}D_-}(-y)^{2J_3} .$$

  This reduces to $g_{\rm class}$ in the limit $\alpha_{ij} \to \infty$.
- When $\mathcal{M}_n$ is compact and J_3 has only isolated fixed points, it can be evaluated by Atiyah-Bott Lefschetz fixed point formula:

$$g_{\rm ref}(\{\alpha_i\},\{c_i\};y) = \sum_{\rm fixed\ pts} \frac{y^{2J_3}}{\det((-y)^L - (-y)^{-L})}$$

  where L is the matrix of the action of J_3 on the holomorphic tangent space around the fixed point.

Q. mech. of multi-centered black holes IV

- E.g. for $n = 2$, $\mathcal{M}_2 = S^2$, $J_3 = \alpha_{12} \cos\theta$:

$$g_{\rm ref} = \frac{(-1)^{\alpha_{12}+1}}{(y - 1/y)} \left(\underbrace{y^{+\alpha_{12}}}_{\text{North pole}} - \underbrace{y^{-\alpha_{12}}}_{\text{South pole}} \right)$$
$$= \mathrm{Tr}_{j=\frac{1}{2}(\alpha_{12}-1)}\, y^{2J_3}$$

if $c_1 \alpha_{12} > 0$, zero otherwise.

Refined index from localization I

- The fixed points of the action of J_3 are collinear multi-centered configurations along the z-axis, such that

$$\sum_{j\neq i}^{n} \frac{\alpha_{ij}}{|z_i - z_j|} = c_i \,, \quad J_3 = \frac{1}{2} \sum_{i<j} \alpha_{ij} \, \mathrm{sign}(z_j - z_i) \,.$$

- Equivalently, fixed points are critical points of the 'superpotential'

$$W(\{z_i\}) = -\sum_{i<j} \mathrm{sign}[z_j - z_i] \, \alpha_{ij} \, \ln|z_j - z_i| - \sum_i c_i z_i$$

- The determinant turns out to be $(y - 1/y)^{n-1}$ times a sign $s(p) = -\mathrm{sign}(\det W'')$ where W'' is the Hessian of W

Refined index from localization II

- After the dust settles, one finds the Coulomb branch formula

$$g_{\rm ref}(\{\alpha_i\},\{c_i\};y)=\frac{(-1)^{\sum_{i<j}\alpha_{ij}+n-1}}{(y-y^{-1})^{n-1}}\sum_p s(p)\,y^{\sum_{i<j}\alpha_{ij}\,{\rm sign}(z_j-z_i)}$$

  where the sum runs over all collinear solutions of Denef's equations.
- The contribution of each fixed point is singular at $y=1$, but the sum over fixed points is guaranteed to produce a symmetric polynomial in y and $1/y$, as long as $\mathcal{M}_n$ is compact. Fortunately, this is always the case for configurations involved in wall-crossing.

Coulomg branch wall-crossing formula I

- Having computed the index of the quantum mechanics of n centers, we can apply the same logic as before and write (naively),

$$\Delta\Omega(\gamma) \stackrel{??}{=} \sum_{n\geq 2} \sum_{\gamma=\alpha_1+\cdots+\alpha_n} g(\{\alpha_i\}) \prod_{i=1}^{n} \Omega(\alpha_i)$$

  where $\gamma = M\gamma_1 + N\gamma_2, \alpha_i = M_i\gamma_1 + N_i\gamma_2$, and $\Omega(\alpha_i)$ is the index on the side where the bound state does not exist.
- This is almost right, but it overlooks the issue of statistics.

Statistics I

- If the centers were classical, distinguishable objects, Maxwell- Boltzmann statistics would require a symmetry factor

$$\Delta\Omega(\gamma) \stackrel{!?}{=} \sum_{n\geq 2} \sum_{\gamma=\alpha_1+\dots+\alpha_n} \frac{g(\{\alpha_i\})}{|\mathrm{Aut}(\{\alpha_i\})|} \prod_{i=1}^{n} \Omega(\alpha_i)$$

  where $\mathrm{Aut}(\{\alpha_i\})$ is the subgroup of the permutation group leaving $\{\alpha_i\}$ invariant.

