BPS black holes and generalized error functions

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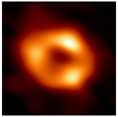
based on work with Rishi Raj, arXiv:2507.08551

+earlier works with S. Alexandrov, J. Manschot, S. Banerjee, ...

Introduction

- A central goal for any theory of quantum gravity is to provide a microscopic explanation of the thermodynamical entropy of black holes in General Relativity *[Bekenstein'72, Hawking'74]*

$$S_{BH} = \frac{A}{4G_N}$$



$$S_{BH} \overset{?}{=} \log \Omega$$

- As shown by *[Strominger Vafa'96,...]*, String Theory provides a quantitative description in the case of BPS black holes in vacua with extended SUSY: at weak string coupling, black hole micro-states arise as bound states of D-branes wrapped on cycles of the internal manifold.
- Besides confirming the consistency of string theory as a theory of quantum gravity, this has opened up many fruitful connections with mathematics.

BPS indices and Donaldson-Thomas invariants

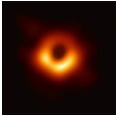
- In the context of type IIA strings compactified on a Calabi-Yau three-fold X, BPS states are described mathematically by stable objects in the derived category of coherent sheaves $\mathcal{C} = D^b\mathrm{Coh}X$. The Chern character $\gamma = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2, \mathrm{ch}_3)$ is identified as the electromagnetic charge, or D6-D4-D2-D0-brane charge.
- The problem becomes a question in Donaldson-Thomas theory: for fixed $\gamma \in K(X)$, compute the generalized DT invariant $\Omega_z(\gamma)$ counting (semi)stable objects of class γ, and determine its growth as $|\gamma| \to \infty$.
- Importantly, $\Omega_z(\gamma)$ depends on the moduli of X, or more generally on a choice of Bridgeland stability condition $\sigma \in \mathrm{Stab}\,\mathcal{C}$. In particular, it can jump on real-codimension one loci known as walls of marginal stability. The jump is governed by a universal wall-crossing formula *[Joyce Song'08, Kontsevich Soibelman'08]*.

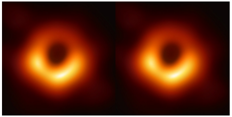
Wall-crossing and black hole bound states

- Walls correspond to loci where the decay $\gamma \to \sum_i \gamma_i$ is energetically possible. Since $m_i = |Z(\gamma_i)|$, this is possible only when the phases of all $Z(\gamma_i)$ are aligned.
- In the simplest 'primitive' case $\gamma \to \gamma_1 + \gamma_2$, the jump is

$$\Delta\Omega_\sigma(\gamma_1+\gamma_2) = \langle\gamma_1,\gamma_2\rangle\,\Omega_\sigma(\gamma_1)\,\Omega_\sigma(\gamma_2)$$

easily reproduced from the SUSY quantum mechanics of the electron-monopole problem (more on this later).





- More generally, the jump can involve an arbitrary number of constituents. The dynamics is complicated, but the index is computable using localization *[Manschot BP Sen'10]*.

Modularity of D4-D2-D0 indices I

- Viewing Type IIA string theory as the reduction of M-theory on a circle, allows to make very non-trivial physical predictions about generalized DT invariants. E.g. the GW/DT relation of *[MNOP'03]*.
- In particular, D4-D2-D0 black holes turn out to be black strings in disguise, obtained by wrapping an M5-brane on a divisor $\mathcal{D}$. This indicates that suitable generating series of rank 0 DT invariants should have specific modular properties *[Maldacena Strominger Witten'97]*. This gives very good control on their asymptotic growth, and allows to test agreement with the BH prediction $\Omega_z(\gamma) \simeq e^{S_{BH}(\gamma)}$.
- More precisely, D4-D2-D0 indices occur as Fourier coefficients in the elliptic genus $\mathcal{I}(\tau, z^a) = \mathrm{Tr}(-1)^F q^{L_0 - \frac{c_L}{24}} e^{2\pi i q_a z^a}$ of the two-dimensional superconformal field theory with $(0, 4)$ SUSY.

Modularity of D4-D2-D0 indices II

- Using the spectral flow symmetry, the elliptic genus has a theta series decomposition

$$\mathcal{I} = \sum_{\mu \in \Lambda^*/\Lambda} h_{p,\mu}(\tau)\, \Theta(\tau, z^a)$$

where Λ^*/Λ is the finite discriminant group associated to $\Lambda = (H_4(X,\mathbb{Z}), \kappa_{ab} := \kappa_{abc} p^c)$, and

$$h_{p,\mu}(\tau) := \sum_n \bar{\Omega}(0, p, \mu, n)\, q^{n - \frac{\chi(\mathcal{D})}{24} + \frac{1}{2}\mu^2 - \frac{1}{2} p\mu}$$

- If the SCFT has a discrete spectrum, $h_{p,\mu}(\tau)$ must be a vector-valued, weakly holomorphic modular form in the (dual) Weil representation attached to Λ.

