Quivers, black holes and attractor indices

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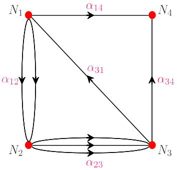
based on 1804.06928 with Sergei Alexandrov (prerequisite for 1808.08479)

and on earlier work 2011-15 with Jan Manschot and Ashoke Sen

Introduction I

- Moduli spaces of quiver representations play a prominent role in representation theory and algebraic geometry.



$$\alpha_{ab} = \begin{pmatrix} 0 & 2 & -1 & 1 \\ -2 & 0 & 3 & 0 \\ 1 & -3 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}$$

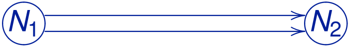
Given a quiver Q with K vertices, adjacency matrix $\alpha_{ij} = -\alpha_{ji}$, dimension vector $\gamma = (N_1, \dots N_K)$ and stability parameters $\zeta = (\zeta_1, \dots \zeta_K)$ such that $\sum_{i=1}^{K} N_i \zeta_i = 0$, the quiver moduli space $\mathcal{M}_Q(\gamma, \zeta)$ is the set of equivalence classes of stable linear maps $\Phi_{ij,k} : \mathbb{C}^{N_i} \to \mathbb{C}^{N_j}$, for each (i, j) such that $\alpha_{ij} > 0$, $k = 1, \dots \alpha_{ij}$, modulo conjugation by $\prod GL(N_i)$ (and subject to algebraic relations $\partial_\Phi \mathcal{W} = 0$ when the quiver has oriented loops)

Introduction II

- In physics, they control the vacuum structure of certain supersymmetric gauge theories with product gauge groups in various dimensions.
- More surprisingly, they also govern the spectrum of BPS dyons in a large class of 4D, $\mathcal{N}=2$ field theories, and the spectrum of BPS black holes in $\mathcal{N}=2$ string vacua, at least in certain sectors.

Douglas Moore '96, Fiol '00, Alim Cecotti Cordova Espahbodi Rastogi Vafa '11

E.g. for $SU(2)$ SYM, BPS states of charge $(2N_1, N_2 - N_1)$ are in 1-1 correspondence with harmonic forms on the moduli space of the Kronecker quiver with $m = 2$ arrows:



For $N_1 = N_2 = 1, m = 2$, the moduli space is $\mathbb{P}^1$ supports two harmonic forms, corresponding to the massive W-bosons.

Introduction III

- This can be traced to the fact that the quantum mechanics of BPS charged particles in $D = 3+1, \mathcal{N} = 2$ field/string theories is described by a $0+1$-dimensional supersymmetric gauge theory with product gauge group, whose Higgs branch coincides with quiver moduli space $\mathcal{M}_Q(\zeta)$ of stable representations.
- The same $D = 0+1$ gauge theory also has a Coulomb branch, which can be interpreted as the phase space $\mathcal{M}_n$ of a system of n BPS particles in $\mathbb{R}^3$, with Coulomb and Lorentz interactions.

  E.g. for the Kronecker quiver with m arrows, the Coulomb branch is $\mathcal{M}_2 = (S^2, m \cos\theta \mathrm{d}\theta \mathrm{d}\phi)$*, supporting m harmonic spinors.*
- Using physics intuition about the dynamics of BPS particles and black holes, one can learn new facts about the cohomology of quiver moduli spaces.

Introduction IV

- In particular, the Joyce-Song or Kontsevich-Soibelman wall-crossing formulae, which govern the jump in the Euler number (or more generally, Poincaré polynomial) of $\mathcal{M}_Q(\gamma, \zeta)$ when the stability condition is varied, can be derived by quantizing the BPS phase space $\mathcal{M}_n$ and using localization.

de Boer at al '08; Manschot BP Sen '10

- More generally, the Coulomb branch formula expresses the Poincaré polynomial of $\mathcal{M}_Q(\gamma, \zeta)$ for any stability condition ζ in terms of new quiver indices, which are independent of ζ. Physically they should count single centered black holes, but their mathematical definition has remained mysterious.