Statistics II

- Instead, the centers are quantum objects, with Bose statistics if $\Omega(\alpha_i) > 0$, or Fermi statistics if $\Omega(\alpha_i) < 0$. One can show that the Maxwell-Boltzmann prescription nevertheless works, provided one replaces everywhere the index $\Omega(\gamma)$ with its rational counterpart

$$\bar{\Omega}(\gamma) \equiv \sum_{d|\gamma} \frac{1}{d^2} \Omega(\gamma/d)$$

or similarly for the refined index,

$$\bar{\Omega}(\gamma, y) \equiv \sum_{d|\gamma} \frac{(y-y^{-1})}{d(y^d-y^{-d})} \Omega(\gamma/d, y^d)$$

Statistics III

- Consider for example $\Delta(\gamma_1 + 2\gamma_2)$: it receives contributions from bound states $\{\gamma_1+\gamma_2, \gamma_2\}$, $\{\gamma_1, 2\gamma_2\}$, $\{\gamma_1, \gamma_2, \gamma_2\}$.
- Taking into account Bose-Fermi statistics for $\{\gamma_1, \gamma_2, \gamma_2\}$,

$$\begin{aligned}\Delta\Omega(\gamma_1 + 2\gamma_2) =& (-1)^{\gamma_{12}}\gamma_{12}\Omega^+(\gamma_2)\,\Omega^+(\gamma_1+\gamma_2)\\ &+ 2\gamma_{12}\,\Omega^+(2\gamma_2)\,\Omega^+(\gamma_1)\\ &+ \tfrac{1}{2}\gamma_{12}\,\Omega^+(\gamma_2)\left(\gamma_{12}\Omega^+(\gamma_2)+1\right)\Omega^+(\gamma_1)\,.\end{aligned}$$

In terms of the rational invariant $\bar{\Omega}(2\gamma_2) = \Omega(2\gamma_2) + \frac{1}{4}\Omega(\gamma_2)$, charge conservation is restored:

$$\begin{aligned}\Delta\bar{\Omega}(\gamma_1 + 2\gamma_2) =& (-1)^{\gamma_{12}}\gamma_{12}\bar{\Omega}^+(\gamma_2)\,\bar{\Omega}^+(\gamma_1+\gamma_2)\\ &+ 2\gamma_{12}\,\bar{\Omega}^+(2\gamma_2)\,\bar{\Omega}^+(\gamma_1)\\ &+ \tfrac{1}{2}[\gamma_{12}\,\bar{\Omega}^+(\gamma_2)]^2\bar{\Omega}^+(\gamma_1)\,.\end{aligned}$$

Coulomb branch wall crossing formula I

- We have finally arrived at the Coulomb branch wall-crossing formula:

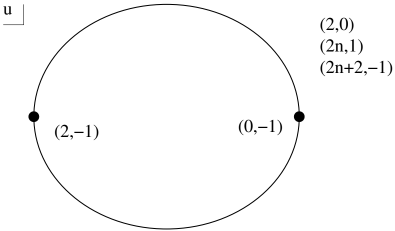
$$\Delta\bar{\Omega}(\gamma, y) = \sum_{n\geq 2} \sum_{\gamma=\alpha_1+\cdots+\alpha_n} \frac{g_{\rm ref}(\{\alpha_i\}), y}{|{\rm Aut}(\{\alpha_i\})|} \prod_{i=1}^{n} \bar{\Omega}(\alpha_i, y)$$

- Remarkably, the formula agrees with the mathematical formulae established in the context of Donaldson-Thomas invariants for the derived category of Abelian sheaves.

Kontsevich-Soibelman;Joyce-Song

Coulomb branch wall crossing formula II

- The formula reproduces for example the weak coupling spectrum of pure $SU(2)$ gauge theory found by Seiberg and Witten:



- In the next lecture, I shall address the question of computing the total index $\Omega(\gamma; t)$ (not only its variation) in terms of the indices of elementary constituents, such as the monopole and dyon, and connections with quivers.