Modularity of D4-D2-D0 indices III

- When $\mathcal{D}$ is very ample and irreducible, there are no walls extending to large volume, so the choice of chamber is irrelevant. The central charges are given by *[Maldacena Strominger Witten'97]*

$$\begin{cases} c_L = & p^3 + c_2(TX) \cdot p = \chi(\mathcal{D}) \, , \\ c_R = & p^3 + \frac{1}{2} c_2(TX) \cdot p = 6\chi(\mathcal{O}_{\mathcal{D}}) \end{cases}$$

  Cardy's formula predicts a growth $\Omega(0, p, \beta, n \to \infty) \sim e^{2\pi\sqrt{p^3\, n}}$ in perfect agreement with Bekenstein-Hawking formula !

- Moreover, since the space of vector-valued weakly holomorphic modular form has finite dimension, the full series is completely determined by its polar coefficients, with $n + \frac{1}{2}\mu^2 - \frac{1}{2} p\mu < \frac{\chi(\mathcal{D})}{24}$.

Mock modularity of rank 0 DT invariants

- When $\mathcal{D}$ is reducible, the generating series $h_{p^a,\mu_a}(\tau)$ in a suitable ("large volume attractor") chamber is expected to be a mock modular form of higher depth *[Alexandrov BP Manschot'16-20]*)
- Namely, there exists explicit, universal non-holomorphic theta series $\Theta_n(\{p_i\},\tau,\bar\tau)$ such that (ignoring the μ's for simplicity)

$$\widehat{h}_p(\tau,\bar\tau) = h_p(\tau) + \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \Theta_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^{n} h_{p_i}(\tau)$$

transforms as a modular form. The completed series satisfy the holomorphic anomaly equation,

$$\partial_{\bar\tau}\widehat{h}_p(\tau,\bar\tau) = \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \widehat{\Theta}_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^{n} \widehat{h}_{p_i}(\tau,\bar\tau)$$

Crash course on indefinite theta series

- Θ_n and $\widehat{\Theta}_n$ belongs to the class of indefinite theta series

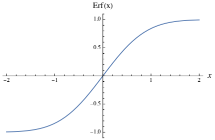
$$\vartheta_{\Phi,q}(\tau,\bar\tau) = \tau_2^{-\lambda} \sum_{k\in\Lambda+q} \Phi\left(\sqrt{2\tau_2}k\right) e^{-\mathrm{i}\pi\tau Q(k)}$$

  where (Λ, Q) is an even lattice of signature $(r, d-r)$, $q \in \Lambda^*/\Lambda$, $\lambda \in \mathbb{R}$. The series converges if $f(x) \equiv \Phi(x) e^{\frac{\pi}{2}Q(x)} \in L_1(\Lambda\otimes\mathbb{R})$.
- Theorem (Vignéras, 1978): $\{\vartheta_{\Phi,q}, q \in \Lambda^*/\Lambda\}$ transforms as a vector-valued modular form of weight $(\lambda + \frac{d}{2}, 0)$ provided
  - $R(x)f, R(\partial_x)f \in L_2(\Lambda\otimes\mathbb{R})$ for any polynomial $R(x)$ of degree ≤ 2
  - $\left[\partial_x^2 + 2\pi(x\partial_x - \lambda)\right]\Phi = 0$ [*]
- The relevant lattice for Θ_n and $\widehat{\Theta}_n$ is $\Lambda = H^2(X,\mathbb{Z})^{\oplus(n-1)}$, with signature $(r, d-r) = (n-1)(1, b_2(X)-1)$.

Indefinite theta series

- Example 1 (Siegel): $\Phi = e^{\pi Q(x_+)}$, where x_+ is the projection of x on a fixed plane of dimension r, satisfies [*] with $\lambda = -n$. ϑ_Φ is then the usual (non-holomorphic) Siegel-Narain theta series.
- Example 2 (Zwegers): In signature $(1, d-1)$, choose C, C' two vectors such that $Q(C), Q(C'), (C, C') > 0$, then

$$\widehat{\Phi}(x) = \mathrm{Erf}\left(\frac{(C,x)\sqrt{\pi}}{\sqrt{Q(C)}}\right) - \mathrm{Erf}\left(\frac{(C',x)\sqrt{\pi}}{\sqrt{Q(C')}}\right)$$



satisfies [*] with $\lambda = 0$. As $|x| \to \infty$, or if $Q(C) = Q(C') = 0$,

$$\widehat{\Phi}(x) \to \Phi(x) := \mathrm{sgn}(C, x) - \mathrm{sgn}(C', x)$$

- The theta series $\Theta_2(\{p_1, p_2\})$, $\widehat{\Theta}_2(\{p_1, p_2\})$ fall in this class. The generalization to $n \geq 3$ involves generalized error functions.