Manschot BP Sen '11-14; Lee Wang Yi '12-13

Introduction V

- In this talk, I want to explain the flow tree formula, which instead expresses the Poincaré polynomial of $\mathcal{M}_Q(\gamma, \zeta)$ in terms of attractor indices. Like the quiver invariants, the attractor indices are independent of ζ, but they have a clear mathematical definition.
- The physics intuition behind the flow tree formula is split attractor flow conjecture, which represents bound states of n black holes as hierarchies of two-particle bound states. This conjecture was originally made by Denef in the context of $\mathcal{N} = 2$ supergravity, but it can be formulated purely in the framework of quiver moduli, and leads to a mathematical precise statement.

Outline

Outline

1. Quiver quantum mechanics and multi-centered solutions
2. The Coulomb branch formula
3. The flow tree formula

Quiver quantum mechanics I

- Pointlike particles in $\mathcal{N}=2$ field theories and string vacua on $\mathbb{R}^{3,1}$ carry electromagnetic charges $\gamma \in \Gamma$ in a lattice equipped with a symplectic pairing $\langle \gamma, \gamma' \rangle \in \mathbb{Z}$ known as the DSZ product.
- BPS particles of charge γ have mass $M = |Z_\gamma(u)|$, where the central charge $Z_\gamma(u)$ is linear in γ, but depends on the moduli u. BPS bound states are counted (with sign) by the BPS index

$$\Omega(\gamma, u) = \mathrm{Tr}_{\mathcal{H}'_1(\gamma,u)}(-1)^{2J_3} \quad \in \mathbb{Z} ,$$

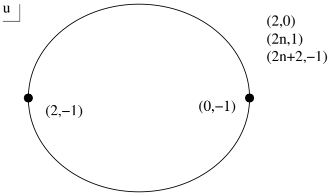
In $\mathcal{N}=2$ field theories, the refined index $\Omega(\gamma, y, u)$ defined with insertion of $y^{2(J_3+I_3)}$ is also protected. *[Gaiotto Moore Neitzke '10]*

- The index may jump on walls of marginal stability, where

$$W(\gamma_L, \gamma_R) = \{u \,/\, \arg Z_{\gamma_L}(u) = \arg Z_{\gamma_R}(u)\}$$

such that $\gamma = M_L\gamma_L + M_R\gamma_R$ for some positive integers M_L, M_R. The jump is due to the (dis)appearance of BPS bound states of constituents with charges $\gamma_i = M_{L,i}\gamma_L + M_{R,i}\gamma_R$ in the positive cone spanned by γ_L, γ_R.

Cecotti Vafa 1992; Seiberg Witten 1994



Quiver quantum mechanics III

- The quantum mechanics of n non-relativistic particles with charges $\{\gamma_i\}_{i=1}^n$ is described by $\mathcal{N}=4$ quiver quantum mechanics: if all γ_i's are distinct, this is a 0+1-dimensional gauge theory with n Abelian vector multiplets $\vec{r}_i$ and chiral multiplets $\phi_{ij,\alpha}$, $\alpha = 1,\dots \langle\gamma_i,\gamma_j\rangle$ with charge $(1,-1)$ under $U(1)_i \times U(1)_j$, for all i,j such that $\langle\gamma_i,\gamma_j\rangle > 0$.
- If some of the charges coincide, e.g. if $\{\gamma_i\}$ consists of N_1 copies of $\alpha_1,\dots,N_K$ copies of α_K with all α_j distinct, then the gauge group is $\prod_{j=1\dots K} U(N_j)$ and the chiral multiplets $\phi_{ij,k}$, $\alpha = 1,\dots,\alpha_{ij}$ are in the representation $(N_i,\bar{N}_j)$ whenever $\alpha_{ij} \equiv \langle\alpha_i,\alpha_j\rangle > 0$.
- The Fayet-Iliopoulos parameters depend on the moduli u via $\zeta_i = 2\,\mathrm{Im}\left[e^{-i\psi} Z_{\alpha_i}(u)\right]$ where $\psi = \arg Z_{\sum_i N_i\alpha_i}(u)$ such that $\sum_i N_i\zeta_i = 0$.
- If the quiver has oriented loops, there is also a gauge invariant superpotential $\mathcal{W}(\phi)$.