Outline

1. Introduction
2. Quantum mechanics of multi-centered BHs
3. **Index from single-centered degeneracies**
4. Applications to quivers

Summary of first lecture I

- We have studied the refined index $\Omega(\gamma, t, y) = \text{Tr}(-y)^{2J_3}$ in $N = 2$ gauge theories and string vacua in the vicinity of a wall of marginal stability $W(\gamma_1, \gamma_2)$.
- For $\gamma = M\gamma_1 + N\gamma_2$, $\Omega(\gamma)$ receives contributions from BPS bound states of more elementary constituents with charge $\alpha_i = M_i\gamma_1 + N_i\gamma_2$, which exist on one side of the wall only.
- The contribution of each bound state factorizes into a product of internal degeneracies $\Omega(\alpha_i)$ and the index $g_{\rm ref}(\{\alpha_i\}, y)$ of the quantum mechanics of n-centers.

Summary of first lecture II

- $g_{\rm ref}$ is equal to the equivariant Dirac index of the BPS phase space $\mathcal{M}_n$, a compact symplectic space of dimension $2n-2$. The index can be evaluated by localization:

$$g_{\rm ref}(\{\alpha_i\},\{c_i\};y) = \frac{(-1)^{\sum_{i<j}\alpha_{ij}+n-1}}{(y-y^{-1})^{n-1}} \sum_p s(p)\, y^{\sum_{i<j}\alpha_{ij}\,{\rm sign}(z_j-z_i)}$$

where the sum runs over all solutions of

$$\sum_{j\neq i}^{n} \frac{\alpha_{ij}}{|z_i-z_j|} = c_i\,, \qquad c_i = \Lambda \sum_{j\neq i} \alpha_{ij}$$

and $s(p)=\pm 1$ is a computable sign.

Summary of first lecture III

- Although constituents with identical charge α_i satisfy Bose-Fermi statistics, they can be effectively treated as Bolzmannian at the expense of replacing the index $\Omega(\alpha_i, y)$ by the rational refined index

$$\bar{\Omega}(\gamma, y) \equiv \sum_{d|\gamma} \frac{(y-y^{-1})}{d(y^d-y^{-d})} \Omega(\gamma/d, y^d)$$

- The total variation of $\Omega(\gamma)$ is then given by the Coulomb branch wall-crossing formula

$$\Delta\bar{\Omega}(\gamma, y) = \sum_{n\geq 2} \sum_{\gamma=\alpha_1+\dots+\alpha_n} \frac{g_{\mathrm{ref}}(\{\alpha_i\}), y)}{|\mathrm{Aut}(\{\alpha_i\})|} \prod_{i=1}^{n} \bar{\Omega}(\alpha_i, y)$$

Outline of this lecture I

- Wall-crossing shows that some of the BPS states contributing to $\Omega(\gamma)$ are not elementary, but can be decomposed into more elementary constituents. However these constituents may still decay elsewhere.
- This suggests that there may exist a set of truly elementary, absolutely stable BPS states such that any other BPS state would arise as a bound state of those.
- This is realized in pure $SU(2)$ gauge theorie, where all states arise as bound states of the monopole and dyon. In SUGRA, single centered black holes should play the role of elementary constituents.

Outline of this lecture II

- The first purpose of this lecture is to propose a master formula which expresses the total index $\Omega_{\rm ref}(\gamma; t, y)$ in terms of the indices $\Omega^{\rm S}_{\rm ref}(\gamma; y)$ associated to the elementary/single centered constituents.
- The second purpose is to test the validity of the formula in the context of quiver quantum mechanics, which describe certain D-brane bound states (as well as the BPS spectrum of certain gauge theories).