Generalized error functions I

- Extend the representations

$$E_1(x) = \int_{\mathbb{R}} e^{-\pi(x-x')^2}\,\mathrm{sign}(x')\,\mathrm{d}x' = \mathrm{Erf}(x\sqrt{\pi}),$$
$$M_1(x) = \frac{\mathrm{i}}{\pi}\int_{\mathbb{R}-\mathrm{i}x} e^{-\pi z^2-2\pi\mathrm{i}z}\frac{\mathrm{d}z}{z} = -\,\mathrm{sign}(x)\,\mathrm{Erfc}(|x|\sqrt{\pi}),$$

to $\mathbf{x}\in\mathbb{R}^r$, $\mathcal{M}\in\mathbb{R}^{r\times r}$ *[Alexandrov Banerjee Manschot BP'16, Nazaroglu'16]*

$$E_r(\mathcal{M},\mathbf{x}) = \int_{\mathbb{R}^r}\mathrm{d}^r\mathbf{z}\,e^{-\pi(\mathbf{x}-\mathbf{z})^T(\mathbf{x}-\mathbf{z})}\prod_{i=1}^{r}\mathrm{sign}\left(\mathcal{M}^T\mathbf{z}\right)_i,$$
$$M_r(\mathcal{M};\mathbf{x}) = \left(\frac{\mathrm{i}}{\pi}\right)^r|\det\mathcal{M}|^{-1}\int_{\mathbb{R}^r-\mathrm{i}\mathbf{x}}\mathrm{d}^r\mathbf{z}\,\frac{e^{-\pi\mathbf{z}^T\mathbf{z}-2\pi\mathrm{i}\mathbf{z}^T\mathbf{x}}}{\prod_{i=1}^{r}\left(\mathcal{M}^{-1}\mathbf{z}\right)_i}$$

Generalized error functions II

- Both E_r and M_r are annihilated by Vignéras operator $\partial_{\mathbf{x}}^2 + 2\pi \mathbf{x} \partial_{\mathbf{x}}$.
- $E_r(\mathcal{M}, \mathbf{x})$ is a C^∞ function of $\mathbf{x}$, which asymptotes to $\prod_{i=1}^r \text{sgn}(\mathcal{M}^T \mathbf{x})_i$ as $|\mathbf{x}| \to \infty$
- $M_r(\mathcal{M}, \mathbf{x})$ is a C^∞ function away from the hyperplanes $(\mathcal{M}^{-1}\mathbf{x})_i = 0$, exponentially suppressed as $|\mathbf{x}| \to \infty$
- Both E_r and M_r are invariant under rescaling the columns of $\mathcal{M}$ by arbitrary positive factors or permuting them, and under rotating $(\mathcal{M}, \mathbf{x}) \mapsto (O\,\mathcal{M}, O\,\mathbf{x})$ by $O \in O(r)$.

Generalized error functions III

- For any collection $\mathcal{V} = (\mathbf{v}_1, \dots, \mathbf{v}_r)$ of r vectors in $\mathbb{R}^d$ with fixed quadratic form $\mathcal{Q}$ of signature $(r, d-r)$, such that the Gram matrix $\mathcal{V}^T \mathcal{Q} \mathcal{V}$ is positive definite, define the boosted error functions

$$\begin{aligned} \Phi_r^E(\mathcal{Q}, \{\mathbf{v}_i\}, \mathbf{x}) &= E_r(\mathcal{B}\mathcal{Q}\mathcal{V}, \mathcal{B}\mathcal{Q}\mathbf{x}) \\ \Phi_r^M(\mathcal{Q}, \{\mathbf{v}_i\}, \mathbf{x}) &= M_r(\mathcal{B}\mathcal{Q}\mathcal{V}, \mathcal{B}\mathcal{Q}\mathbf{x}) \end{aligned}$$

  where $\mathcal{B}$ is an orthonormal basis of $\langle \mathcal{V} \rangle$, $\mathcal{B}\mathcal{Q}\mathcal{B}^T = 1$.
- $\Phi_r^E(\mathcal{Q}, \mathcal{V}, \mathbf{x})$ is a C^∞ function of $\mathbf{x}$ which asymptotes to $\mathrm{sgn}(\mathcal{V}^T \mathcal{Q}\mathbf{x})$ as $|\mathbf{x}| \to \infty$. $\Phi_r^M(\mathcal{Q}, \mathcal{V}, \mathbf{x})$ is exponentially suppressed as $|\mathbf{x}| \to \infty$.
- Φ_r^E can be expressed in terms of Φ_r^M, and vice-versa,