Quiver quantum mechanics IV

- Classically, the space of vacua consists of
  - the Higgs branch, where all $\vec{r}_i$ coincide and G is broken to $U(1)$;
  - the Coulomb branch, where all $\phi_{ij,\alpha}$ vanish, $\vec{r}^i$ are diagonal matrices and G is broken to $U(1)^K$;
  - possibly mixed branches.
- Quantum mechanically, the wave function spreads over both branches. At small string coupling g_s, it is mostly supported on the Higgs branch, while at strong g_s, it is mainly supported on the Coulomb branch. *[Denef '02]*
- BPS states on the Higgs branch are described by harmonic forms on quiver moduli spaces. They should admit an alternative Coulomb branch description in terms of multi-centered black hole bound states.

Higgs branch and quiver moduli I

- The space of SUSY vacua on the Higgs branch is the set $\mathcal{M}_Q(\gamma,\zeta)$ of gauge-inequivalent solutions of the F- and D-term equations

$$\forall i: \sum_{j;\alpha=1}^{\alpha_{ij}>0} \phi_{ij,\alpha}^{\dagger}\,\phi_{ij,\alpha} - \sum_{j;\alpha=1}^{-\alpha_{ij}>0} \phi_{ji,\alpha}^{\dagger}\,\phi_{ji,\alpha} = \zeta_i\,\mathbb{I}_{N_i\times N_i} \quad \text{[D]}$$

$$\forall i,j,\alpha:\ \partial_{\phi_{ij,\alpha}} W = 0 \quad \text{[F]}$$

- Equivalently, $\mathcal{M}_Q(\gamma,\zeta)$ is the moduli space of quiver representations with potential, i.e. the space of stable solutions of the F-term equations, modulo the complexified gauge group $\prod_i GL(N_i,\mathbb{C})$.
- Here 'stable' means that $\mu(\gamma') < \mu(\gamma)$ for any proper subrepresentation, where $\gamma=(N_1,\dots N_K)$ is the charge vector and $\mu(\gamma)=(\sum c_\ell N_\ell)/\sum N_\ell$ is the slope. *[King 94; Reineke 03]*

Higgs branch and quiver moduli II

- BPS states on the Higgs branch correspond to harmonic forms on $\mathcal{M}_Q(\zeta)$, in 1-1 correspondence with Dolbeault cohomology classes in $H^{p,q}(\mathcal{M}_Q(\gamma,\zeta),\mathbb{Z})$. The form degree $2J_3^L = p+q-d$ is identified with the Cartan of $SO(3)$, while $2J_3^R = p-q$ is the Cartan of $SU(2)_R$.
- It is convenient to package the Hodge numbers $h_{p,q}$into the Hodge 'polynomial', a symmetric Laurent polynomial in y, t:

$$g_Q(\zeta; y, t) = \sum_{p,q=0}^{2d} h_{p,q}(\mathcal{M}_Q(\gamma,\zeta))\,(-y)^{p+q-d}\,t^{p-q}$$

This reduces to the Poincaré polynomial for $t=1$; to the Hirzebruch polynomial, or χ_{y^2}-genus, for $t=1/y$; to the Euler number for $y=t=1$.

Higgs branch and quiver moduli III

- The Poincaré polynomial $g_Q(\gamma, \zeta; y, 1)$ can be computed – at least for primitive charge vector, and no loop – by counting points over finite fields and using the Weil conjectures, proven by Deligne.

Reineke '02

- The Hirzebruch polynomial $g_Q(\gamma, \zeta; y, 1/y)$ can be computed using localization, in terms of as a Jeffrey-Kirwan residue.

