BPS indices, Vafa-Witten invariants and quivers

Boris Pioline

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based on arXiv:2004.14466 with Guillaume Beaujard and Jan Manschot

and earlier work with Sergei Alexandrov and Ashoke Sen

Introduction

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- Subleading corrections to the black hole entropy, both on the macroscopic and microscopic side, were studied in many subsequent works at increasing level of detail, uncovering new physics (e.g. wall-crossing phenomena) and new mathematics (e.g. connections to enumerative geometry and modular forms).

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- For string vacua with $\mathcal{N} = 4$ supersymmetry, the exact (signed) number of BPS black hole microstates $\Omega(\gamma, t)$ is now known for any charge vector γ and any moduli t, in terms of Fourier coefficients of a particular Siegel modular form (or a weak Jacobi form for $\mathcal{N} = 8$).

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- From mathematical point of view, the BPS indices correspond to generalized Donaldson-Thomas invariants of $K3 \times T^2$ (and orbifolds thereof), which can be computed rigorously.

Bryan Oberdieck Pandharipande Yin 2015-18

Introduction

- In contrast, for $\mathcal{N}=2$ string vacua, precision counting of black hole microstates is much less advanced. The (0,4) SCFT describing D4-D2-D0 bound states as wrapped M5-branes is poorly understood (beyond its central charges c_L, c_R), and the BPS index $\Omega(\gamma, t)$ is known exactly only for very special cases.

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- Part of the difficulty is that $\Omega(\gamma, t)$ depends on the moduli t in a very intricate way, due to wall-crossing phenomena associated to BPS bound states with arbitrary number of constituents. The moduli space itself receives quantum corrections, unlike in $\mathcal{N} \geq 4$.

Introduction

- While the generating function $h_p(\tau) \sim \sum_n \Omega(0, p^a, q_a, q_0) q^n$ of BPS indices for D4-D2-D0 black holes in a certain (large volume attractor) chamber is expected to be related to the elliptic genus of the (0,4) SCFT, it cannot be modular covariant in general, except when the divisor $P = p^a \omega_a$ wrapped by the D4-brane is irreducible.

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- In recent work, we have characterized the modular anomaly of $h_p(\tau)$, in terms of the generating functions h_{P_i} for all possible splittings $p = \sum_{i=1}^n p_i$. In other words, h_p is a mock modular form of depth $n-1$ with specified shadow.

Alexandrov Banerjee Manschot BP 2016-2019

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- This anomaly follows from existence of isometric action of S-duality on the moduli space after circle compactification, but remains to be understood from the SCFT viewpoint, or from the mathematics of generalized Donaldson-Thomas invariants.

Summary

- In this talk, I will consider D4-D2-D0 bound states in type II string compactified on a local Calabi-Yau manifold K_S, the total space of the canonical bundle over a complex Fano surface S. The BPS index for D4-D2-D0 branes supported on S is then given by the Euler number of the moduli space of stable coherent sheaves on S.

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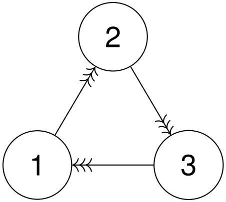
- For $[D4] = N[S]$, $\Omega(\gamma, t)$ coincides with the Vafa-Witten invariants of S, computed by topologically twisted $\mathcal{N} = 4$ SYM with gauge group $U(N)$. S-duality implies that generating functions should be (mock) modular. The precise modular anomaly follows from our results for DT invariants on a compact CY.

Vafa Witten 1994; Minahan Nemeschansky Vafa Warner 1998

Summary

- Our main tool will be the equivalence between the derived category of coherent sheaves on a Fano surface S, and the derived category of representations of a certain quiver (Q, W). E.g. for local $\mathbb{P}^2$, or equivalently $\mathbb{C}^3/\mathbb{Z}_3$,



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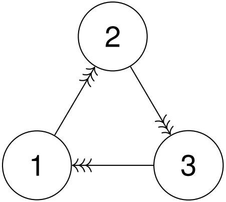

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- In general, the nodes of the quiver correspond to certain rigid sheaves E_i on S forming an exceptional collection.