$$\Phi_r^E(\mathcal{Q}, \{\mathbf{v}_i\}, \mathbf{x}) = \sum_{\mathcal{I} \subset \{1,\dots r\}} \Phi_{|\mathcal{I}|}^M(\mathcal{Q}, \{\mathbf{v}_i\}_{i\in\mathcal{I}}, \mathbf{x}) \prod_{j \notin \mathcal{I}} \mathrm{sign}(\mathbf{v}_{j\perp\mathcal{I}} \mathcal{Q}\mathbf{x}),$$

  where $\mathbf{v}_{j\perp\mathcal{I}}$ is the projection of $\mathbf{v}_j$ orthogonal to the vectors $\mathbf{v}_{i\in\mathcal{I}}$.

Multi-black hole quantum mechanics I

- Consider the Lagrangian with 4 real supercharges

$$L = \sum_{i=1}^{n} \frac{m_i}{2} \left(\dot{\vec{x}}_i^2 + D_i^2 + 2\mathrm{i}\bar{\lambda}_i \dot{\lambda}_i \right) + \sum_{i=1}^{n} (-U_i D_i + \vec{A}_i \cdot \dot{\vec{x}}_i) + \sum_{i,j=1}^{n} \vec{\nabla}_i U_j \cdot \bar{\lambda}_i \vec{\sigma} \lambda_j$$

where

$$U_i = -\frac{1}{2}\Big(\sum_{j\neq i} \frac{\gamma_{ij}}{|\vec{x}_i - \vec{x}_j|} - c_i\Big), \quad \vec{\nabla}_i U_j = \frac{1}{2}(\vec{\nabla}_i \times \vec{A}_j + \vec{\nabla}_j \times \vec{A}_i).$$

- Eliminating the auxiliary fields D_i, one generates a potential

$$V = \sum_{i=1}^{n} \frac{U_i^2}{2m_i}.$$

Multi-black hole quantum mechanics II

- Supersymmetric ground states satisfy Denef's equations:

$$\sum_{j \neq i} \frac{\gamma_{ij}}{|\vec{x}_i - \vec{x}_j|} = c_i$$

These are the same equations which determine the relative positions of the centers in stationary BPS solutions of $\mathcal{N} = 2$ supergravity, provided $c_i = 2\mathrm{Im}(e^{-i\phi} Z(\gamma_i)), \phi = \arg(Z(\gamma))$.

- The moduli space

$$\mathcal{M}_n(\{\gamma_i, c_i\}) = \left\{ (\vec{x}_i) \in \mathbb{R}^{3n},\ \forall i \ \sum_{j \neq i} \frac{\gamma_{ij}}{|\vec{x}_i - \vec{x}_j|} = c_i \right\} / \mathbb{R}^3$$

carries a natural $SO(3)$-invariant symplectic structure,

$$\omega = \frac{1}{4} \sum_{i<j} \epsilon_{abc} \frac{\gamma_{ij}}{r_{ij}^3} \, \mathrm{x}_{ij}^a \, \mathrm{dx}_{ij}^b \wedge \mathrm{dx}_{ij}^c, \quad \vec{J} = \frac{1}{2} \sum_{i<j} \gamma_{ij} \frac{\vec{x}_{ij}}{r_{ij}},$$

- The Witten index localizes on time-independent configurations *[Girardello Imimbo Mukhi'83]*

$$\mathcal{I}_n = \int \prod_{i=1}^{n-1} \frac{\mathrm{d}^3\vec{x}_i\,\mathrm{d}\bar{\lambda}_i\,\mathrm{d}\lambda_i\,\mathrm{d}D_i}{4\pi^2\beta}\, e^{-\beta\left(\sum_{i=1}^{n}(\mathrm{i}U_iD_i+\sum_{i,j}\left(\frac12 M_{ij}D_iD_j+\vec{\nabla}_jU_i\bar{\lambda}_i\vec{\sigma}\lambda_j\right)\right)}$$

  where $M_{ij}=m_i\delta_{ij}-\frac{m_im_j}{m_{\rm tot}}$ is the reduced mass matrix.
- Integrating out the fermions produces

$$\int \prod_{i=1}^{n-1} \mathrm{d}\bar{\lambda}_i\,\mathrm{d}\lambda_i\, e^{-\beta\sum_{i,j=1}^{n-1}\vec{\nabla}_jU_i\,\bar{\lambda}_i\vec{\sigma}\lambda_j} = (\beta^2)^{n-1}\,\det(\vec{\nabla}_jU_i\otimes\vec{\sigma}),$$