Benini Eager Hori Tachikawa '13; Hori Kim Yi '14

Coulomb branch and multi-centered black holes I

- On the Coulomb branch, after integrating out the massive chiral multiplets, supersymmetric vacua are solutions of Denef's equations

$$\forall i \ : \ \sum_{j \neq i} \frac{\alpha_{ij}}{|\vec{r}_i - \vec{r}_j|} = \zeta_i(u) \qquad (\alpha_{ij} := \langle \alpha_i, \alpha_j \rangle)$$

- The same equations describe multi-centered supersymmetric solutions in $\mathcal{N} = 2$ supergravity !

Denef 2000, Denef Bates 2003

Coulomb branch and multi-centered black holes II

- For fixed charges α_i and moduli u, the space of solutions modulo overall translations is a symplectic manifold $\mathcal{M}_n(\{\alpha_i, \zeta_i\})$ of dimension $2n-2$, carrying a symplectic action of $SO(3)$:

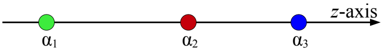
$$\omega = \frac{1}{2}\sum_{i<j} \alpha_{ij} \sin\theta_{ij}\, \mathrm{d}\theta_{ij} \wedge \mathrm{d}\phi_{ij} \ , \quad \vec{J} = \frac{1}{2}\sum_{i<j} \alpha_{ij} \frac{\vec{r}_{ij}}{|r_{ij}|}$$

de Boer El Showk Messamah Van den Bleeken 2008

- Given a symplectic manifold, geometric quantization produces a Hilbert space $\mathcal{H}$, the space of harmonic spinors for the Dirac operator D coupled to ω. The Coulomb index $g_C(\{\alpha_i, \zeta_i\}, y) \equiv \mathrm{Tr}(-y)^{2J_3}$ in the SUSY quantum mechanics is equal to the equivariant index of (D, ω). *[Manschot BP Sen '11]*

The Coulomb index from localization I

- At least when the quiver has no loop and ζ is generic, $\mathcal{M}_n$ is compact. Since $\mathcal{M}_n$ admits a $U(1)$ action, the equivariant index can be computed by localization. *[Atiyah Bott, Berline Vergne]*
- For any n, the fixed points of the action of J_3 are collinear multi-centered configurations along the z-axis:



$$\forall i \, , \quad \sum_{j\neq i} \frac{\alpha_{ij}}{|z_i - z_j|} = \zeta_i \, , \qquad J_3 = \frac{1}{2} \sum_{i<j} \alpha_{ij} \, \mathrm{sign}(z_j - z_i) \, .$$

- These fixed points are isolated, and classified by permutations σ:

$$\boxed{g_C(\{\alpha_i, \zeta_i\}, y) = \frac{(-1)^{\sum_{i<j} \alpha_{ij} + n - 1}}{(y - y^{-1})^{n-1}} \sum_{\sigma} s(\sigma) \, y^{\sum_{i<j} \alpha_{\sigma(i)\sigma(j)}}} \, , \qquad s(\sigma) \in \mathbb{Z}$$

MPS '10

The Coulomb index from localization II

- E.g. for $n = 2$, $\mathcal{M}_2 = S^2$, $J_3 = \alpha_{12} \cos\theta$:

$$g_C(\{\alpha_i, \zeta_i\}, y) = \frac{(-1)^{\alpha_{12}+1}}{y - 1/y} \left(\underbrace{y^{+\alpha_{12}}}_{\text{North pole}} - \underbrace{y^{-\alpha_{12}}}_{\text{South pole}} \right) \stackrel{y\to 1}{\longrightarrow} \pm\alpha_{12}$$

- E.g. for $n = 3$ with $\alpha_{12} > \alpha_{23}$, there are 4 collinear configurations:

$$g_C(\{\alpha_i, \zeta_i\}, y) = \frac{(-1)^{\alpha_{13}+\alpha_{23}+\alpha_{12}}}{(y-1/y)^2} \times$$
$$\Big[\underbrace{y^{\alpha_{13}+\alpha_{23}+\alpha_{12}}}_{(123)} - \underbrace{y^{-\alpha_{13}-\alpha_{23}+\alpha_{12}}}_{(312)} - \underbrace{y^{\alpha_{13}+\alpha_{23}-\alpha_{12}}}_{(213)} + \underbrace{y^{-\alpha_{13}-\alpha_{23}-\alpha_{12}}}_{(321)} \Big]$$
$$\stackrel{y\to 1}{\longrightarrow} \pm\langle\alpha_1, \alpha_2\rangle \, \langle\alpha_1 + \alpha_2, \alpha_3\rangle$$

