From black holes to quivers

Boris Pioline

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based on work with J. Manschot and A. Sen,

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Introduction I

In 4D string vacua with $\mathcal{N} = 4$ or $\mathcal{N} = 8$ SUSY, the spectrum of BPS black holes is essentially completely understood:

- Detailed microscopic derivation of the index from D1-D5-KKM
- Agreement with the Bekenstein-Hawking-Wald entropy, including subleading quantum corrections.
- Moduli dependence of index entirely accounted by two-centered bound states
- Partition functions have nice modular properties, even after subtracting multi-centered contributions
- Single centered (non-polar) degeneracies are positive, consistently with $\mathrm{Tr}(-1)^F = \mathrm{Tr}\,1 \geq 0$ for spherically symmetric BH.

Dijkgraaf Verlinde^2; de Wit et al; David Jatkar Sen; Dabholkar et al;...

Introduction II

In vacua with $\mathcal{N} = 2$ SUSY, such as type IIA string theory on a CY threefold, the situation is far less understood and much richer:

- Microscopic description for vanishing D6 and primitive D3-brane charges only, at large volume only. In general, $\Omega(\gamma, z^a)$ is a generalized DT invariant, hard to (define and) compute.
- Subleading quantum corrections are more involved, and include e.g. all topological couplings $R^2 F^{2h-2}$.
- Bound states of $n \geq 3$ centers can and do contribute. Some of them (scaling solutions) seem to be stable everywhere.
- Modular properties are unclear, although invariance under monodromies and type IIB $SL(2, \mathbb{Z})$ should be there somehow.

Maldacena Strominger Witten; Ooguri Strominger Vafa; Denef Moore; ...

Introduction III

- In this talk I will describe techniques to compute contributions of multi-centered black holes to the BPS index in generic $\mathcal{N} = 2$ string vacua.
- One of the goals is to extract single-centered black hole contributions from the total index. This can in principle be compared with the path integral in $AdS_2 \times S^2$ (or quantum entropy function), which can be computed by localization.
- On the microscopic side, BPS states can sometimes be described by quivers. The analogue of single-centered black holes are then middle cohomology states. The macroscopic description suggests that these states are robust under wall-crossing, and that they determine the complete cohomology of the quiver moduli space.

Outline

Outline

Preliminaries I

- Consider $\mathcal{N} = 2$ supergravity in 4 dimensions. Let $\Gamma = \Gamma_e \oplus \Gamma_m$ be the lattice of electric and magnetic charges, with antisymmetric (Dirac-Schwinger- Zwanziger) integer pairing

$$\langle \gamma, \gamma' \rangle = q_\Lambda p'^\Lambda - q'_\Lambda p^\Lambda \in \mathbb{Z}$$

- BPS states preserve 4 out of 8 supercharges, and saturate the bound $M \geq |Z(\gamma, t^a)|$ where $Z(\gamma, t^a) = \langle Y(t^a), \gamma \rangle$ is the central charge.
- The index $\Omega(\gamma; t^a) = \mathrm{Tr}_{\mathcal{H}'_\gamma(t^a)}(-1)^{2J_3}$ (where $\mathcal{H}'_\gamma(t^a)$ is the Hilbert space of one-particle states with charge $\gamma \in \Gamma$ in the vacuum with vector moduli t^a) receives contributions from short multiplets only.

Preliminaries II

- In $N = 2$ gauge theories (but not SUGRA/string vacua), there is an additional $SU(2)_R$ symmetry, and the spin character

$$\Omega(\gamma; t, y) = \mathrm{Tr}(-1)^{2J_3}\, y^{2(I_3+J_3)}$$

is protected. The 'no exotics' conjecture asserts that all states have $I_3 = 0$, and the PSC coincides with the refined index

$$\Omega(\gamma; t, y) = \mathrm{Tr}(-1)^{2J_3}\, y^{2J_3}$$

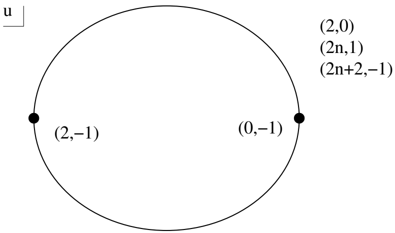
Gaiotto Neitzke Moore

- In $N = 2$ SUGRA/string vacua, one can still define $\Omega(\gamma; t, y)$ but it is no longer protected, hence could get contributions from non-BPS states and depend on HM moduli.