Witten index from localization II

- Key observation: The bosonic configuration space $\mathbb{R}^{3n-3}$ is foliated by the phase spaces $\mathcal{M}_n(\{\gamma_i, u_i\})$ with $u_i \in \mathbb{R}^{n-1}$. The flat integration measure on $\mathbb{R}^{3n-3}$ combines with the fermionic determinant to produce the Liouville measure on $\mathcal{M}_n$, times flat measure on $\mathbb{R}^{n-1}$,

$$\boxed{\prod_{i=1}^{n-1} \mathrm{d}^3\vec{x}_i \, \det(\vec{\nabla}_i U_j \otimes \vec{\sigma}) = \frac{(-1)^{n-1}}{2^{n-1}(n-1)!} \left(\prod_{i=1}^{n-1} \mathrm{d}u_i\right) \, \omega^{n-1},}$$

Proof: follows from $\det(Q + \mathrm{i}M) = \mathrm{pf}\begin{pmatrix} M & Q \\ -Q & M \end{pmatrix}$.

- For $n = 2$, this boils down to $r^2 \mathrm{d}r \mathrm{d}\Omega_2 \times -\frac{\kappa^2}{4r^4} = \frac{1}{2}\kappa \mathrm{d}\rho \times \frac{1}{2}\kappa \mathrm{d}\Omega_2$.

Witten index from localization III

- Integrating over D_i, we get

$$\mathcal{I}_n = \sqrt{\det \tfrac{\beta}{8\pi M}} \int \prod_{i=1}^{n-1} \mathrm{d}u_i \ \mathrm{Vol}(\{\gamma_i, u_i\}) \, e^{-\frac{\beta}{8}(u_i - c_i) M_{ij}^{-1} (u_j - c_j)}$$

where $\mathrm{Vol}(\{\gamma_i, u_i\}) = \frac{(-1)^{\sum_{i<j} \gamma_{ij} - n+1}}{(2\pi)^{n-1}(n-1)!} \int_{\mathcal{M}_n(\{\gamma_i, u_i\})} \omega^{n-1}$.

- The refined index is expected to be given by a similar formula, replacing the symplectic volume with the equivariant Dirac index,

$$\mathcal{I}_n = \sqrt{\det \tfrac{\beta}{8\pi M}} \int \prod_{i=1}^{n-1} \mathrm{d}u_i \ \mathrm{Ind}(\{\gamma_i, u_i\}, y) \, e^{-\frac{\beta}{8}(u_i - c_i) M_{ij}^{-1} (u_j - c_j)}$$

- At zero temperature, this is dominated by $u_i = c_i$, hence reduces to $\mathrm{Ind}(\{\gamma_i, c_i\}, y)$ counting supersymmetric bound states.

Witten index from localization IV

- Both $\mathrm{Vol}(\{\gamma_i, u_i\})$ and $\mathrm{Ind}(\{\gamma_i, u_i\}, y)$ are locally constant functions of u_i, away from walls of marginal stability. Thus the Witten index is a linear combination of generalized error functions !
- Moreover, both are computable using Duistermaat-Heckman / Atiyah-Bott localization with respect to $U(1) \subset SO(3)$. *[Manschot BP Sen'11]*:

$$\mathrm{Ind}(\{\gamma, c\};\, y) = \frac{(-1)^{\sum_{i<j} \gamma_{ij} - n+1}}{(y - 1/y)^{n-1}} \sum_{\sigma \in S_n} F_n(\{\gamma_{\sigma(i)}, c_{\sigma(i)}\})\, y^{\sum_{i<j} \gamma_{\sigma(i)\sigma(j)}}$$

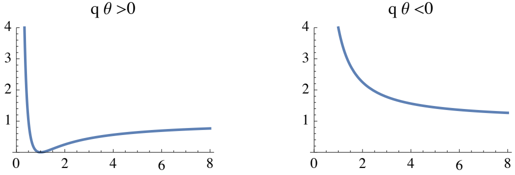
$$\mathrm{Vol}(\{\gamma_i, c_i\}) = \lim_{y \to 1} \mathrm{Ind}(\{\gamma, c\};\, y)$$

Two-body electron-monopole problem I

- Consider a non-relativistic particle of electric charge $q = \frac{1}{2}\langle\gamma_1, \gamma_2\rangle$ in the field of a Dirac monopole of unit magnetic charge:

$$H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 - \frac{q}{2m}\vec{B}\cdot\vec{\sigma}\otimes(1_2 - \sigma_3) + \frac{1}{2m}\left(\vartheta - \frac{q}{r}\right)^2$$

$$\vec{\nabla}\wedge\vec{A} = \vec{B} = \frac{\vec{r}}{r^3}\,, \quad m = \frac{|Z_{\gamma_1}||Z_{\gamma_2}|}{|Z_{\gamma_1}|+|Z_{\gamma_2}|}\,, \quad \frac{\vartheta^2}{2m} = |Z_{\gamma_1}| + |Z_{\gamma_2}| - |Z_{\gamma_1+\gamma_2}|$$



Two-body electron-monopole problem II

- H describes two bosonic degrees of freedom with helicity $h = 0$, and one helicity $h = \pm 1/2$ fermionic doublet with gyromagnetic ratio $g = 4$.