- *For any n, one can compute $s(\sigma)$ by replacing α_{ij} by $\lambda\alpha_{ij}$ whenever $|i - j| > 1$ and studying the jumps as λ is varied from $\lambda = 0$ (nearest neighbor interactions) to $\lambda = 1$* *[MPS '13]*

Outline

The Coulomb branch formula I

- For Abelian quivers without loops, the Coulomb index turns out to coincide with the Hodge polynomial of the moduli space of stable quiver representations, known from Reineke's formula:

$$g_Q(\gamma, \zeta; y, t) = g_C(\{\alpha_i, \zeta_i\}, y)$$

  In this case the elementary constituents carry charge α_i and no internal degrees of freedom, $\Omega(\alpha_i) = 1$.
- For non-Abelian quivers, one must take into account that some of the centers are indistinguishable, and apply Bose-Fermi statistics.
- Equivalently, one can apply Boltzmann statistics, provided one includes constituents with charge vector $r\alpha_i, r \geq 1$, each of them weighted with the rational index *[Joyce Song '08; MPS' 10]*

$$\bar{\Omega}(\gamma, y) := \sum_{d|\gamma} \frac{1}{d} \frac{y - 1/y}{y^d - y^{-d}} \Omega(\gamma/d, y^d)$$

The Coulomb branch formula II

- For non-Abelian quivers with no loop, one can show that Reineke's formula agrees with the Coulomb branch formula

$$\bar{g}_Q(\gamma,\zeta;y,t) = \sum_{\gamma=\sum\gamma_i} \frac{g_C(\{\gamma_i,c_i\};y)}{|\mathrm{Aut}(\{\gamma_i\})|} \prod_i \bar{\Omega}(\gamma_i,y)$$

where $\bar{\Omega}(\gamma)=0$ unless $\gamma=r\alpha_i$ is a multiple of the vectors α_i attached to the nodes, in which case $\Omega(r\alpha_i)=\delta_{r,1}, c_i=r\zeta_i$.

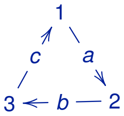
- This effectively reduces the original non-Abelian quiver to a combination of Abelian quivers. E.g. for the Kronecker quiver with m arrows, dimension vector $\gamma=(N_1,N_2)=(2,1)$,

$$g_Q[2 \xrightarrow{m} 1] = \frac{g_Q[1 \xrightarrow{2m} 1]}{2(y+1/y)} + \frac{1}{2} g_Q[1 \xrightarrow{m} 1 \xleftarrow{m} 1]$$

corresponding to bound states $\{2\gamma_1,\gamma_2\}$ and $\{\gamma_1,\gamma_1,\gamma_2\}$. *[MPS '11]*

The Coulomb branch formula III

- In presence of loops, this relation breaks down. The Coulomb index $g_C(\{\gamma_i, c_i\}; y)$ computed by localization is no longer a symmetric Laurent polynomial, but a rational function, due to the fact that the phase space $\mathcal{M}_n$ is in general non compact.
- E.g., consider the 3-node quiver



$$0 < a < b + c$$
$$0 < b < c + a$$
$$0 < c < a + b$$

For any ζ, there exist scaling solutions of Denef's equations

$$\frac{a}{r_{12}} - \frac{c}{r_{13}} = \zeta_1 , \quad \frac{b}{r_{23}} - \frac{a}{r_{12}} = \zeta_2 , \quad \frac{c}{r_{31}} - \frac{b}{r_{23}} = \zeta_3 ,$$

with $r_{12} \sim a\,\epsilon, r_{23} \sim b\,\epsilon, r_{13} \sim c\,\epsilon$ as $\epsilon \to 0$.