The Master Formula (first pass) I

- Let $\Omega_S(\alpha_i, y)$ be the index of elementary/single centered states with charge α_i. The total index $\Omega(\gamma, z^a; y)$, or rather its rational counterpart, should be a sum over all possible bound states

$$\bar{\Omega}(\gamma, z^a; y) = \sum_{n\geq 1} \sum_{\sum \alpha_i = \gamma} \frac{g_{\rm ref}(\{\alpha_i\},\{c_i\},y)}{|\mathrm{Aut}(\{\alpha_i\})|} \bar{\Omega}_S(\alpha_i, y)$$

- Unlike in wall-crossing case, there are potentially infinitely many possible sets $\{\alpha_i\}$ such that $\gamma = \sum \alpha_i$ and $\bar{\Omega}_S(\alpha_i, y) \neq 0$. Hopefully the regularity condition $e^{2U} > 0$ rules out all but a finite number of them...

Scaling solutions I

- Ignoring this issue, another serious concern is that, unlike in wall-crossing case, the phase space $\mathcal{M}_n$ need not be compact. As a result, $g_{\rm ref}(\{\alpha_i\},\{c_i\},y)$ and hence $\Omega(\gamma,z^a;y)$ may not be symmetric Laurent polynomials.
- To illustrate this, take $n=3$,$\alpha_{12}=a$, $\alpha_{23}=b$, $\alpha_{31}=c$ satisfying triangular inequalities $0<a<b+c$, etc, there exist solutions of Denef's equations

$$\frac{a}{r_{12}}-\frac{c}{r_{13}}=c_1\,,\ \frac{b}{r_{23}}-\frac{a}{r_{12}}=c_2$$

with $r_{12}\sim a\epsilon, r_{23}\sim b\epsilon, r_{13}\sim c\epsilon, \vec{J}^2\sim\epsilon^2$ as $\epsilon\to 0$.

Scaling solutions II

- For $c_1, c_2 > 0$, the only collinear configurations are (123) and (321), leading to

$$g_{\rm ref} = \frac{(-1)^{a+b+c}(y^{a+b-c} + y^{-a-b+c})}{(y-1/y)^2}$$

- This is not a polynomial in y, in particular it is singular as $y \to 1$. Still the (equivariant) volume of $\mathcal{M}_n$ is finite.
- This can be repaired by adding by hand a term with $J_3 \simeq 0$, attributed to scaling solutions:

$$\tilde{g}_{\rm ref} = \frac{(-1)^{a+b+c}\left(y^{a+b-c} + y^{-a-b+c} - \left[\begin{array}{cc} 2 & a+b+c \text{ even} \\ y+1/y & a+b+c \text{ odd} \end{array}\right.\right.}{(y-1/y)^2}$$

Scaling solutions III

- More generally, scaling regions in $\mathcal{M}_n(\{\alpha_i\})$ arise whenever there exist a subset A and vectors $\vec{r}_i \in \mathbb{R}^3$, $i \in A$ such that

$$\forall i \in A\,, \quad \sum_{j\in A} \frac{\alpha_{ij}}{|\vec{r}_{ij}|} = 0\,.$$

  This is independent of the c_i's, so scaling solutions cannot be removed by changing the moduli.

- One can give a general prescription to compactify the BPS phase space $\mathcal{M}_n$, and complete the sum over collinear fixed points $g_{\rm ref}$ into a symmetric Laurent polynomial $\tilde{g}_{\rm ref}$, however this is not quite sufficient to ensure that the $\Omega(\gamma, t)$ resulting from the Master formula is sensible.

Master formula (second pass) I

- Instead, we postulate

$$\bar{\Omega}(\gamma,y)=\sum_{n\geq 1}\sum_{\{\alpha_i\},\sum\alpha_i=\gamma}\frac{g_{\rm ref}(\{\alpha_i\},\{c_i\},y)}{|{\rm Aut}(\{\alpha_i\})|}$$

$$\times\prod_{i=1}^{n}\left\{\sum_{m_i|\alpha_i}\frac{y-1/y}{m_i(y^{m_i}-y^{-m_i})}\left[\Omega_{\rm S}(\alpha_i/m_i,y^{m_i})+\Omega_{\rm scaling}(\alpha_i/m_i,y^{m_i})\right]\right\}$$