- $\Omega(\gamma; t)$ is locally constant, but may jump on codimension 1 loci in VM moduli space called 'walls of marginal stability', where the bound state spectrum mixes with the continuum:

$$W(\gamma_1, \gamma_2) = \{t \in \mathcal{M} \ : \ Z(\gamma_1, t)/Z(\gamma_2, t) \in \mathbb{R}^+\}$$

- For e.g. in pure $N = 2$, $D = 4$ SYM with $G = SU(2)$,



Seiberg Witten; Bilal Ferrari

Wall-crossing II

- Basic mechanism: Some of the BPS states with charge $\gamma = M\gamma_1 + N\gamma_2$ are bound states of more elementary BPS states with charge $\alpha_i = M_i\gamma_1 + N_i\gamma_2$, and these bound states exist only on one side of the wall.
- Rk1: for a given wall, we can always choose the basis γ_1, γ_2 such that $\Omega(M\gamma_1 + N\gamma_2)$ has support on the cone $MN \geq 0$. The constituents have $M_i \geq 0, N_i \geq 0$, so only a finite number of bound states can occur.
- Rk2: The index of states with $\gamma \notin \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$ is constant across the wall. So are $\Omega(\gamma_1)$ and $\Omega(\gamma_2)$.
- Rk3: The index $\Omega(M\gamma_1 + N\gamma_2)$ may contain contributions from bound states of constituents with charges lying outside the lattice $\mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$. But those are insensitive to the wall.

Primitive wall-crossing I

- For $\langle \gamma_1, \gamma_2 \rangle \neq 0$, there exists a two-centered BPS solution of charge $\gamma = \gamma_1 + \gamma_2$, angular momentum $\vec{J} = \frac{1}{2}\langle \gamma_1, \gamma_2 \rangle \, \vec{u}$:

$$|x_1 - x_2| = \sqrt{G_4}\frac{\langle \Gamma_1, \Gamma_2 \rangle}{2} \frac{|Z(\Gamma_1 + \Gamma_2, t)|}{\mathrm{Im}(Z(\Gamma_1, t)\bar{Z}(\Gamma_2, t))}$$

Denef 2002

- The solution exists only on one side of the wall. As t approaches the wall, the distance r_{12} diverges and the bound state decays into its constituents γ_1 and γ_2.

Primitive wall-crossing II

- Near the wall, the two centers can be treated as pointlike particles with $\Omega(\gamma_i)$ internal degrees of freedom, interacting via Newton, Coulomb, Lorentz, scalar exchange forces.
- The classical BPS phase space $\mathcal{M}_2$ for the two-particle system is the two-sphere, with symplectic form

$$\omega = \frac{1}{2}\gamma_{12} \sin\theta \, d\theta d\phi \ , \quad \gamma_{12} \equiv \langle \gamma_1, \gamma_2 \rangle$$

  such that rotations ∂_ϕ are generated by $J_3 = \frac{1}{2}\gamma_{12}\cos\theta$.
- Quantum mechanically, one obtains $|\gamma_{12}|$ states transforming as a spin $J = \frac{1}{2}(|\gamma_{12}| - 1)$ multiplet under rotations.

Primitive wall-crossing III

- Near the wall, the internal degrees of freedom decouple from the configurational degrees of freedom. The index of the two-particle bound state is then

$$\Omega_{\rm bound} = \underbrace{(-1)^{\gamma_{12}+1}\gamma_{12}}_{\substack{\text{angular}\\ \text{momentum}}} \times \underbrace{\Omega(\gamma_1)}_{\substack{\text{internal}\\ \text{states of 1}}} \times \underbrace{\Omega(\gamma_2)}_{\substack{\text{internal}\\ \text{states of 2}}}$$

- These are the only bound states of charge $\gamma = \gamma_1 + \gamma_2$ which (dis)appear across the wall. Thus

$$\Delta\Omega(\gamma_1+\gamma_2) = (-1)^{\gamma_{12}+1}\,\gamma_{12}\,\Omega(\gamma_1)\,\Omega(\gamma_2)\,,$$

Denef Moore

Primitive wall-crossing IV

- Similarly, the variation of the refined index is

$$\Delta\Omega(\gamma_1+\gamma_2;y) = \frac{(-y)^{\gamma_{12}} - (-y)^{-\gamma_{12}}}{y-1/y}\,\Omega(\gamma_1;y)\,\Omega(\gamma_2;y)$$

Diaconescu Moore; Dimofte Gukov

- Let us try to extend this reasoning to compute $\Delta\Omega(M\gamma_1+N\gamma_2)$ for general (M,N).

Multi-centered solutions I

- The most general stationary, BPS solution of $\mathcal{N}=2$ SUGRA is

$$\mathrm{d}s^2 = -e^{2U}\left(\mathrm{d}t+\mathcal{A}\right)^2 + e^{-2U}\mathrm{d}\vec{r}^2$$

$$2\,e^{-U(\vec{r})}\mathrm{Im}\left[e^{-\mathrm{i}\phi}Y\left(t^a(\vec{r})\right)\right] = \beta + \sum_{i=1}^{n}\frac{\alpha_i}{|\vec{r}-\vec{r}_i|},$$

$$\phi = \arg Z_\gamma\ , \quad \gamma = \alpha_1+\cdots+\alpha_n\ , \quad \beta = 2\,\mathrm{Im}\left[e^{-\mathrm{i}\phi}\,Y(t_\infty)\right]$$