The Coulomb branch formula IV

- The formula can be repaired by
  - allowing constituents with charge $\alpha = \sum n_i \alpha_i$ supported on nodes linked by a closed loop,
  - weighting each constituent by

$$\Omega_{\rm tot}(\alpha; y) = \Omega_S(\alpha; y) + \sum_{\substack{\{\beta_i \in \Gamma\}, \{m_i \in \mathbb{Z}\} \\ m_i \geq 1, \sum_i m_i \beta_i = \alpha}} H(\{\beta_i\}; \{m_i\}; y) \prod_i \Omega_S(\beta_i; y^{m_i})$$

where $\Omega_S(\alpha; y)$ are new quiver invariants counting single centered solutions, and $H(\{\beta_i\}; \{m_i\}; y)$ are rational functions taking into account scaling solutions. Both are independent of the stability conditions.

MPS '13, '14

The Coulomb branch formula V

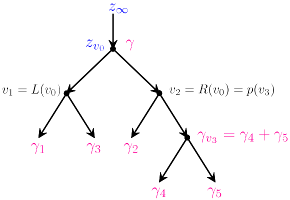
- $H(\{\beta_i\}; \{m_i\}; y)$ is fixed recursively by the minimal modification hypothesis.
  - H is symmetric under $y \to 1/y$,
  - H vanishes at $y \to 0$,
  - the coefficient of $\prod_i \Omega_S(\beta_i; y^{m_i})$ in the expression for $\Omega(\sum_i m_i \beta_i; y)$ is a Laurent polynomial in y.

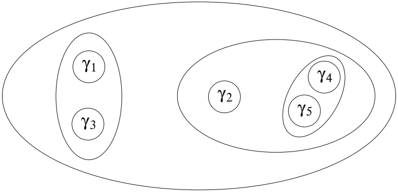
  The formula is implemented in MATHEMATICA: `CoulombHiggs.m`
- Since they are supposed to count single centered, spherically symmetric black holes, the quiver invariants $\Omega_S(\alpha, y)$ are conjectured to be independent of y (though they can depend on t).
- Moreover, they typically grow exponentially with the entries of the adjacency. E.g. $\Omega_S(\alpha_1 + \alpha_2 + \alpha_3) \sim 2^{a+b+c}$ for the Abelian 3-node quiver. *[Denef Moore '07]*

Outline

The flow tree formula I

- Rather than computing the Coulomb index of $\mathcal{M}_n$ by localization, one may instead apply the split attractor flow conjecture, which posits that all BPS states can be constructed from nested two-particle bound states:

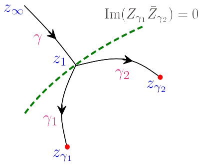




Denef '00; Denef Green Raugas '01; Denef Moore'07

The flow tree formula II

- Along each edge flowing into a vertex $\gamma \to \gamma_L + \gamma_R$, the moduli flow as in a spherically black hole, $\partial_r u^a = g^{a\bar{b}} \partial_{\bar{u}^b} |Z_\gamma(u)|$, until they hit the wall of marginal stability for the decay $\mathrm{Im} Z_{\gamma_L} \bar{Z}_{\gamma_R}(u_1) = 0$, and bifurcate into two flows with charges γ_L and γ_R.



- In order for the bound state to exist, one requires at each vertex

$$\langle \gamma_{L(v)}, \gamma_{R(v)} \rangle \, \mathrm{Im}\left[Z_{\gamma_{L(v)}} \bar{Z}_{\gamma_{R(v)}}(u_{p(v)}) \right] > 0 \quad \& \quad \mathrm{Re}\left[Z_{\gamma_{L(v)}} \bar{Z}_{\gamma_{R(v)}}(u_v) \right] > 0$$

  In the limit where quiver quantum mechanics is valid, the second condition is automatic.

