A Coulomb branch viewpoint on quiver indices

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based on work with J. Manschot and A. Sen,

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Introduction I

Unlike the $\mathcal{N} = 4$ case, the spectrum of BPS states in $\mathcal{N} = 2$ string vacua, such as type IIA string / CY_3, is not yet under complete control:

- Microscopically: they are stable objects in the category of coherent sheaves. The BPS indices $\Omega(\gamma, z^a)$ are generalized Donaldson-Thomas invariants, hard to (define and) compute.

  Douglas, Kontsevich, Aspinwall, Bridgeland...

- Macroscopically: in general multi-centered BPS black holes. For given γ, it is hard to determine what $\{\alpha_i\}$ with $\sum \alpha_i = \gamma$ can contribute.

  Denef; Denef Moore

- For vanishing D6-brane charge and in a certain chamber, a SCFT description analogous to the D1-D5 system exists, but it is not well understood.

  Maldacena Strominger Witten

Introduction II

One aspect *is* very well-understood: discontinuities of the BPS index $\Omega(\gamma, z^a)$ across walls of marginal stability where $Z(\gamma_1)/Z(\gamma_2) \in \mathbb{R}^+$, are determined by a universal wall-crossing formula.

- On the microscopic side, this is a mathematical statement about DT invariants for the category of coherent sheaves, or for the category of quiver representations.

  Kontsevich, Joyce-Song

- On the macroscopic side, this corresponds to the (dis)appearance of multi-centered BPS black hole solutions.

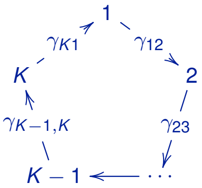
  Denef Moore; Andriyash Denef Jafferis Moore...

- The same wall-crossing formula applies in $\mathcal{N} = 2$ gauge theories, with BPS dyons playing the role of BPS black holes.

  Lee Yi; Kim Park Wang Yi

Introduction III

- Both descriptions can be derived from $\mathcal{N} = 4$ quiver quantum mechanics, which describes the low energy dynamics of BPS bound states in regions of moduli space where the central charges of the constituents nearly align.



- The Higgs branch relates the quiver representations, while the Coulomb branch reproduces the moduli space of multi-centered BPS black hole solutions. The topology of both branches jumps across the wall in agreement with the wall-crossing formula.

Introduction IV

Denef

- In fact, for certain D-brane systems, the quiver quantum mechanics description applies more generally, e.g. for fractional D-branes near orbifold singularities, or for coherent sheaves built from exceptional collections.

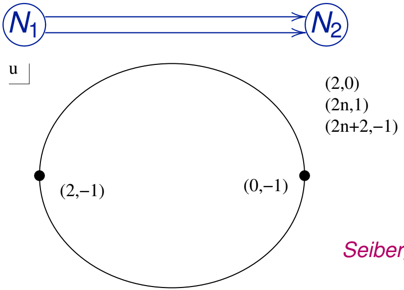
Douglas Moore, Aspinwall, ...

- By decoupling gravity, one can show that, for a large class of $D = 4$, $\mathcal{N} = 2$ gauge theories, the full BPS spectrum can be obtained as bound states of a set of elementary, absolutely stable constituents, described by $N = 4$ quiver quantum mechanics.

Fiol; Cecotti Neitzke Vafa;Alim Cecotti Cordova Espahbodi Rastogi Vafa;Chuang Diaconescu, Manschot Moore Soibelman,...

Introduction V

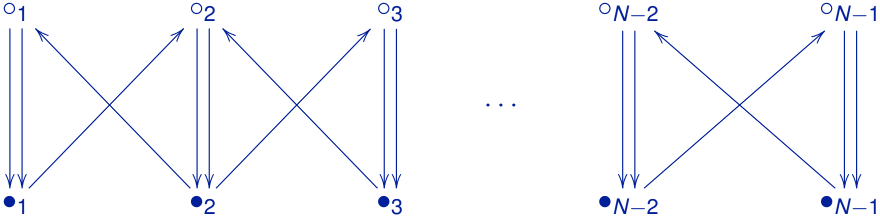
- E.g. for $SU(2)$ with no flavor, states of charge $(2N_1, N_2 - N_1)$ arise as bound states of N_1 monopoles and N_2 dyons, described by the Kronecker quiver:



Seiberg Witten; Bilal Ferrari

Introduction VI

- In general, the quiver can involve loops, e.g for $SU(N)$ with no flavor,



- In the presence of loops, the quiver quantum mechanics requires a choice of superpotential. For generic superpotential (and coprime dimension vector), the Higgs branch is compact, but the Coulomb branch is in general non-compact, as centers can approach to arbitrary small distance.