- The integrability condition for $\mathcal{A}$ requires

$$[*] \qquad \sum_{\substack{j=1\\ j\neq i}}^{n}\frac{\langle\alpha_i,\alpha_j\rangle}{|\vec{r}_i-\vec{r}_j|} = c_i\ , \qquad c_i \equiv 2\,\mathrm{Im}\left(e^{-\mathrm{i}\phi}Z_{\alpha_i}\right)$$

Denef; Bates Denef

Multi-centered solutions II

- This provides $n-1$ conditions on $3n$ locations $\vec{r}_i$. Modding out by translations in $\mathbb{R}^3$, we define $\mathcal{M}_n(\{\alpha_i\},\{c_i\})$ to be the $2n-2$ dimensional space of solutions to $[*]$.
- For the solution to be regular, one must also ensure

$$e^{-2U(\vec{r})} = \frac{1}{\pi} S\left(\beta + \sum_{i=1}^{n} \frac{\alpha_i}{|\vec{r}-\vec{r}_i|}\right) > 0\,, \qquad \forall\, \vec{r} \in \mathbb{R}^3\ .$$

  This may remove some connected components in $\mathcal{M}_n$.
- When all charges α_i lie in a two-dimensional lattice $\mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$ and satisfy the cone condition $MN \geq 0$, this condition appears to be automatically satisfied. Moreover $c_i = \Lambda \sum_{j\neq i} \alpha_{ij}$ with $\Lambda \to 0$ at the wall.

Q. mech. of multi-centered black holes I

- At least when the centers are well-separated, the dynamics of the bound state is described by $\mathcal{N} = 4$ quantum mechanics with $3n$ bosonic coordinates, $4n$ fermionic coordinates with Lagrangian

$$\mathcal{L} = \sum_i [W_i(\vec{r}_i)]^2 + \sum_i \vec{A}_i \dot{\vec{r}}_i + \sum_{i,j} \gamma_{ij} \dot{\vec{r}}_i \dot{\vec{r}}_j + \dots$$

$$W_i = \sum_{j \neq i} \frac{\langle \alpha_i, \alpha_j \rangle}{|\vec{r}_i - \vec{r}_j|} - c_i \ , \qquad \vec{A}_i = \sum_{j \neq i} \langle \alpha_i, \alpha_j \rangle \, \vec{A}_{\mathrm{Dirac}}(\vec{r}_i - \vec{r}_j)$$

Denef;Kim Park Wang Yi

Q. mech. of multi-centered black holes II

- The classical ground state dynamics is then first order quantum mechanics on the BPS phase space $\mathcal{M}_n = \{W_i = 0\}$ equipped with the symplectic form

$$\omega = \frac{1}{2} \sum_{i<j} \alpha_{ij} \sin\theta_{ij}\, \mathrm{d}\theta_{ij} \wedge \mathrm{d}\phi_{ij} \;, \quad \vec{J} = \frac{1}{2} \sum_{i<j} \alpha_{ij} \frac{\vec{r}_{ij}}{r_{ij}}$$

de Boer El Showk Messamah Van den Bleeken

- Semiclassically, the number of states is equal to the (equivariant) symplectic volume

$$g_{\mathrm{class}}(\{\alpha_i\}, \{c_i\}; y) = \frac{(-1)^{\sum_{i<j} \alpha_{ij} - n + 1}}{(2\pi)^{n-1} (n-1)!} \int_{\mathcal{M}_n} \omega^{n-1}\, y^{2J_3}$$

Q. mech. of multi-centered black holes III

- Quantum mechanically, the refined index is equal to the equivariant Dirac index of the symplectic manifold $(\mathcal{M}_n, \omega)$,

$$g(\{\alpha_i\}, \{c_i\}; y) = \mathrm{Tr}_{\mathrm{Ker}D_+}(-y)^{2J_3} - \mathrm{Tr}_{\mathrm{Ker}D_-}(-y)^{2J_3} \ .$$

  This reduces to g_{class} in the limit $\alpha_{ij} \to \infty$.

- When $\mathcal{M}_n$ is compact and J_3 has only isolated fixed points, it can be evaluated by Atiyah-Bott Lefschetz fixed point formula:

$$g(\{\alpha_i\}, \{c_i\}; y) = \sum_{\mathrm{fixed\ pts}} \frac{y^{2J_3}}{\det((-y)^L - (-y)^{-L})}$$

  where L is the matrix of the action of J_3 on the holomorphic tangent space around the fixed point.

- E.g. for $n = 2$, $\mathcal{M}_2 = S^2$, $J_3 = \alpha_{12} \cos\theta$:

$$g = \frac{(-1)^{\alpha_{12}+1}}{(y - 1/y)} \left(\underbrace{y^{+\alpha_{12}}}_{\text{North pole}} - \underbrace{y^{-\alpha_{12}}}_{\text{South pole}} \right)$$
$$= \mathrm{Tr}_{j=\frac{1}{2}(\alpha_{12}-1)}\, y^{2J_3}$$

if $c_1 \alpha_{12} > 0$, zero otherwise.