D'Hoker Vinet 1985; Denef 2002; Avery Michelson 2007;Lee Yi 2011

- H commutes with 4 supercharges – here $\vec{\Pi} = \vec{p} - q\vec{A}$:

$$Q_4 = \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 & -\mathrm{i}\left(\vartheta - \frac{q}{r}\right) + \vec{\sigma}\cdot\vec{\Pi} \\ \mathrm{i}\left(\vartheta - \frac{q}{r}\right) + \vec{\sigma}\cdot\vec{\Pi} & 0 \end{pmatrix}$$

$$Q_a = \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 & -\left(\vartheta - \frac{q}{r}\right)\vec{\sigma} - \mathrm{i}\vec{\Pi} + \vec{\Pi}\wedge\vec{\sigma} \\ -\left(\vartheta - \frac{q}{r}\right)\vec{\sigma} + \mathrm{i}\vec{\Pi} + \vec{\Pi}\wedge\vec{\sigma} & 0 \end{pmatrix}.$$

$$\{Q_m, Q_n\} = 2H\,\delta_{mn}$$

Two-body electron-monopole problem III

- Going to a basis of monopole spherical harmonics, the Schrödinger equation with energy $E = k^2/(2m)$ becomes

$$\left[-\frac{1}{r}\partial_r^2\, r + \frac{\nu^2 - q^2 - \frac14}{r^2} + \left(\vartheta - \frac{q}{r} \right)^2 \right] \Psi(r) = k^2 \Psi \,,$$

  where

$$\nu = j + \frac{1}{2} + h \,, \quad j = |q| + h + \ell \,, \ell \in \mathbb{N} \,.$$

- Supersymmetric bound states exist for $q\vartheta > 0$, $h = -1/2$, $\ell = 0$, and form a multiplet of spin $j = |q| - \frac12$, with $2j + 1 = |\langle \gamma_1, \gamma_2 \rangle|$.

Denef 2002

Two-body electron-monopole problem IV

- The S-matrix for partial waves is similar to that of H-atom,

$$S_\nu(k) = \frac{\Gamma\left(\frac{1}{2}+\nu+\mathrm{i}\frac{q\vartheta}{\sqrt{k^2-\vartheta^2}}\right)}{\Gamma\left(\frac{1}{2}+\nu-\mathrm{i}\frac{q\vartheta}{\sqrt{k^2-\vartheta^2}}\right)} = e^{2\mathrm{i}\delta_\nu(k)}.$$

BP, arXiv:1501.01643

- The contribution of the continuum to $\mathrm{Tr}(-1)^F\, e^{-2\pi R H}$ is thus

$$\sum_{h=0^2,\pm\frac{1}{2}}(-1)^{2h}\sum_{\ell=0}^{\infty}\int_{k=\vartheta}^{\infty}\frac{\mathrm{d}k\,\partial_k}{2\pi\mathrm{i}}\,\log\frac{\Gamma\left(|q|+\ell+2h+1+\mathrm{i}\frac{q\vartheta}{\sqrt{k^2-\vartheta^2}}\right)}{\Gamma\left(|q|+\ell+2h+1-\mathrm{i}\frac{q\vartheta}{\sqrt{k^2-\vartheta^2}}\right)}\,e^{-\frac{\pi R k^2}{m}}$$

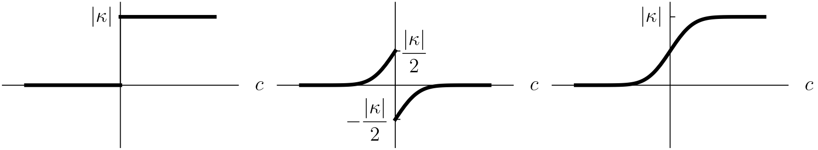
Two-body electron-monopole problem V

- Terms with $\ell > 0$ cancel, leaving the contribution from $\ell = 0$ only:

$$\mathrm{Tr}(-1)^F\, e^{-2\pi R H} = -\,|2q|\,\Theta(q\vartheta) - \frac{2q\vartheta}{\pi}\int_{k=|\vartheta|}^{\infty} \frac{\mathrm{d}k}{k\sqrt{k^2-\vartheta^2}}\, e^{-\frac{\pi R k^2}{m}}$$