- $\Omega_{\rm scaling}$ is determined recursively in terms of $\Omega_{\rm S}$ by the minimal modification hypothesis:

$$\Omega_{\rm scaling}(\alpha;y)=\sum_{\substack{\{\beta_i\in\Gamma\},\{m_i\in\mathbb{Z}\}\\ m_i\geq 1,\ \sum_i m_i\beta_i=\alpha}} H(\{\beta_i\};\{m_i\};y)\prod_i\Omega_{\rm S}(\beta_i;y^{m_i})$$

Master formula (second pass) II

- $H(\{\beta_i\};\{m_i\};y)$ is uniquely determined by the conditions
  - H is symmetric under $y \to 1/y$,
  - H vanishes at $y \to 0$,
  - the coefficient of $\prod_i \Omega^{\rm S}_{\rm ref}(\beta_i;y^{m_i})$ in the expression for $\Omega(\sum_i m_i\beta_i;y)$ is a Laurent polynomial in y.
- E.g, for the 3-center configuration discussed above,

$$H(\{\gamma_1,\gamma_2,\gamma_3\};\{1,1,1\};y) = \begin{cases} -2\,(y-y^{-1})^{-2} & ,\, a+b+c \text{ even} \\ (y+y^{-1})\,(y-y^{-1})^{-2} & ,\, a+b+c \text{ odd} \end{cases}$$

so the prescription reduces to $g_{\rm ref} \to \tilde{g}_{\rm ref}$, but not in general...

Master formula (second pass) III

- The Master Formula expresses the set of indices $\Omega_{\rm ref}(\gamma; t, y)$ in terms of a new set of indices $\Omega^S_{\rm ref}(\gamma, y)$. What have we gained ?
- First, $\Omega^S_{\rm ref}(\gamma)$ no longer depend on the moduli. The formula is by construction consistent with wall-crossing.
- Second, in SUGRA we expect $\Omega^S_{\rm ref}(\gamma, y)$ to count micro-states of single-centered BPS black holes, and therefore to be y-independent. This gives a non-trivial constraint on the total refined index $\Omega_{\rm ref}(\gamma; t, y)$.

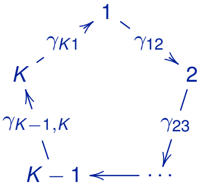
Outline

Quiver quantum mechanics I

- The $\mathcal{N}=4$ quantum mechanics of multi-centered solitons/black holes arises as the Coulomb branch of a more complicated $\mathcal{N}=4$ matrix quantum mechanics, whose matter content is captured by a quiver.
- Each node $\ell = 1...K$ represents a $U(N_\ell)$ vector multiplet $(\vec{r}_\ell, D_\ell)$, each arrow represents a chiral multiplet $\phi_{k,\ell}$ in $(N_\ell, \bar{N}_k)$ representation of $U(N_\ell) \times U(N_k)$. The set $\{N_\ell\}$ is called the dimension vector.
- In addition, one must specify Fayet-Iliopoulos terms c_ℓ such that $\sum_\ell N_\ell c_\ell = 0$, and (in presence of closed oriented loops) a gauge invariant superpotential W.
- Quivers also describe SUSY gauge theories in higher dimension, but here we focus on $D = 1$.

Quiver quantum mechanics II

- In the case of $N = \sum_{\ell=1\dots K} N_\ell$ centers, N_ℓ of which carrying charge γ_ℓ, the quiver has K nodes and $\gamma_{\ell k}$ arrows from node ℓ k.



- The FI term c_ℓ depends on the VM moduli, while the coefficients of W in general depend on HM moduli.

Quiver quantum mechanics III

- The Coulomb branch description arises by integrating out the chiral multiplets, and reproduces Denef's equations

$$\forall \ell \, , \quad \sum_{k \neq \ell} \frac{\gamma_{\ell k}}{|\vec{r}_\ell - \vec{r}_k|} = c_\ell$$

  It is valid when the centers are far apart.
- The Higgs branch description arises by integrating out the vector multiplets. It is valid for large values of the chiral multiplet scalars $\phi_{k\ell}$.
- If both branches are regular, one expects the two descriptions to be dual and have the same BPS states. This can break down if the Coulomb and Higgs branches mix.