Refined index from localization I

- The fixed points of the action of J_3 are collinear multi-centered configurations along the z-axis, such that

$$\sum_{j\neq i}^{n} \frac{\alpha_{ij}}{|z_i - z_j|} = c_i \,, \quad J_3 = \frac{1}{2} \sum_{i<j} \alpha_{ij} \, \mathrm{sign}(z_j - z_i) \,.$$

- Equivalently, fixed points are critical points of the ‘superpotential’

$$W(\{z_i\}) = -\sum_{i<j} \mathrm{sign}[z_j - z_i] \, \alpha_{ij} \, \ln |z_j - z_i| - \sum_i c_i z_i$$

- The determinant turns out to be $(y - 1/y)^{n-1}$ times a sign $s(p) = -\mathrm{sign}(\det W'')$ where W'' is the Hessian of W

Refined index from localization II

- After the dust settles, one finds the Coulomb branch formula

$$g(\{\alpha_i\},\{c_i\};y)=\frac{(-1)^{\sum_{i<j}\alpha_{ij}+n-1}}{(y-y^{-1})^{n-1}}\sum_p s(p)\,y^{\sum_{i<j}\alpha_{ij}\,\mathrm{sign}(z_j-z_i)}$$

  where the sum runs over all collinear solutions of Denef's equations.
- The contribution of each fixed point is singular at $y=1$, but the sum over fixed points is guaranteed to produce a symmetric polynomial in y and $1/y$, as long as $\mathcal{M}_n$ is compact.
- Fortunately, this is always the case for charge configurations $\alpha_i = M_i\gamma_1 + N_i\gamma_2$ involved in wall-crossing, away from the walls.

Coulomg branch wall-crossing formula I

- Having computed the index of the quantum mechanics of n centers, we can apply the same logic as before and write (naively),

$$\Delta\Omega(\gamma) \overset{??}{=} \sum_{n\geq 2} \sum_{\gamma=\alpha_1+\dots+\alpha_n} g(\{\alpha_i\}) \prod_{i=1}^{n} \Omega^+(\alpha_i)$$

where $\gamma = M\gamma_1 + N\gamma_2, \alpha_i = M_i\gamma_1 + N_i\gamma_2$, and $\Omega^+(\alpha_i)$ is the index on the side where the bound state does not exist.

- This is almost right, but it overlooks the issue of statistics.

Statistics I

- If the centers were classical, indistinguishable objects, Maxwell-Boltzmann statistics would require a symmetry factor

$$\Delta\Omega(\gamma) \overset{!?}{=} \sum_{n\geq 2} \sum_{\gamma=\alpha_1+\dots+\alpha_n} \frac{g(\{\alpha_i\})}{|\mathrm{Aut}(\{\alpha_i\})|} \prod_{i=1}^{n} \Omega^+(\alpha_i)$$

where $\mathrm{Aut}(\{\alpha_i\})$ is the subgroup of the permutation group leaving $\{\alpha_i\}$ invariant.

- Instead, the centers are quantum objects, with Bose statistics if $\Omega(\alpha_i) > 0$, or Fermi statistics if $\Omega(\alpha_i) < 0$. One can show that the Maxwell-Boltzmann prescription nevertheless works, provided one replaces everywhere the index $\Omega(\gamma)$ with the rational index

$$\bar{\Omega}(\gamma) \equiv \sum_{d|\gamma} \frac{1}{d^2} \Omega(\gamma/d) , \quad \bar{\Omega}(\gamma, y) \equiv \sum_{d|\gamma} \frac{(y-y^{-1})}{d(y^d-y^{-d})} \Omega(\gamma/d, y^d)$$

Statistics II

- Consider for example $\Delta(\gamma_1 + 2\gamma_2)$: it receives contributions from bound states $\{\gamma_1 + \gamma_2, \gamma_2\}$, $\{\gamma_1, 2\gamma_2\}$, $\{\gamma_1, \gamma_2, \gamma_2\}$.
- Taking into account Bose-Fermi statistics for $\{\gamma_1, \gamma_2, \gamma_2\}$,

$$\begin{aligned}\Delta\Omega(\gamma_1 + 2\gamma_2) =& (-1)^{\gamma_{12}}\gamma_{12}\Omega^+(\gamma_2)\,\Omega^+(\gamma_1+\gamma_2)\\ &+ 2\gamma_{12}\,\Omega^+(2\gamma_2)\,\Omega^+(\gamma_1)\\ &+ \tfrac{1}{2}\gamma_{12}\,\Omega^+(\gamma_2)\,\big(\gamma_{12}\Omega^+(\gamma_2)+1\big)\Omega^+(\gamma_1)\ .\end{aligned}$$