$$= -\,2|q|\,\Theta(q\vartheta) + |q|\,\mathrm{sgn}(q\vartheta)\,\mathrm{Erfc}\left(|\vartheta|\sqrt{\frac{\pi R}{m}}\right)$$

$$= -\,|q| - q\,\mathrm{Erf}\left(\vartheta\sqrt{\frac{\pi R}{m}}\right) .$$



Two-body electron-monopole problem VI

- Using localization, this is easily reproduced:

$$
\begin{aligned}
\mathcal{I}_2 &= -\frac{\beta\kappa^2}{4\pi}\int_0^\infty \frac{r^2\mathrm{d}r}{r^4}\int_{-\infty}^\infty \mathrm{d}D\; e^{-\beta\left(-\frac{i}{2}D\left(\frac{\kappa}{r}-c\right)+\frac{m}{2}D^2\right)} \\
&= -\kappa^2\left(\frac{\beta}{8\pi m}\right)^{\frac12}\int_0^\infty \mathrm{d}\rho\; e^{-\frac{\beta}{8m}(\kappa\rho-c)^2} \\
&= -\frac{\kappa^2}{2}\left(\frac{\beta}{8\pi m}\right)^{\frac12}\int_{-\infty}^\infty \mathrm{d}\rho(1+\operatorname{sign}\rho)\; e^{-\frac{\beta}{8m}(\kappa\rho-c)^2} \\
&= -\frac{\kappa}{2}\left[\operatorname{sign}(\kappa)+E_1\left(c\sqrt{\frac{\beta}{8\pi m}}\right)\right]
\end{aligned}
$$

with $\rho = 1/r$.

BPS index from localization I

- When $\mathcal{M}_n(\{\gamma_i, u_i\})$ is compact, its equivariant volume or Dirac index is computable by localization with respect to $U(1) \subset SO(3)$. Fixed points are collinear configurations subject to

$$\sum_{j\neq i} \frac{\gamma_{ij}}{|z_i - z_j|} = c_i ,$$

i.e. critical points of $W = -\sum_{i<j} \gamma_{ij} \log|z_j - z_i| - \sum_i c_i z_i$.

- Solutions are classified by the ordering of the centers along the z-axis, weighted by $\operatorname{sign}\det \partial^2 W$:

$$\mathrm{Ind}_C(\{\gamma, c\};\, y) = \frac{(-1)^{\sum_{i<j}\gamma_{ij} - n+1}}{(y-1/y)^{n-1}} \sum_{\sigma\in S_n} F_{C,n}(\{\gamma_{\sigma(i)}, c_{\sigma(i)}\})\, y^{\sum_{i<j}\gamma_{\sigma(i)\sigma(j)}}$$

BPS index from localization II

- When $\mathcal{M}_n(\{\gamma_i, u_i\})$ is not compact, i.e. in the presence of scaling solutions, there are additional boundary contributions which ensure that $\mathrm{Ind}(\{\gamma, c\};\, y)$ is a symmetric Laurent polynomial in y.
- Alternatively, use the flow tree formula.

$$\mathrm{Ind}_{\rm tree}(\{\gamma, c\};\, y) = \frac{(-1)^{\sum_{i<j}\gamma_{ij}-n+1}}{(y-1/y)^{n-1}} \sum_{\sigma\in S_n} F_{{\rm tree},n}(\{\gamma_{\sigma(i)}, c_{\sigma(i)}\})\, y^{\sum_{i<j}\gamma_{\sigma(i)\sigma(j)}}$$

where $F_{{\rm tree},n}(\{\gamma_i, c_i\})$ counts planar attractor flow trees. E.g.

$$F_{\rm tree,2} = \frac{1}{2}\Big[\mathrm{sgn}(c_1) + \mathrm{sgn}(\gamma_{12})\Big]$$

$$\begin{aligned} F_{\rm tree,3} = \frac{1}{4}\, & [(\mathrm{sgn}(c_1) + \mathrm{sgn}(\gamma_{12}))\,(\mathrm{sgn}(c_1 + c_2) + \mathrm{sgn}(\gamma_{23})) \\ & - (\mathrm{sgn}(\gamma_{2+3,1}) + \mathrm{sgn}(\gamma_{12}))\,\big(\mathrm{sgn}(\gamma_{3,1+2}) + \mathrm{sgn}(\gamma_{23})\big)\big]\,, \end{aligned}$$