Baer-Bondal-Rickart 1989-90, Herzog Walcher 2003; Aspinwall Melnikov 2004

Summary

- The BPS index $\Omega(\gamma, t)$ is equal to the Euler number $\Omega(\vec{N}, \vec{\zeta})$ of the moduli space of semi-stable quiver representations with dimension vector $\vec{N}$ and FI parameters $\vec{\zeta}$ determined from (γ, t).

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- Unless Q has no loops, the BPS index $\Omega(\vec{N}, \vec{\zeta})$ is in general difficult to compute. However, quivers coming from exceptional collections on Fano surfaces are special: the 'attractor index'

$$\Omega_*(\vec{N}) = \Omega(\vec{N}, \vec{\zeta}_*(\vec{N}))$$

vanishes, unless $\vec{N}$ is supported on a single node, or $\vec{N} \propto \vec{N}_{D0}$. Here $\vec{\zeta}_*(\vec{N})$ is the 'attractor' or 'self-stability condition'.

Beaujard Manschot BP 2020

Summary

- The BPS index elsewhere can be computed by using the flow tree formula, which expresses $\Omega(\vec{N}, \vec{\zeta})$ in terms of $\Omega_*(\vec{N}_i)$ for all decompositions $\vec{N} = \sum_i \vec{N}_i$.

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- In particular, the large volume attractor chamber on CY side corresponds to the anti-attractor chamber, or 'canonical chamber' $\vec{\zeta}^c = -\vec{\zeta}_*(\vec{N})$ on the quiver side !
- Modular properties of VW invariants imply that the generating functions of quiver indices $Z_{\vec{N}_0}(\tau) \sim \sum_n \Omega_c(\vec{N}_0 + n\vec{N}_{D0})q^n$ should be a mock modular form !

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- This gives an efficient way of computing BPS indices / VW invariants for any Fano surface, not necessarily toric, and possibly for any rational surface.
- In the rest of this talk, I will explain some background about exceptional collections, toric surfaces, quivers, etc, and demonstrate how the method works in simple examples.

Outline

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D-branes and coherent sheaves

- At large volume, D-branes on a Calabi-Yau threefold X are described by coherent sheaves E on X: morally, a vector bundle whose fiber dimension may jump. A D6-brane is supported on all of X, a D4-brane on a divisor, a D2-brane on a curve and a D0-brane on a point.

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- The D-brane charge can be read off from the Chern character $\mathrm{ch}(E) = [\mathrm{rk}, \mathrm{ch}_1, \mathrm{ch}_2, \mathrm{ch}_3] \in H^{\mathrm{even}}(X, \mathbb{Q})$.

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- The spectrum of open strings between D-branes associated to coherent sheaves E, E' is determined from the extension groups $\mathrm{Ext}^k_X(E, E')$. Ext^0_X corresponds to tachyons (projected out when $E = E'$), Ext^1_X to nearly massless states, $\mathrm{Ext}^{k\geq 2}_X$ to massive strings irrelevant at low energy.

D-branes and coherent sheaves

- When $X = K_S$, the total space of the canonical bundle K_S over a smooth complex surface S, D4-branes supported on S are obtained by lifting coherent sheaves $E \to i_*(E)$ from S to X.

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- The Ext groups on X are related to those on S by

$$\mathrm{Ext}^k_X(i_*E, i_*E') = \mathrm{Ext}^k_S(E, E') \oplus \mathrm{Ext}^{3-k}_S(E, E')$$

Thus, light open strings originate both from Ext^1_S and Ext^2_S, while Ext^0_S and Ext^3_S lead to tachyons.

D-branes and coherent sheaves

- The dimension of Ext groups can be inferred from the Euler form

$$\chi(E,E') := \sum_{k\geq 0} (-1)^k \dim \mathrm{Ext}^k_S(E,E')$$

By the Riemann-Roch formula, it depends only on the Chern characters $\gamma(E) = [\mathrm{rk}(E), c_1(E), \mathrm{ch}_2(E)]$,

$$\chi(E,E') = \mathrm{rk}(E)\,\mathrm{rk}(E') + \mathrm{rk}(E)\,\mathrm{ch}_2(E') + \mathrm{rk}(E')\,\mathrm{ch}_2(E) - c_1(E)c_1(E') + \frac{1}{2}\left[\mathrm{rk}(E)\,\deg(E') - \mathrm{rk}(E')\,\deg(E)\right]$$

where $\deg(E) = c_1(E)\cdot c_1(S)$.