In terms of the rational invariant $\bar\Omega(2\gamma_2) = \Omega(2\gamma_2) + \frac{1}{4}\Omega(\gamma_2)$, charge conservation is manifest:

$$\begin{aligned}\Delta\bar\Omega(\gamma_1 + 2\gamma_2) =& (-1)^{\gamma_{12}}\gamma_{12}\bar\Omega^+(\gamma_2)\,\bar\Omega^+(\gamma_1+\gamma_2)\\ &+ 2\gamma_{12}\,\bar\Omega^+(2\gamma_2)\,\bar\Omega^+(\gamma_1) + \tfrac{1}{2}[\gamma_{12}\,\bar\Omega^+(\gamma_2)]^2\bar\Omega^+(\gamma_1)\ .\end{aligned}$$

Coulomb branch wall crossing formula I

- We have finally arrived at the Coulomb branch wall-crossing formula:

$$\Delta\bar{\Omega}(\gamma, y) = \sum_{n\geq 2} \sum_{\gamma=\alpha_1+\cdots+\alpha_n} \frac{g(\{\alpha_i\}, y)}{|\mathrm{Aut}(\{\alpha_i\})|} \prod_{i=1}^{n} \bar{\Omega}^{+}(\alpha_i, y)$$

- Remarkably, the formula agrees with the mathematical formulae established in the context of Donaldson-Thomas invariants for the derived category of Abelian sheaves.

Kontsevich-Soibelman;Joyce-Song

- The formula reproduces for example the weak coupling spectrum of pure $SU(2)$ gauge theory found by Seiberg and Witten.

Outline

Towards elementary constituents I

- Wall-crossing shows that some of the BPS states contributing to $\Omega(\gamma)$ are not elementary, but can be decomposed into more elementary constituents. However these constituents may still decay elsewhere.
- This suggests that the existence of a set of truly elementary, absolutely stable BPS states such that any other BPS state would arise as a bound state of those.
- This is realized in pure $SU(2)$ gauge theorie, where all states arise as bound states of the monopole and dyon. In SUGRA, single centered black holes should play the role of elementary constituents.

- In the remainder we shall propose a master formula which expresses the total index $\Omega(\gamma; t, y)$ in terms of the indices $\Omega_S(\gamma; y)$ associated to the elementary/single centered constituents.
- We shall test the validity of the formula in the context of quiver quantum mechanics, which describe certain D-brane bound states (as well as the BPS spectrum of certain gauge theories).

The Master Formula (first pass) I

- Let $\Omega_S(\alpha_i, y)$ be the index of elementary/single centered states with charge α_i. The total index $\Omega(\gamma; t, y)$, or rather its rational counterpart, should be a sum over all possible bound states

$$\bar{\Omega}(\gamma; t, y) = \sum_{n\geq 1} \sum_{\sum \alpha_i = \gamma} \frac{g(\{\alpha_i\}, \{c_i\}, y)}{|\mathrm{Aut}(\{\alpha_i\})|} \bar{\Omega}_S(\alpha_i, y)$$

- Unlike in wall-crossing case, there are potentially infinitely many possible sets $\{\alpha_i\}$ such that $\gamma = \sum \alpha_i$ and $\bar{\Omega}_S(\alpha_i, y) \neq 0$. Hopefully the regularity condition $e^{2U} > 0$ rules out all but a finite number of them...

Scaling solutions I

- Ignoring this issue, another serious concern is that, unlike in wall-crossing case, the phase space $\mathcal{M}_n$ may be non-compact. As a result, $g(\{\alpha_i\}, \{c_i\}, y)$ and hence $\Omega(\gamma; t, y)$ may not be symmetric Laurent polynomials.
- To illustrate this, take $n = 3$,$\alpha_{12} = a$, $\alpha_{23} = b$, $\alpha_{31} = c$ satisfying triangular inequalities $0 < a < b + c$, etc, there exist scaling solutions of Denef's equations

$$\frac{a}{r_{12}} - \frac{c}{r_{13}} = c_1 \,, \quad \frac{b}{r_{23}} - \frac{a}{r_{12}} = c_2$$

with $r_{12} \sim a\epsilon, r_{23} \sim b\epsilon, r_{13} \sim c\epsilon, \vec{J}^2 \sim \epsilon^2$ as $\epsilon \to 0$.

Scaling solutions II

- For $c_1, c_2 > 0$, the only collinear configurations are (123) and (321), leading to

$$g = \frac{(-1)^{a+b+c}(y^{a+b-c} + y^{-a-b+c})}{(y - 1/y)^2}$$

- This is not a polynomial in y, in particular it is singular as $y \to 1$. Still the (equivariant) volume of $\mathcal{M}_n$ is finite.
- This could be repaired by adding by hand a term with $J_3 \simeq 0$, attributed to scaling solutions:

$$\tilde{g} = \frac{(-1)^{a+b+c}\left(y^{a+b-c} + y^{-a-b+c} - \begin{bmatrix} 2 & a+b+c \text{ even} \\ y+1/y & a+b+c \text{ odd} \end{bmatrix}\right)}{(y - 1/y)^2}$$

Scaling solutions III

- More generally, scaling regions in $\mathcal{M}_n(\{\alpha_i\})$ arise whenever there exist a subset A and vectors $\vec{r}_i \in \mathbb{R}^3$, $i \in A$ such that

$$\forall i \in A\,, \quad \sum_{j\in A} \frac{\alpha_{ij}}{|\vec{r}_{ij}|} = 0\,.$$

  This is independent of the c_i's, so scaling solutions cannot be removed by changing the moduli.