Introduction VII

- The goal of this talk is to try and remedy this problem and give an effective Coulomb branch description of the spectrum for an arbitrary quiver with loops.
- The resulting formula provides a parametrization of the cohomology of the Higgs branch in terms of a set of "single centered invariants", or "intrinsic Higgs states", which are independent of the Fayet-Iliopoulos parameters.

Bena Berkooz El Showk de Boer van den Bleeken; Manschot BP Sen; Lee Wang Yi

Quiver quantum mechanics I

- Consider an $\mathcal{N}=4$ gauge theory in $0+1$ dimensions, obtained by reducing $\mathcal{N}=1$ gauge theory in $3+1$ dimension. Each vector multiplet consists of $(\vec{r}, A_t, \lambda, D)$, each chiral multiplet consists of (ϕ, ψ, F). The R-symmetry group is $SO(4) = SU(2)_L \times SU(2)_R$.
- We restrict to gauge theories whose content is encoded in a quiver: each node $\ell = 1...K$ represents a $U(N_\ell)$ vector multiplet, each arrow from k to ℓ represents a chiral multiplet in $(N_\ell, \bar{N}_k)$ representation of $U(N_\ell) \times U(N_k)$.
- In addition, one must specify Fayet-Iliopoulos terms c_ℓ such that $\sum_\ell N_\ell c_\ell = 0$, and (in presence of closed oriented loops) a gauge invariant superpotential W.
- The ranks $\{N_\ell\}$ are encoded in a dimension vector $\gamma = \sum N_\ell \gamma_\ell$ in a lattice Γ, endowed with an antisymmetric pairing $\langle \gamma_k, \gamma_\ell \rangle = \alpha_{k\ell}$, the number of arrows from node k to node ℓ (counted with sign).

Quiver quantum mechanics II

Classically, the space of vacua consists of

- the Higgs branch, where all X^i vanish and G is broken to $U(1)$;
- the Coulomb branch, where all $\phi_{k,\ell,\alpha}$ vanish, r^i are diagonal matrices and G is broken to $U(1)^K$;
- possibly mixed branches.

Quantum mechanically, the wave functions spread over both the Coulomb and Higgs branches.

- In the large $\phi_{k,\ell,\alpha}$ region, one can integrate out the vector multiplets, leading to the Higgs branch description;
- In the large $|\vec{r}_k - \vec{r}_\ell|$ region, one can integrate out the chiral multiplets, leading to the Coulomb branch description;

At small g_s, the wave function is mostly supported on the Higgs branch, while at strong g_s, it is mainly supported on the Coulomb branch.

Quantum mechanics on the Higgs branch I

- The moduli space $\mathcal{M}_H$ of SUSY vacua on the Higgs branch is the set of gauge-inequiv. solutions of the F-term and D-term equations

$$\forall \ell : \sum_{\gamma_{\ell k}>0} \phi^*_{\ell k} \, T^a \, \phi_{\ell k} - \sum_{\gamma_{k\ell}>0} \phi^*_{k\ell} \, T^a \, \phi_{k\ell,\alpha} = c_\ell \, \mathrm{Tr}(T^a)$$

$$\forall k, \ell, \alpha : \ \partial_{\phi_{k\ell,\alpha}} W = 0$$

- Equivalently, $\mathcal{M}_H$ is the moduli space of quiver representations with potential, i.e. the space of stable solutions of the F-term equations, modulo the complexified gauge group $\prod_\ell GL(N_\ell, \mathbb{C})$.
- Here 'stable' means that $\mu(\gamma') < \mu(\gamma)$ for any proper subrepresentation of γ, where $\mu(\gamma) = (\sum c_\ell N_\ell)/\sum N_\ell$ is the slope.