Quantum mechanics on the Higgs branch I

- The moduli space of SUSY vacua on the Higgs branch $\mathcal{M}_H$ is the set of solutions of the F-term $\partial_\phi W = 0$ and D-term equations

$$\forall \ell : \sum_{\gamma_{\ell k} > 0} \phi^*_{\ell k} \, T^a \, \phi_{\ell k} - \sum_{\gamma_{k \ell} > 0} \phi^*_{k \ell} \, T^a \, \phi_{k \ell, \alpha, s' t} = c_\ell \, \mathrm{Tr}(T^a)$$

  modulo the action of $\prod_\ell U(N_\ell)$.
- Equivalently, $\mathcal{M}_H$ is the space of semi-stable solutions of $\partial_\phi W = 0$ modulo $\prod_\ell GL(N_\ell, \mathbb{C})$.

Quantum mechanics on the Higgs branch II

- BPS states on the Higgs branch correspond to cohomology classes in $H^*(\mathcal{M}_{\rm H},\mathbb{Z})$. They transform under $SU(2)$ according to the Lefschetz action

$$J_+ \cdot h = \omega \wedge h\,, \quad J_- = \omega \llcorner h\,, \quad J_3 \cdot h = \tfrac{1}{2}(n-d)h\,.$$

where d is the complex dimension of $\mathcal{M}_H$, n the degree of h.

- The refined index on the Higgs branch is given by the Poincaré polynomial

$$Q(\mathcal{M}_H;y) = \mathrm{Tr}'(-y)^{2J_3} = \sum_{p=1}^{2d} b_p(\mathcal{M}_H)\,(-y)^{p-d}$$

Q is a polynomial in $y, 1/y$, symmetric under $y \to 1/y$.

Quantum mechanics on the Higgs branch III

- For example, for the 2-node (Kronecker) quiver with k arrows, dimension vector $(1,1)$, $c_1 > 0$,

$$1 \xrightarrow{\quad k \quad} 2$$

$$\mathcal{M}_H = \mathbb{C}^k/\mathbb{C}^\times = \mathbb{P}^{k-1} \;\Rightarrow\; Q(\mathcal{M}_H; y) = \frac{(-y)^k - (-y)^{-k}}{y - 1/y}$$

  Remarkably, for $k = \gamma_{12}$ this agrees with the index of the QM of a 2-centered solution (γ_1, γ_2) with $\Omega(\gamma_1) = \Omega(\gamma_2) = 1$!

- For $k = 2$ and dimension vector (M, N), one can show that $Q(\mathcal{M}_H; y)$ is the number of BPS states of charge $M\gamma\gamma_2$ in pure $SU(2)$ Seiberg-Witten theory, where γ_1, γ_2 are the monopole and dyon charges !

Quantum mechanics on the Higgs branch IV

- More generally, the BPS spectrum of many (if not all) $\mathcal{N}=2$, $D=4$ gauge theories is governed by a quiver. Any BPS state arises as a bound state of the BPS states which occur in the strong coupling chamber.

Alim Cecotti Cordova Espahbodi Rastogi Vafa; Cecotti del Zotto; ...

- In general, the computation of $Q(\mathcal{M}_H; y)$ is a hard problem. For quivers without oriented closed loop, there is a general formula based on Harder-Narasimhan filtrations. For non-Abelian quivers with loops and generic superpotential, there is no general way to compute $Q(\mathcal{M}_H; y)$ at present.