- One could give a general prescription to compactify the BPS phase space $\mathcal{M}_n$, and complete the sum over collinear fixed points g into a symmetric Laurent polynomial $\tilde{g}$, however this is not quite sufficient to ensure that the $\Omega(\gamma, t)$ resulting from the Master formula is sensible.

Master formula (second pass) I

- Instead, we postulate

$$\bar{\Omega}(\gamma, y) = \sum_{n\geq 1} \sum_{\{\alpha_i\}, \sum \alpha_i = \gamma} \frac{g(\{\alpha_i\},\{c_i\},y)}{|\mathrm{Aut}(\{\alpha_i\})|} \times \prod_{i=1}^{n} \left\{ \sum_{m_i|\alpha_i} \frac{y-1/y}{m_i(y^{m_i}-y^{-m_i})} \left[\Omega_S(\alpha_i/m_i, y^{m_i}) + \Omega_{\rm scaling}(\alpha_i/m_i, y^{m_i}) \right] \right\}$$

- $\Omega_{\rm scaling}$ is determined recursively in terms of Ω_S by the minimal modification hypothesis:

$$\Omega_{\rm scaling}(\alpha; y) = \sum_{\substack{\{\beta_i \in \Gamma\}, \{m_i \in \mathbb{Z}\} \\ m_i \geq 1, \sum_i m_i \beta_i = \alpha}} H(\{\beta_i\}; \{m_i\}; y) \prod_i \Omega_S(\beta_i; y^{m_i})$$

Master formula (second pass) II

- $H(\{\beta_i\};\{m_i\};y)$ is uniquely determined by the conditions
  - H is symmetric under $y \to 1/y$,
  - H vanishes at $y \to 0$,
  - the coefficient of $\prod_i \Omega_S(\beta_i; y^{m_i})$ in the expression for $\Omega(\sum_i m_i\beta_i; y)$ is a Laurent polynomial in y.
- E.g, for the 3-center configuration discussed above,

$$H(\{\gamma_1,\gamma_2,\gamma_3\};\{1,1,1\};y) = \begin{cases} -2\,(y-y^{-1})^{-2} & , a+b+c \text{ even} \\ (y+y^{-1})\,(y-y^{-1})^{-2} & , a+b+c \text{ odd} \end{cases}$$

so the prescription reduces to $g \to \tilde{g}$, but not in general...

Master formula (second pass) III

- The Master Formula expresses the set of indices $\Omega(\gamma; t, y)$ in terms of a new set of indices $\Omega^S(\gamma, y)$. What have we gained ?
- First, $\Omega^S(\gamma)$ no longer depend on the moduli. The formula is by construction consistent with wall-crossing.
- Second, in SUGRA we expect $\Omega^S(\gamma, y)$ to count micro-states of single-centered BPS black holes, and therefore to be y-independent. This gives a non-trivial constraint on the total refined index $\Omega(\gamma; t, y)$.

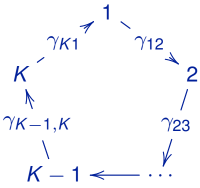
Outline

Quiver quantum mechanics I

- The $\mathcal{N} = 4$ quantum mechanics of multi-centered solitons/black holes arises as the Coulomb branch of a more complicated $\mathcal{N} = 4$ matrix quantum mechanics, whose matter content is captured by a quiver.
- Each node $\ell = 1...K$ represents a $U(N_\ell)$ vector multiplet $(\vec{r}_\ell, D_\ell)$, each arrow represents a chiral multiplet $\phi_{k,\ell}$ in $(N_\ell, \bar{N}_k)$ representation of $U(N_\ell) \times U(N_k)$. The set $\{N_\ell\}$ is called the dimension vector.
- In addition, one must specify Fayet-Iliopoulos terms c_ℓ such that $\sum_\ell N_\ell c_\ell = 0$, and (in presence of closed oriented loops) a gauge invariant superpotential W.
- Quivers also describe SUSY gauge theories in higher dimension, but here we focus on $D = 1$.

Quiver quantum mechanics II

- In the case of $N = \sum_{\ell=1\dots K} N_\ell$ centers, N_ℓ of which carrying charge γ_ℓ, the quiver has K nodes and $\gamma_{\ell k}$ arrows from node ℓ to k.



- The FI terms c_ℓ depend on the VM moduli, while the coefficients of W in general depend on HM moduli.