King; Reineke

Quantum mechanics on the Higgs branch II

- BPS states on the Higgs branch correspond to Dolbeault cohomology classes in $H^{p,q}(\mathcal{M}_H, \mathbb{Z})$. The Cartan generators of $SU(2)_L \times SU(2)_R$ are represented by

$$J_3^L = (p+q-2)/2, \quad J_3^R = (p-q)/2 \ , \quad d = \dim_{\mathbb{C}} \mathcal{M}_H$$

  J_3^L is part of the Lefschetz action on any kähler manifold, $J_+^L = \omega\wedge, J_-^L = \omega\llcorner$ while $J_\pm^R$ are ill understood.

- It is convenient to encode the Hodge numbers $h_{p,q}$ into the Hodge polynomial

$$Q(\gamma; c_i; y, t) = \mathrm{Tr}'(-y)^{2J_3} = \sum_{p,q=0}^{2d} h_{p,q}\,(-y)^{p+q-d}\, t^{p-q}$$

  Q is a Laurent polynomial in y, t. For $t=1$, it reduces to the Poincaré polynomial; for $t = 1/y$, to the Hirzebruch polynomial, or χ_{y^2}-genus; for $y=t=1$, to the Euler number.

Quantum mechanics on the Higgs branch III

- For example, for the Kronecker quiver with k arrows, $c_1 > 0$,

$$N_1 \xrightarrow{\quad k \quad} N_2$$

- For $N_1 = N_2 = 1$,

$$\mathcal{M}_H = \mathbb{C}^k/\mathbb{C}^\times = \mathbb{P}^{k-1} \;\Rightarrow\; Q(\gamma_1+\gamma_2; y, t) = \frac{(-y)^k - (-y)^{-k}}{y - 1/y}$$

  corresponding to a $SU(2)$ multiplet of spin $J = (k-1)/2$.
- For $N_1 = 1$, $N_2 = N$, $\mathcal{M}_H$ is the Grassmannian of N-planes in $\mathbb{C}^k$,

$$Q(\gamma_1 + N\gamma_2; y, t) = \frac{(-y)^{N(k-N)}[k,y]!}{[N,y]!\,[k-N,y]!}$$

  where $[N, y]! = [1, y][2, y]\ldots[N, y]$ is the deformed factorial, with $[N, y] = (y^{2N} - 1)/(y^2 - 1)$.

Quantum mechanics on the Higgs branch IV

- For quivers without oriented closed loop, and primitive dimension vector, Reineke has computed the Hodge polynomial:

$$Q(\gamma, c; y, t) = (-y)^{\sum_{i,j} N_k N_\ell \max(\alpha_{k\ell},0) - 1 + \sum_\ell N_\ell} (y^2 - 1)^{1-\sum_\ell N_\ell} \sum (-1)^{s-1} \, y^{2\sum_{a\leq b}\sum_{k,\ell} \max(\alpha_{k\ell},0) - N_k^b N_\ell^a} \prod_{a,\ell} \frac{1}{[N_\ell^a, y]!}$$

where the sum runs over all $(N_1, \dots, N_K) = \sum_{a=1}^{s} (N_1^a, \dots, N_K^a)$

$$\forall b = 1 \dots s-1 \, , \, \sum_{a=1}^{b} \sum_{\ell=1}^{K} N_\ell^a c_a > 0 \, .$$

The result is independent of t, i.e. all states are singlets under $SU(2)_R$. The formula can be generalized to non-primitive dimension vector, although $\mathcal{M}_H$ is singular in that case.

Quantum mechanics on the Higgs branch V

- Recently, it has become possible to compute the Hirzebruch polynomial $Q(\mathcal{M}_H; y, t = 1/y)$ for quivers with loop and generic superpotential using localization. The result is expressed as a Jeffrey-Kirwan residue, schematically

$$Q(\gamma; y, t = 1/y) = \frac{1}{|W|} \sum_p \mathrm{JK-Res}(Q(p), c) \, [g(u, y) \, \mathrm{d}^r u]$$

where $g(u, y)$ is a product of one-loop factors associated to vector and chiral multiplets, p runs over all isolated intersections of singular hyperplanes, and $Q(p)$ is the corresponding set of charges.

Benini Eager Hori Tachikawa; Hori Kim Yi

- Unlike the case of 2D GLSM, $Q(\gamma; y)$ is discontinuous across the walls.