Reineke; Joyce

Quantum mechanics on the Higgs branch V

- For Abelian quivers with loops, $\mathcal{M}_H$ is a complete intersection in a product of projective spaces, and its cohomology can be computed using the Lefschetz hyperplane theorem

$$b_p(\mathcal{M}_H) = \begin{cases} b_p(\mathcal{M}_{\rm amb}) & p < d \\ b_{2d-p}(\mathcal{M}_{\rm amb}) & p > d \end{cases},$$

and the Riemann-Roch theorem to compute $\chi(\mathcal{M}_H)$ and hence the middle cohomology $b_d(\mathcal{M}_H)$.

Denef Moore; Bena Berkooz El Showk de Boer van den Bleeken

Higgs branch vs Coulomb branch I

- For the Kronecker quiver with dimension vector (M, N), the refined index on the Higgs branch is given by Reineke's formula. It can be recast as

$$\bar{\Omega}(\gamma, y) = \sum_{n\geq 2} \sum_{\gamma=\alpha_1+\dots+\alpha_n} \frac{g_{\rm ref}(\{\alpha_i\}), y)}{|{\rm Aut}(\{\alpha_i\})|} \prod_{i=1}^n \bar{\Omega}(\alpha_i, y)$$

$$g_{\rm Higgs}(\{\alpha_i\}, y) = \frac{(-1)^{\sum \alpha_{ij}+n-1}}{(y-1/y)^{n-1}} \sum_{\sigma} N(\{\alpha_i\}, \sigma)\, y^{\sum_{i<j} \alpha_{\sigma(i)\sigma(j)}}$$

$$N(\{\alpha_i\}, \sigma) = \prod_{\substack{k=2\dots n \\ \sigma(k)<\sigma(k-1)}} \Theta(\langle \gamma, \sum_{i=k}^{n} \alpha_{\sigma(i)} \rangle) \prod_{\substack{k=2\dots n \\ \sigma(k)>\sigma(k-1)}} \Theta(\langle \sum_{i=k}^{n} \alpha_{\sigma(i)}, \gamma \rangle)$$

Higgs branch vs Coulomb branch II

- Amazingly, this agrees with the Coulomb branch formula for $\gamma = M\gamma_1 + N\gamma_2$, $k = \gamma_{12}$, $\Omega(\gamma_1) = \Omega(\gamma_2) = 1$!

Manschot BP Sen; Sen

- More generally, as long as the Coulomb branch and Higgs branch don't mix, one expects a 1-1 map between BPS states on the Higgs branch and BPS states on the Coulomb branch, assuming that each node carries unit degeneracy.
- This breaks down in presence of scaling solutions, where the centers on the Coulomb branch can come arbitrarily close. In such cases, there can be vastly more states on the Higgs branch than on the Coulomb branch !

Higgs branch vs Coulomb branch III

- In the presence of scaling configurations, the Coulomb branch describes only a subset of the states on the Higgs branch. For example, consider the three-node quiver

$$
\begin{tikzcd}
 & 1 \arrow[dr, "a"] & \\
3 \arrow[ur, "c"] & & 2 \arrow[ll, "b"]
\end{tikzcd}
\qquad
\begin{array}{l}
a, b, c \geq 0 \\
N_1 = N_2 = N_3 = 1 \\
c_1, c_2 > 0, c_3 < 0
\end{array}
$$

- The generating function of the Higgs branch indices

$$Q(x_1, x_2, x_3; y) = \sum_{a \geq 0, b \geq 0, c \geq 0} (x_1)^a (x_2)^b (x_3)^c \, Q(\mathcal{M}_{a,b,c}, y)$$

can be computed using Lefschetz and Riemann-Roch.

Higgs branch vs Coulomb branch IV

- it decomposes into $Q = Q_{\rm C} + Q_{\rm S}$, where

$$Q_{\rm C} = \frac{x_1 x_2 \left\{ 1 - x_1 x_2 + x_1 x_2 x_3 \left(x_1 + x_2 + y + y^{-1} \right) \right\}}{(1-x_1 x_2)(1-x_1 x_3)(1-x_2 x_3)(1+x_1/y)(1+x_1 y)(1+x_2/y)(1+x_2 y)}$$
$$Q_{\rm S} = \frac{x_1^2 x_2^2 x_3^2}{(1-x_1 x_2)(1-x_2 x_3)(1-x_1 x_3)[1-x_1 x_2 - x_2 x_3 - x_1 x_3 - 2 x_1 x_2 x_3]}$$