The flow tree formula III

- Remarkably, the first condition can be checked in terms of asymptotic stability parameters $c_i = \mathrm{Im} Z_{\gamma_i} \bar{Z}_{\sum \gamma_i}(u_\infty)$, without integrating the flow along each edge ! It suffices to apply the discrete attractor flow *[Alexandrov BP '18]*

$$c_{v,i} = c_{p(v),i} - \frac{\langle \gamma_v, \gamma_i \rangle}{\langle \gamma_v, \gamma_{L(v)} \rangle} \sum_{j=1}^{n} m^j_{L(v)} c_{p(v),j}$$

where m^i_v are the components of $\gamma_v = \sum_{i=1}^n m^i_v \gamma_i$. This ensures $\sum m^i_{L(v)} c_{v,i} = \sum m^i_{R(v)} c_{v,i} = 0$ for each of the two subquivers.

The flow tree formula IV

- At the leaves of the tree, the attractor flow reaches the attractor point z_{γ_i} such that no further splittings are allowed. An analogue of the point z_γ which makes sense in the context of quiver moduli is the attractor stability condition

$$\boxed{\zeta_i(\gamma) = -\sum_{j=1}^{K} \alpha_{ij} N_j} \,, \qquad \gamma = \sum_{i=1}^{K} N_i \alpha_i$$

We denote the Hodge polynomial at this point, or attractor index, by $\Omega_\star(\gamma, y, t) = g_Q(\gamma, \zeta(\gamma); y, t)$, and its rational counterpart by $\bar{\Omega}_\star(\gamma, y, t)$.

The flow tree formula V

- The flow tree formula then states

$$g_Q(\gamma,\zeta,y,t) = \sum_{\gamma=\sum_{i=1}^n \gamma_i} \frac{g_{\rm tr}(\{\gamma_i,c_i\},y)}{|{\rm Aut}\{\gamma_i\}|} \prod_{i=1}^n \bar\Omega_*(\gamma_i,y,t)$$

where the sum over $\{\gamma_i\}$ runs over unordered decompositions of γ into sums of positive vectors $\gamma_i \in \Lambda_+$, and $g_{\rm tr}$ is the tree index

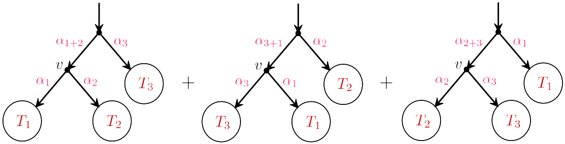
$$g_{\rm tr}(\{\gamma_i,c_i\},y) = \sum_{T\in\mathcal{T}_n(\{\gamma_i\})} \Delta(T)\,\kappa(T)$$

$$\Delta(T) = \frac{1}{2^{n-1}} \prod_{v\in V_T} \left[{\rm sgn}\,\Big(\sum_i m^i_{L(v)} c_{v,i}\Big) + {\rm sgn}(\gamma_{L(v)R(v)})\right].$$

$$\kappa(T) \equiv (-1)^{n-1} \prod_{v\in V_T} \kappa(\gamma_{L(v)R(v)})\,, \qquad \kappa(x) = (-1)^x \frac{y^x - y^{-x}}{y-y^{-1}}$$

The flow tree formula VI

- The formula tree flow is consistent with the wall-crossing formula across walls of marginal stability. Since it trivially holds in the attractor chamber, it must hold everywhere.
- It appears to have additional discontinuities across fake walls associated to the inner bound states, but these cancel after summing over trees, due to $\kappa(\gamma_{12})\,\kappa(\gamma_{1+2,3}) + \mathrm{cycl} = 0$.



- Unlike the Coulomb index g_C, the tree index g_{tr} is always a symmetric Laurent polynomial in y (away from walls of marginal stability), whether or not the quiver has loops. The price to pay is that the attractor indices $\bar{\Omega}_*(\gamma_i, y, t)$ are in general y, t-dependent.