Wall-crossing formula I

- Across walls in the space of FI parameters where $\mu(\gamma_1) = \mu(\gamma_2)$, the topology of $\mathcal{M}_H$ for dimension vectors $\gamma = M\gamma_1 + N\gamma_2$ can change, leading to a jump in $Q(\gamma; y, t)$.
- To state the wall-crossing formula, it is convenient to introduce the rational invariants

$$\bar{Q}(\gamma; y, t) = \sum_{m|\gamma} \frac{1}{m} \frac{y - 1/y}{y^m - 1/y^m} Q(\gamma/m, y^m, t^m)$$

Wall-crossing formula II

- Across the wall where $\mu(\gamma_1) = \mu(\gamma_2)$, the rational invariants $Q(\gamma; y, t)$ change according to

$$\bar{Q}(\gamma, c_+; y, t) = \sum_{\gamma=\sum \alpha_i} \frac{g_H(\{\alpha_i\}, y)}{|\mathrm{Aut}(\{\alpha_i\})|} \prod_i \bar{Q}(\alpha_i, c_-; y, t)$$

  where $\alpha_i = M_i \gamma_1 + N_i \gamma_2$, and $g_H(\{\alpha_i\}, y)$ is the Poincaré polynomial associated to an Abelian quiver consisting of one node for each α_i, and $\langle \alpha_i, \alpha_j \rangle$ arrows from α_i to α_j.
  E.g.: $\Delta \bar{Q}(\gamma_1 + \gamma_2) = (-1)^{\gamma_{12}} \frac{y^{\gamma_{12}} - y^{-\gamma_{12}}}{y - 1/y} \bar{Q}(\gamma_1) \bar{Q}(\gamma_2)$
- As we shall see, this formula has a transparent interpretation on the Coulomb branch.

Quantum mechanics on the Coulomb branch I

- The Coulomb branch description arises by integrating out the chiral multiplets, leading to (here i runs over $N = \sum N_\ell$ values (ℓ, m) with $m = 1 \dots N_\ell$; $\alpha_{ij} = \alpha_{k\ell}, c_i = c_\ell$)

$$\mathcal{L} = \sum_i [W_i(\vec{r}_i)]^2 + \sum_i \vec{A}_i \dot{\vec{r}}_i + \sum_{i,j} \gamma_{ij} \dot{\vec{r}}_i \dot{\vec{r}}_j + \dots$$

$$W_i = \sum_{j \neq i} \frac{\alpha_{ij}}{|\vec{r}_i - \vec{r}_j|} - c_i \ , \qquad \vec{A}_i = \sum_{j \neq i} \alpha_{ij} \vec{A}_{\mathrm{Dirac}}(\vec{r}_i - \vec{r}_j)$$

Denef;Kim Park Wang Yi

- The supersymmetric vacua are given by

$$\forall i \ , \quad \sum_{j \neq i} \frac{\alpha_{ij}}{|\vec{r}_i - \vec{r}_j|} = c_\ell$$

Quantum mechanics on the Coulomb branch II

- Identifying $\alpha_{ij} = \langle \alpha_i, \alpha_j \rangle$ and $c_i = 2\mathrm{Im}(e^{-2\mathrm{i}\phi} Z_{\alpha_i})$, these are exactly the equations obeyed by multi-centered black holes with charges $\{\alpha_i\}$!
- Modding out by translations in $\mathbb{R}^3$, we define $\mathcal{M}_N(\{\alpha_i\}, \{c_i\})$ to be the $2N-2$ dimensional space of solutions to $[*]$, equipped with the symplectic form

$$\omega = \frac{1}{2} \sum_{i<j} \alpha_{ij} \sin\theta_{ij}\, \mathrm{d}\theta_{ij} \wedge \mathrm{d}\phi_{ij}\ , \quad \vec{J} = \frac{1}{2} \sum_{i<j} \alpha_{ij} \frac{\vec{r}_{ij}}{r_{ij}}$$

such that $\vec{J}$ is the moment map for $SU(2)$ rotations.

de Boer El Showk Messamah Van den Bleeken

- E.g. for $N = 2$, $\mathcal{M}_2 = S^2$, $\omega = \frac{1}{2} \sin\theta \mathrm{d}\theta \mathrm{d}\phi$, $J_3 = \alpha_{12} \cos\theta$.

Quantum mechanics on the Coulomb branch III

- Quantum mechanically, BPS states are zero modes of the Dirac operator on $(\mathcal{M}_N, \omega)$. They are counted by the equivariant Dirac index

$$g_C(\{\alpha_i\}, \{c_i\}; y) = \mathrm{Tr}_{\mathrm{Ker} D_+}(-y)^{2J_3} - \mathrm{Tr}_{\mathrm{Ker} D_-}(-y)^{2J_3} \ .$$

where D is the Dirac operator on $\mathcal{M}_N$, twisted by ω.