D-branes and coherent sheaves

- Stable D-branes correspond to Gieseker-stable coherent sheaves on S. The sheaf E is stable if all proper subsheaves E' have

$$\begin{cases} \nu_J(E') < \nu_J(E) \\ \nu_J(E') = \nu_J(E) \quad \text{and} \quad \frac{\mathrm{ch}_2(E')}{\mathrm{rk}(E')} < \frac{\mathrm{ch}_2(E)}{\mathrm{rk}(E)} \end{cases}$$

where $\nu_J(E) = \frac{c_1(E)\cdot J}{\mathrm{rk}(E)}$ is the slope and J the Kähler form.

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  where $\nu_J(E) = \frac{c_1(E)\cdot J}{\mathrm{rk}(E)}$ is the slope and J the Kähler form.
- The moduli space of G-stable sheaves of Chern vector γ has expected dimension

$$d_{\mathbb{C}}(\mathcal{M}^S_{\gamma,J}) = 1 - \chi(E,E)$$

  and is invariant under tensoring with a line bundle $\mathcal{L}$,

$$c_1 \to c_1 + Nc_1(\mathcal{L}) \,, \quad \mathrm{ch}_2 \to \mathrm{ch}_2 - Nc_1(\mathcal{L})\cdot c_1 + \frac{1}{2} N\, c_1(\mathcal{L})^2$$

Exceptional collections

- An exceptional sheaf is one such that

$$\mathrm{Ext}^0_S(E,E) \simeq \mathbb{C}, \quad \mathrm{Ext}^k_S(E,E) = 0 \quad \forall k > 0$$

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- An exceptional collection is an ordered set $\mathcal{C} = (E_1, \ldots, E_r)$ of exceptional sheaves such that

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- A full exceptional collection collection is one such that the Chern characters $\{\mathrm{ch}\, E_i, i = 1 \dots r\}$ span the lattice $K(S)$. For a simply connected surface S, $r = \chi(S)$.

Exceptional collections

- Full exceptional collections satisfying the no-tachyon condition

$$\mathrm{Ext}^0_S(E_i, E_j) = \mathrm{Ext}^3_S(E_i, E_j) = 0 \quad \forall i \neq j$$

can be constructed from a strongly cyclic exceptional collection $\mathcal{C}^\vee = (E^1_\vee, \dots, E^r_\vee)$, such that $\chi(E_i, E^j_\vee) = \delta^i{}_j$.

Aspinwall Melnikov 2004; Herzog Karp 2006

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- Note that $E_i, E^i_\vee$ are denoted $E^\vee_i, E^i$ in our paper !

Exceptional collections and quivers

- To any such collection one associates a quiver Q with nodes $i \in Q_0$ corresponding to E_i. Arrows come from $\mathrm{Ext}^1_S(E_j, E_i)$ (morphisms $\Phi_{ij\alpha}$) and $\mathrm{Ext}^2_S(E_j, E_i)$ (constraints $C_{ij\alpha}$)

$$\begin{tikzcd} i \arrow[r, "\mathrm{Ext}^1_S(E_j, E_i)"'] & j \arrow[l, bend right, dotted, "\mathrm{Ext}^2_S(E_i, E_j)"'] \end{tikzcd}$$

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- The data (Q, W) specifies a 0+1 dimensional supersymmetric gauge theory with 4 supercharges, a.k.a. quiver quantum mechanics. *[Denef 2002]*

Coherent sheaves and quiver representations

- The net number of arrows is given by the antisymmetrized Euler form

$$\kappa_{ij} = S_{ji} - S_{ij} = \langle E_i, E_j \rangle$$

where

$$\begin{aligned} \langle E, E' \rangle &= \chi(E, E') - \chi(E', E) \\ &= \mathrm{rk}(E)\,\deg(E') - \mathrm{rk}(E')\,\deg(E) \end{aligned}$$

This coincides with the Dirac-Schwinger-Zwanziger product $\langle \gamma_i, \gamma_j \rangle$ of electromagnetic charges of the constituents.