The flow tree formula VII

- Similarly to the Coulomb index, one may decompose $g_{\rm tr}$ as a sum

$$g_{\rm tr}(\{\gamma_i, c_i\}, y) = \frac{n!(-1)^{n-1+\sum_{i<j}\gamma_{ij}}}{(y-y^{-1})^{n-1}} \,\mathrm{Sym}\, \left[F_{\rm tr}(\{\gamma_i, c_i\})\, y^{\sum_{i<j}\gamma_{ij}}\right]$$

where the partial tree index $F_{\rm tr}(\{\gamma_i, c_i\})$ is defined by

$$F_{\rm tr}(\{\gamma_i, c_i\}) = \sum_{T\in\mathcal{T}_n^{\rm pl}(\{\gamma_i\})} \Delta(T),$$

Here the sum runs over the set of planar flow trees with n leaves carrying ordered charges $\gamma_1, \ldots, \gamma_n$.

The flow tree formula VIII

- The partial tree index satisfies the obvious recursion

$$F_{\rm tr}(\{\gamma_i, c_i\}) = \frac{1}{2}\sum_{\ell=1}^{n-1}\left(\operatorname{sgn}(S_\ell) - \operatorname{sgn}(\Gamma_{n\ell})\right) \times F_{\rm tr}(\{\gamma_i, c_i^{(\ell)}\}_{i=1}^{\ell})\, F_{\rm tr}(\{\gamma_i, c_i^{(\ell)}\}_{i=\ell+1}^{n}),$$

where $c_i^{(\ell)} = c_i - \frac{\beta_{ni}}{\Gamma_{n\ell}} S_\ell$ and

$$S_k = \sum_{i=1}^{k} c_i, \quad \beta_{k\ell} = \sum_{i=1}^{k} \gamma_{i\ell}, \quad \Gamma_{k\ell} = \sum_{i=1}^{k}\sum_{j=1}^{\ell} \gamma_{ij}$$

- Due to appearance of $c_i^{(\ell)}$, this produces signs with arguments which are linear in c_i but polynomial in γ_{ij}.

The flow tree formula IX

- The partial index also satisfies another, less obvious recursion,

$$F_{\rm tr}(\{\gamma_i, c_i\}) = F_n^{(0)}(\{c_i\}) - \sum_{\substack{n_1+\dots+n_m=n \\ n_k\geq 1,\ m<n}} F_{\rm tr}(\{\gamma'_k, c'_k\}_{k=1}^m) \prod_{k=1}^{m} F_{n_k}^{(\star)}(\gamma_{j_{k-1}+1}, \dots, \gamma_{j_k}),$$

where the sum runs over ordered partitions of n with m parts,

$$j_0 = 0, \qquad j_k = n_1 + \dots + n_k, \qquad \gamma'_k = \gamma_{j_{k-1}+1} + \dots + \gamma_{j_k}.$$

$$F_n^{(0)}(\{c_i\}) = \frac{1}{2^{n-1}} \prod_{i=1}^{n-1} \mathrm{sgn}(S_i), \quad F_n^{(\star)}(\{\gamma_i\}) = \frac{1}{2^{n-1}} \prod_{i=1}^{n-1} \mathrm{sgn}(\Gamma_{ni}).$$

There are no longer any fake walls, and all arguments of sign are linear in γ_{ij}. *Trick:* $\mathrm{sgn}(x_1 + x_2)\,[\mathrm{sgn} x_1 + \mathrm{sgn} x_2] = 1 + \mathrm{sgn} x_1\, \mathrm{sgn} x_2$.

Conclusion I

- We now have two different ways of expressing the Poincaré polynomial of quiver moduli in terms of invariants which do not depend on stability conditions:
  - the quiver invariants $\Omega_S(\gamma, t)$, which count single centered black holes and are y-independent, but are defined only recursively. *Useful for holography !*
  - the attractor indices $\Omega_*(\gamma, y, t)$, which have a clear mathematical definition but count both single centered black holes and scaling solutions. *Useful for modularity !*
- For quivers without loops, the two invariants are identical and trivial: $\Omega_*(\gamma, y, t) = \Omega_S(\gamma) = 1$ if γ is a basis vector, 0 otherwise.
- For quivers with loops, the two invariants differ, and can be related by evaluating the Coulomb branch formula at the attractor point. It would be interesting to find ways to compute them directly.

Thank you for your attention !