Quiver quantum mechanics III

- The Coulomb branch description arises by integrating out the chiral multiplets, and reproduces Denef's equations

$$\forall \ell \ , \quad \sum_{k\neq\ell} \frac{\gamma_{\ell k}}{|\vec{r}_\ell - \vec{r}_k|} = c_\ell$$

  It is valid when the centers are far apart.
- The Higgs branch description arises by integrating out the vector multiplets. It is valid for large values of the chiral multiplet scalars $\phi_{k\ell}$.
- If both branches are regular, one expects the two descriptions to be dual and have the same BPS states. This can break down if the Coulomb and Higgs branches mix.

Quantum mechanics on the Higgs branch I

- The moduli space $\mathcal{M}_H$ of SUSY vacua on the Higgs branch is the set of solutions of the F-term $\partial_\phi W = 0$ and D-term equations

$$\forall \ell : \sum_{\gamma_{\ell k}>0} \phi^*_{\ell k}\, T^a\, \phi_{\ell k} - \sum_{\gamma_{k\ell}>0} \phi^*_{k\ell}\, T^a\, \phi_{k\ell,\alpha,s't} = c_\ell\, \mathrm{Tr}(T^a)$$

  modulo the action of $\prod_\ell U(N_\ell)$.
- Equivalently, $\mathcal{M}_H$ is the space of semi-stable solutions of $\partial_\phi W = 0$ modulo $\prod_\ell GL(N_\ell, \mathbb{C})$.

Quantum mechanics on the Higgs branch II

- BPS states on the Higgs branch correspond to cohomology classes in $H^*(\mathcal{M}_H, \mathbb{Z})$. They transform under $SU(2)$ according to the Lefschetz action

$$J_+ \cdot h = \omega \wedge h\,, \quad J_- = \omega \llcorner h\,, \quad J_3 \cdot h = \tfrac{1}{2}(n-d)h\,.$$

where d is the complex dimension of $\mathcal{M}_H$, n the degree of h.

- The refined index on the Higgs branch is given by the Poincaré polynomial

$$Q(\mathcal{M}_H; y) = \mathrm{Tr}'(-y)^{2J_3} = \sum_{p=1}^{2d} b_p(\mathcal{M}_H)\,(-y)^{p-d}$$

Q is a polynomial in $y, 1/y$, symmetric under $y \to 1/y$.

Quantum mechanics on the Higgs branch III

- For example, for the 2-node (Kronecker) quiver with k arrows, dimension vector $(1,1)$, $c_1 > 0$,

$$1 \xrightarrow{\quad k \quad} 2$$

$$\mathcal{M}_H = \mathbb{C}^k/\mathbb{C}^\times = \mathbb{P}^{k-1} \;\Rightarrow\; Q(\mathcal{M}_H; y) = \frac{(-y)^k - (-y)^{-k}}{y - 1/y}$$

  Remarkably, this agrees with the index of the Coulomb index of 2-centered solutions with $\langle \gamma_1, \gamma_2 \rangle = k$!

- For $k = 2$ and dimension vector (M, N), one can show that $Q(\mathcal{M}_H; y)$ is the number of BPS states of charge $M\gamma_1 + N\gamma_2$ in pure $SU(2)$ Seiberg-Witten theory, where γ_1, γ_2 are the monopole and dyon charges !

Quantum mechanics on the Higgs branch IV

- More generally, the BPS spectrum of many (if not all) $N = 2$, $D = 4$ gauge theories is governed by a quiver. Any BPS state arises as a bound state of the BPS states which occur in the strong coupling chamber.

Alim Cecotti Cordova Espahbodi Rastogi Vafa; Cecotti del Zotto; ...

- In general, the computation of $Q(\mathcal{M}_H; y)$ is a hard problem. For quivers without oriented closed loop, there is a general formula based on Harder-Narasimhan filtrations. This formula reproduces our Coulomb branch formula, albeit in very non-trivial way !

Reineke; Joyce; MPS; Sen

Quantum mechanics on the Higgs branch V

- For non-Abelian quivers with loops and generic superpotential, there is no general way to compute $Q(\mathcal{M}_H; y)$ at present.
- For Abelian quivers with loops, $\mathcal{M}_H$ is a complete intersection in a product of projective spaces $\mathcal{M}_{\rm amb}$, whose cohomology can be computed using the Lefschetz hyperplane theorem

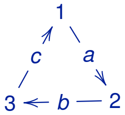
$$b_p(\mathcal{M}_H \subset \mathcal{M}_{\rm amb}) = \begin{cases} b_p(\mathcal{M}_{\rm amb}) & p < d \\ b_{2d-p}(\mathcal{M}_{\rm amb}) & p > d \end{cases},$$

and the Riemann-Roch theorem to compute $\chi(\mathcal{M}_H)$ and hence the middle cohomology $b_d(\mathcal{M}_H)$.