Manschot BP Sen; Kim Park Wang Yi

- When $\mathcal{M}_N$ is compact and J_3 has only isolated fixed points, the index can be evaluated by Atiyah-Bott Lefschetz fixed point formula:

$$g_C(\{\alpha_i\}, \{c_i\}; y) = \sum_{\text{fixed pts}} \frac{y^{2J_3}}{\det((-y)^L - (-y)^{-L})}$$

where L is the action of J_3 on the holomorphic tangent space.

Quantum mechanics on the Coulomb branch IV

- E.g. for $n = 2$, $\mathcal{M}_2 = S^2$, $J_3 = \alpha_{12} \cos\theta$:

$$g_C = \frac{(-1)^{\alpha_{12}+1}}{(y - 1/y)} \left(\underbrace{y^{+\alpha_{12}}}_{\text{North pole}} - \underbrace{y^{-\alpha_{12}}}_{\text{South pole}} \right)$$

if $c_1 \alpha_{12} > 0$, zero otherwise. This is recognized as the character of a $SU(2)$ representation of spin $j = \frac{1}{2}(\alpha_{12} - 1)$, reproducing the Higgs branch result !

Coulomb index from localization I

- The fixed points of the action of J_3 are collinear multi-centered configurations along the z-axis, such that

$$\sum_{j\neq i}^{n} \frac{\alpha_{ij}}{|z_i - z_j|} = c_i \,, \quad J_3 = \frac{1}{2} \sum_{i<j} \alpha_{ij}\, \mathrm{sign}(z_j - z_i)\,.$$

- Equivalently, fixed points are critical points of the 'superpotential'

$$W(\{z_i\}) = -\sum_{i<j} \mathrm{sign}[z_j - z_i]\, \alpha_{ij}\, \ln|z_j - z_i| - \sum_i c_i z_i$$

- The determinant turns out to be $(y - 1/y)^{n-1}$ times a sign $s(p) = -\mathrm{sign}(\det W'')$ where W'' is the Hessian of W

Coulomb index from localization II

- After the dust settles, the equivariant Dirac index is given by

$$g_C(\{\alpha_i\},\{c_i\};y) = \frac{(-1)^{\sum_{i<j}\alpha_{ij}+n-1}}{(y-y^{-1})^{n-1}} \sum_p s(p)\, y^{\sum_{i<j}\alpha_{ij}\,\mathrm{sign}(z_j-z_i)}$$

  where the sum runs over all collinear solutions of Denef's equations.
- The contribution of each fixed point is singular at $y=1$, but the sum over fixed points is guaranteed to produce a symmetric polynomial in y and $1/y$, (provided $\mathcal{M}_n$ is compact).
- Importantly, all states are singlets under $SU(2)_R$, so g_C agrees with the Hodge polynomial (in particular, it is independent of t)

Coulomb index from localization III

- Remarkably, for an Abelian quiver without loop ($N_\ell = 0$ or 1 for all ℓ), $g_C(\{\alpha_i j\}, \{c_i\}, y)$ agrees with Reineke's formula for the Hodge polynomial of the Higgs branch !
- This shows that the BPS states on the Higgs branch can equivalently be viewed as bound states of elementary states γ_ℓ associated to the nodes of the quiver, each carrying unit degeneracy, $\Omega_S(\gamma_\ell) = 1$.
- For non-Abelian quivers without loop, the centers $r_i = r_{\ell,m}$ for $m = 1 \dots N_\ell$ are identified by the Weyl group of $U(N_\ell)$, and must therefore be treated as indistinguishable. Moreover, when several $r_{\ell,m}$ coincide, the off-diagonal gauge bosons can no longer be integrated out.