- $Q_{\rm C}$ corresponds to contributions from 3-centered black holes, each carrying unit degeneracy. It is y dependent, moduli dependent, and grows polynomially with a, b, c.
- $Q_{\rm S}(x_1, x_2, x_3)$ is y-independent, moduli-independent, has support on $\{a < b + c, b < a + c, c < a + b\}$, and grows exponentially with a, b, c. It counts spin zero Lefschetz multiplets, or pure Higgs states for short.

Master formula for quivers I

- We conjecture that all states on the Higgs branch have a generalized Coulomb branch description, as bound states of elementary constituents whose refined index $\Omega_{\rm S}(\alpha_i)$ is both y-independent and invariant under wall-crossing.
- In particular, the master formula should hold for all quivers,

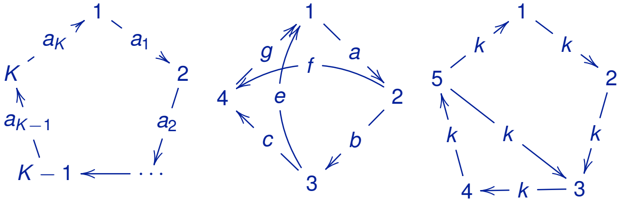
$$\bar\Omega(\gamma,y) = \sum_{n\geq 1} \sum_{\{\alpha_i\}, \sum \alpha_i=\gamma} \frac{g_{\rm ref}(\{\alpha_i\},\{c_i\},y)}{|{\rm Aut}(\{\alpha_i\})|}$$

$$\times \prod_{i=1}^{n} \left\{ \sum_{m_i|\alpha_i} \frac{1}{m_i} \frac{y-1/y}{y^{m_i}-y^{-m_i}} \left[\Omega_{\rm S}(\alpha_i/m_i, y^{m_i}) + \Omega_{\rm scaling}(\alpha_i/m_i, y^{m_i})\right] \right\}$$

where $\Omega_{\rm scaling}$ is determined as before.

Master formula for quivers II

- We have tested the formula on a variety of Abelian quivers:



and on some non-Abelian 3-node quivers: the master formula seems to hold up...

Summary and open problems I

- The total index in $\mathcal{N} = 2$ supersymmetric theories can be written as a sum of contributions from bound states of n elementary/single centered spinless constituents.
- The index Ω_S associated to elementary constituents is both moduli- and y-independent. The existence of bound states does depend on the moduli z^a. Angular momentum is carried by configurational degrees of freedom.
- *The enumeration of all possible decompositions $\gamma = \sum_i \alpha_i$ is currently untractable. Can one find a simple criterium (beyond the attractor flow conjecture) that determines whether a given choice of charges will lead to a regular multi-center solution ?*

Summary and open problems II

- *The computation of the Coulomb branch index $g_{\rm ref}(\{\alpha_i\}, y)$ for general $\{\alpha_i\}$ is currently limited by computer power. Is there a better way to compute the sum over collinear fixed points ?*
- Due to scaling solutions, the contribution of each elementary constituent is corrected by a term $\Omega_{\rm scaling}(y)$ which depends on the index of 'smaller' charges, and ensures that the total index is a Laurent polynomial. *Can one derive this from a detailed study of $N = 4$ quiver quantum mechanics ?*

Summary and open problems III

- In the context of quivers, Ω_S counts the number of Lefschetz singlets in the cohomology of the quiver moduli space. *Why these states are robust under changing the FI parameters ? Is there a general formula for the index for non-Abelian quivers with generic superpotential ?*
- In supergravity, Ω_S corresponds to the degeneracy in the attractor chamber, and should count the degrees of freedom in a single $AdS_2 \times S_2$ throat. *Can one compute it microscopically and match it to the quantum entropy function ?*

Summary and open problems IV

EBXARIΣTO ΠOΛI !