Denef Moore; Bena Berkooz El Showk de Boer van den Bleeken

Higgs branch vs Coulomb branch I

- In the presence of scaling configurations, the Coulomb branch describes only a subset of the states on the Higgs branch. For example, consider the three-node quiver



$$a, b, c \geq 0$$
$$N_1 = N_2 = N_3 = 1$$
$$c_1, c_2 > 0,\ c_3 < 0$$

- The generating function of the Higgs branch indices

$$Q(x_1, x_2, x_3; y) = \sum_{a \geq 0, b \geq 0, c \geq 0} (x_1)^a (x_2)^b (x_3)^c \, Q(\mathcal{M}_{a,b,c}, y)$$

can be computed using Lefschetz and Riemann-Roch.

Higgs branch vs Coulomb branch II

- it decomposes into $Q = Q_{\rm C} + Q_{\rm S}$, where

$$Q_{\rm C} = \frac{x_1 x_2 \left\{1 - x_1 x_2 + x_1 x_2 x_3 \left(x_1 + x_2 + y + y^{-1}\right)\right\}}{(1-x_1 x_2)(1-x_1 x_3)(1-x_2 x_3)(1+x_1/y)(1+x_1 y)(1+x_2/y)(1+x_2 y)}$$

$$Q_{\rm S} = \frac{x_1^2 x_2^2 x_3^2}{(1-x_1 x_2)(1-x_2 x_3)(1-x_1 x_3)[1-x_1 x_2 - x_2 x_3 - x_1 x_3 - 2 x_1 x_2 x_3]}$$

- $Q_{\rm C}$ corresponds to contributions from 3-centered black holes, each carrying unit degeneracy. It is y dependent, moduli dependent, and grows polynomially with a, b, c.
- Instead $Q_{\rm S}$ is y-independent, moduli-independent, has support on $\{a < b + c, b < a + c, c < a + b\}$, and grows exponentially with a, b, c. It counts spin zero Lefschetz multiplets, or pure Higgs states for short.

Master formula for quivers I

- We conjecture that all states on the Higgs branch have a generalized Coulomb branch description, as bound states of elementary constituents whose refined index $\Omega_S(\alpha_i)$ is both y-independent and invariant under wall-crossing.
- In particular, the master formula should hold for all quivers,

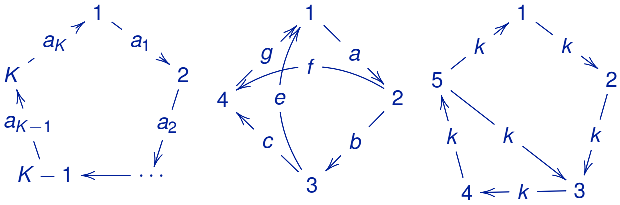
$$\bar{\Omega}(\gamma,y) = \sum_{n\geq 1} \sum_{\{\alpha_i\},\sum\alpha_i=\gamma} \frac{g(\{\alpha_i\},\{c_i\},y)}{|\mathrm{Aut}(\{\alpha_i\})|}$$

$$\times \prod_{i=1}^{n} \left\{ \sum_{m_i|\alpha_i} \frac{1}{m_i} \frac{y-1/y}{y^{m_i}-y^{-m_i}} \left[\Omega_S(\alpha_i/m_i,y^{m_i}) + \Omega_{\rm scaling}(\alpha_i/m_i,y^{m_i})\right] \right\}$$

where $\Omega_{\rm scaling}$ is determined as before.

Master formula for quivers II

- We have tested the formula on a variety of Abelian quivers:



and on some non-Abelian 3-node quivers: the master formula seems to hold up...

Summary and open problems I

- The total index in $\mathcal{N} = 2$ supersymmetric theories can be written as a sum of contributions from bound states of n elementary/single centered spinless constituents.
- The index Ω_S associated to elementary constituents is both moduli- and y-independent. The existence of bound states does depend on the moduli z^a. Angular momentum is carried by configurational degrees of freedom.
- *The enumeration of all possible decompositions $\gamma = \sum_i \alpha_i$ is currently untractable. Can one find a simple criterium (beyond the attractor flow conjecture) that determines whether a given choice of charges will lead to a regular multi-center solution ?*
- *The computation of the Coulomb branch index $g(\{\alpha_i\}, y)$ for general $\{\alpha_i\}$ is currently limited by computer power. Is there a better way to compute the sum over collinear fixed points ?*

Summary and open problems II

- Due to scaling solutions, the contribution of each elementary constituent is corrected by a term $\Omega_{\rm scaling}(y)$ which depends on the index of 'smaller' charges, and ensures that the total index is a Laurent polynomial. *Can one derive this from a detailed study of $N = 4$ quiver quantum mechanics ?*
- In the context of quivers, Ω_S counts the number of Lefschetz singlets in the cohomology of the quiver moduli space. *Why these states are robust under changing the FI parameters ? Is there a general formula for the index for non-Abelian quivers with generic superpotential ?*
- In supergravity, Ω_S corresponds to the degeneracy in the attractor chamber, and should count the degrees of freedom in a single $AdS_2 \times S_2$ throat. *Can one compute it microscopically and match it to the quantum entropy function ?*