- Consider for simplicity bound states $[\gamma_1, \gamma_2, \gamma_2]$ and $[\gamma_1, 2\gamma_2]$: they contribute $\Omega(\gamma_1)$ times

$$\frac{1}{2}\langle\gamma_1, \gamma_2\rangle\Omega(\gamma_2)\ [\langle\gamma_1, \gamma_2\rangle\Omega(\gamma_2) + 1] + \langle\gamma_1, 2\gamma_2\rangle\ \Omega(2\gamma_2)$$

Introducing $\bar{\Omega}(2\gamma_2) = \Omega(2\gamma_2) + \frac{1}{4}\Omega(\gamma_2)$, this can be written as

$$\frac{1}{2}\langle\gamma_1, \gamma_2\rangle^2\Omega(\gamma_2)^2 + \langle\gamma_1, 2\gamma_2\rangle\ \bar{\Omega}(2\gamma_2)$$

More generally, the replacement

$$\Omega(\gamma; y, t) \to \bar{\Omega}(\gamma; y, t) = \sum_{m|\gamma} \frac{1}{m}\frac{y - 1/y}{y^m - 1/y^m}\Omega(\gamma/m, y^m, t^m)$$

allows to turn Bose/Fermi statistics into Boltzmann statistics.

Manschot BP Sen 2010

A Coulomb branch formula for quivers without loops I

- Using this trick, one finds that the BPS states on the Higgs branch states can still be viewed as bound states of elementary states with charge $m\gamma_\ell$, carrying effective degeneracy $\bar{\Omega}_S(m\gamma_\ell) = \frac{1}{m}\frac{y-1/y}{y^m-1/y^m}$ and satisfying Boltzmann statistics:

$$\bar{Q}(\gamma; t, y) = \sum_{\gamma=\sum \alpha_i} \frac{g_C(\{\alpha_i\}, \{c_i\}; y)}{|\mathrm{Aut}(\{\alpha_i\})|} \prod_i \bar{\Omega}_S(\alpha_i; y)$$

Manschot BP Sen; Kim Park Wang Yi

- By construction, this satisfies the wall-crossing formula, but from the Coulomb branch point of view, the fact that $\bar{\Omega}_S(\alpha) \neq 0$ only when α is a multiple of a basis vector γ_ℓ looks artificial.
- Indeed, for quivers with loops, one must relax this assumption, but this is not sufficient...

A Coulomb branch formula for quivers with loops I

- Upon trying to extend the Coulomb branch picture to the case of quivers with loops, one stumbles on the fact that the Coulomb branch is no longer compact, due to the existence of scaling solutions.
- E.g., take a 3-node quiver with $\alpha_{12} = a$, $\alpha_{23} = b$, $\alpha_{31} = c$ satisfying triangular inequalities $0 < a < b + c$, etc. There exist solutions of Denef's equations

$$\frac{a}{r_{12}} - \frac{c}{r_{13}} = c_1 , \qquad \frac{b}{r_{23}} - \frac{a}{r_{12}} = c_2$$

with $r_{12} \sim a\,\epsilon, r_{23} \sim b\,\epsilon, r_{13} \sim c\,\epsilon, \vec{J}^2 \sim \epsilon^2$ as $\epsilon \to 0$.

Denef Moore; Bena Berkooz El Showk de Boer van den Bleeken

A Coulomb branch formula for quivers with loops II

- For $c_1, c_2 > 0$, the only collinear configurations are (123) and (321), leading to

$$g_C = \frac{(-1)^{a+b+c}(y^{a+b-c} + y^{-a-b+c})}{(y-1/y)^2}$$

- This is not a polynomial in y, in particular it is singular as $y \to 1$. Still the phase space $\mathcal{M}_N$ has finite volume.
- This could be repaired by adding by hand a term with $J_3 \simeq 0$, attributed to scaling solutions:

$$\tilde{g} = \frac{(-1)^{a+b+c}\left(y^{a+b-c} + y^{-a-b+c} - \left[\begin{array}{cc} 2 & a+b+c \text{ even} \\ y+1/y & a+b+c \text{ odd} \end{array}\right]\right)}{(y-1/y)^2}$$

A Coulomb branch formula for quivers with loops III

- More generally, scaling regions in $\mathcal{M}_n(\{\alpha_i\})$ arise whenever there exist a subset A and vectors $\vec{r}_i \in \mathbb{R}^3$, $i \in A$ such that

$$\forall i \in A\,, \quad \sum_{j\in A} \frac{\alpha_{ij}}{|\vec{r}_{ij}|} = 0\,.$$

  This is independent of the c_i's, so scaling solutions cannot be removed by changing the moduli.