BPS index from localization III

- Using this, we find e.g. the partial Witten index for 3 centers

$$\begin{aligned}\mathcal{J}_3 = & \frac{1}{4}\Big[E_2\Big(\sqrt{\tfrac{m_1 m_3}{m_2(m_1+m_2+m_3)}}; \tfrac{c_2 m_1 - c_1 m_2}{\sqrt{m_1 m_2(m_1+m_2)}}, \tfrac{c_3}{\sqrt{m_{1+2,3}}}\Big) \\ & - \operatorname{sgn}(\gamma_{1,2+3})\operatorname{sgn}(\gamma_{1+2,3}) \\ & - \Big[E_1\Big(\tfrac{c_3}{\sqrt{m_{1+2,3}}}\Big) - \operatorname{sgn}(\gamma_{1+2,3})\Big]\operatorname{sgn}(\gamma_{12}) \\ & + \Big[E_1\Big(\tfrac{c_1}{\sqrt{m_{1,2+3}}}\Big) - \operatorname{sgn}(\gamma_{2+3,1})\Big]\operatorname{sgn}(\gamma_{23})\Big].\end{aligned}$$

where $m_{1+2,3} = \frac{m_3(m_1+m_2)}{(m_1+m_2+m_3)}$, $\gamma_{1+2,3} = \gamma_1 + \gamma_2$.

- $\mathcal{I}_3(\beta, y)$ is obtained by rescaling $m \mapsto 8\pi m/\beta$, multiplying by $y^{\gamma_{12}+\gamma_{23}+\gamma_{13}}/(y - 1/y)^2$ and summing over permutations.

BPS index from localization IV

- Setting $\beta = 2\pi\tau_2$, this matches the error functions appearing in the 'instanton generating potential', which plays the role of Witten index in 4D *[Alexandrov Moore Neizke BP'14]*

$$\mathcal{G} = \sum_{n=1}^{\infty} \frac{1}{2\pi\sqrt{\tau_2}} \, e^{-\sum_i S^{\rm cl}_{p_i}} \vartheta_{p,\mu}(\Phi^{\rm tot}_n, -1) \left[\prod_{i=1}^{n} \sum_{p_i,\mu_i} h_{p_i,\mu_i} \right]$$

- For one-parameter models, or more generally for collinear magnetic charges, the same error functions evaluated at the large volume attractor point appear in the modular completion of $h_{p,\mu}$. In general however, the x-arguments differ by a rescaling, e.g a factor $\sqrt{\frac{(p^3)(p_1 p_2 p)}{(p_1 p^2)(p_2 p^2)}}$ in the 2 body case.

BPS index from localization V

- At the large volume attractor point, only contributions from the continuum of scattering states remain:

$$\begin{aligned}\mathcal{J}_3^* = \frac{1}{4}\Bigg(& M_2\left(\sqrt{\frac{m_1 m_3}{m_2(m_1+m_2+m_3)}}; -\frac{\sqrt{\tau_2}(m_2\gamma_{1,2+3}-m_1\gamma_{2,1+3})}{\sqrt{m_1 m_2(m_1+m_2)}}, -\frac{\sqrt{\tau_2}\gamma_{1+2,3}}{\sqrt{m_{1+2,3}}}\right) \\ & + M_1\left(\frac{\sqrt{\tau_2}\gamma_{1+2,3}}{\sqrt{m_{1+2,3}}}\right)\left(\mathrm{sgn}\left(m_2\gamma_{1,2+3} - m_1\gamma_{2,1+3}\right) - \mathrm{sgn}\left(\gamma_{12}\right)\right) \\ & + M_1\left(\frac{\sqrt{\tau_2}\gamma_{1,2+3}}{\sqrt{m_{1,2+3}}}\right)\left(\mathrm{sgn}\left(m_2\gamma_{1+2,3} - m_3\gamma_{1+3,2}\right) - \mathrm{sgn}\left(\gamma_{23}\right)\right)\Bigg).\end{aligned}$$

  with $\beta = 2\pi\tau_2$.

- For collinear magnetic charges (e.g. in one-parameter models), the coefficients of M_1 vanish, leaving only the contribution from genuine 3-body scattering.

Summary and open problems I

- Using localization, we managed to evaluate the Witten index of the quantum mechanics of *n* dyons, including both bound state and continuum contributions. Compare with the unsolvable *n*-body problem in Newtonian gravity !
- Our derivation reproduces the non-holomorphic terms in the modular completion of the generating series of D4-D2-D0 invariants, that were predicted earlier using indirect arguments.
- Some details still need to be clarified, e.g. the η-deformation needed to resolve the ambiguity in $\mathrm{sign}(0)$, Kronecker delta contributions supported on walls, a proper derivation of the refined index, etc.
- Currently the modular completion has mainly been tested in one-modulus examples. It would be interesting to study examples with two moduli, e.g. genus-one fibrations or K3 fibrations.

Thanks for your attention !