- Rather than modifying the sum over collinear fixed points g into a symmetric Laurent polynomial $\tilde{g}$, we postulate

$$\bar{\Omega}(\gamma, y) = \sum_{\gamma=\sum \alpha_i} \frac{g_C(\{\alpha_i\},\{c_i\},y)}{|\mathrm{Aut}(\{\alpha_i\})|} \prod_i \bar{\Omega}_{\mathrm{tot}}(\alpha_i; y, t)$$

A Coulomb branch formula for quivers with loops IV

- Here $\Omega_{\rm tot}$ includes contributions both from single centered and scaling solutions:

$$\begin{aligned}\Omega_{\rm tot}(\alpha; y, t) =& \Omega_S(\alpha; y, t) \\ &+ \sum_{\substack{\{\beta_i\in\Gamma\},\{m_i\in\mathbb{Z}\}\\ m_i\geq 1,\, \sum_i m_i\beta_i=\alpha}} H(\{\beta_i\};\{m_i\};y) \prod_i \Omega_{\rm S}(\beta_i; y^{m_i}, t^{m_i})\end{aligned}$$

- $H(\{\beta_i\};\{m_i\};y)$ is uniquely determined by the conditions
  - H is symmetric under $y \to 1/y$,
  - H vanishes at $y \to 0$,
  - the coefficient of $\prod_i \Omega_{\rm S}(\beta_i; y^{m_i})$ in the expression for $\Omega(\sum_i m_i\beta_i; y)$ is a Laurent polynomial in y.

The formula is implemented in mathematica: `CoulombHiggs.m`

A Coulomb branch formula for quivers with loops V

- E.g, for the Abelian 3-node quiver discussed above,

$$H(\{\gamma_1,\gamma_2,\gamma_3\};\{1,1,1\};y) = \begin{cases} -2\,(y-y^{-1})^{-2} & , a+b+c \text{ even} \\ (y+y^{-1})\,(y-y^{-1})^{-2} & , a+b+c \text{ odd} \end{cases}$$

so the prescription gives

$$Q(\gamma_1+\gamma_2+\gamma_3;y,t) = \tilde{g}_C(\{\gamma_1,\gamma_2,\gamma_3\},\{c_i\},y)+\Omega_S(\gamma_1+\gamma_2+\gamma_3;y,t)\,.$$

- Using the Lefschetz hyperplane theorem and Riemann-Roch theorem, one can compute $Q(\gamma_1+\gamma_2+\gamma_3;y,t)$. The resulting $\Omega_S(\gamma_1+\gamma_2+\gamma_3;y,t)$ is found to be independent of y,t, and grow as 2^{a+b+c}, whereas $\tilde{g}_C$ is chamber-dependent and grows polynomially.

Bena Berkooz El Showk de Boer van den Bleeken

A Coulomb branch formula for quivers with loops VI

- The Coulomb branch formula is manifestly consistent with wall-crossing, provided the $\Omega_{\rm S}$'s are independent of the Fayet-Iliopoulos parameters: the $\Omega_{\rm S}$ seem to count absolutely stable constituents, analogous to single-centered black holes !
- For quivers with generic superpotential, we seem to find that the $\Omega_{\rm S}$'s are independent of y, i.e. the corresponding states carry zero angular momentum, just like single-centered black holes.
- The dependence on t only enters though the $\Omega_{\rm S}$'s. Using invariance under mutations, one can show that for the quiver associated to $SU(N)$ gauge theories, the $\Omega_{\rm S}$ all vanish, implying the no-exotics conjecture.

Del Zotto Sen

Summary and open problems I

- The index $\bar{Q}(\gamma)$ in $\mathcal{N}=4$ supersymmetric quantum mechanics can be interpreted as a sum of contributions from bound states of stable constituents with charge α_i such that $\gamma=\sum_i \alpha_i$, interacting via Coulomb/Lorentz forces and carrying $\Omega_S(\alpha_i)$ internal d.o.f.
- For quivers without loops, the elementary constituents are associated to the nodes and have $\Omega_S=1$. For quivers with loop, they are associated to subquivers allowing for scaling solutions.
- This conjecture has been tested on many quivers, it would be nice to derive it, e.g. based on the localization formula of Lee Hori Yi, and understand the mathematical meaning of Ω_S.
- Similarly, we expect that in $N=2$ string vacua, the BPS index $\Omega(\gamma)$ can be reduced to a sum of bound states of single-centered black holes with degeneracy $\Omega_S(\alpha)$ – the quantity to be matched with the quantum entropy function.