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This coincides with the Dirac-Schwinger-Zwanziger product $\langle \gamma_i, \gamma_j \rangle$ of electromagnetic charges of the constituents.
- Different exceptional collections lead to apparently different quiver quantum mechanics, related by Seiberg duality.

Herzog 2004

Example: Local $\mathbb{P}^2$

- The projective plane admits a tachyon-free exceptional collection

$$\mathcal{C} = (\mathcal{O}, \Omega(1)[1], \mathcal{O}(-1)[2])$$

$$\begin{array}{l} \gamma_1 = [1, 0, 0] \\ \gamma_2 = [-2, 1, \frac{1}{2}] \\ \gamma_3 = [1, -1, \frac{1}{2}] \end{array} \qquad S = \begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

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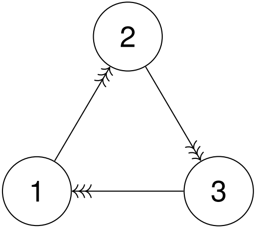
- This leads to the familiar quiver for $\mathbb{C}^3/\mathbb{Z}_3$,



$$W = \sum_{(ijk)\in S_3} \mathrm{sgn}(ijk)\, \Phi^i_{12}\Phi^j_{23}\Phi^k_{31}$$

Douglas Fiol Romelsberger 2000

Coherent sheaves vs. quiver representations

- By the Baer-Bondal-Rickard theorem, given a (full,cyclic, strong) exceptional collection on S, the derived category of coherent sheaves $\mathcal{D}(S)$ is isomorphic to the derived category of quiver representations $\mathcal{D}(Q)$:
$$\mathcal{D}(S) \simeq \mathcal{D}(Q)$$

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$$\mathcal{D}(S) \simeq \mathcal{D}(Q)$$

- $\mathcal{D}(S)$ is graded by the Chern vector $\mathrm{ch}(E) \in K(S)$ while $\mathcal{D}(Q)$ is graded by the dimension vector $\vec{N} \in \mathbb{Z}^{Q_0}$. The two are related by

$$ch(E) = -\sum_i N_i \, \mathrm{ch}(E_i^\vee)$$

with overall minus sign such that $N_i > 0$ for large D0-brane charge.

Coherent sheaves and quiver representations

- The Gieseker stability condition on $\mathcal{D}(S)$ translates into a stability condition $\vec{\zeta}$ on Q,

$$\zeta_i = \lambda\, \mathrm{Im}(Z_{\gamma_i}\, \overline{Z_\gamma})\ , \quad \lambda \in \mathbb{R}^+$$

where $Z_\gamma = -\frac{N}{2}J^2 + J \cdot c_1 - \mathrm{ch}_2$ is the central charge in the large volume limit.

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where $Z_\gamma = -\frac{N}{2}J^2 + J \cdot c_1 - \mathrm{ch}_2$ is the central charge in the large volume limit.

- This automatically satisfies $\sum_i N_i \zeta_i = 0$, and yields, for subrepresentations with dimension vector $\vec{N}' \leq \vec{N}$,

$$\begin{aligned}\sum_i N'_i \zeta_i =& \rho \left[N \int_S J \cdot c_1(E') - N' \int_S J \cdot c_1(E) \right] \\ &+ N'\,\mathrm{ch}_2(E) - N\,\mathrm{ch}_2(E')\end{aligned}$$

where $\rho \gg 1$. The first term is the standard difference of slopes.

Coherent sheaves vs. quiver representations

- Under the assignment $(\mathrm{ch}\, E, J) \to (\vec{N}, \vec{\zeta})$, the moduli spaces of semi-stable objects are expected to be isomorphic. In particular, their dimension should match:

$$
\begin{aligned}
d_{\mathbb{C}}(\mathcal{M}^{S}_{\gamma,J}) =& 1 - \chi(E,E) = 1 - \sum_{i,j} N_i\, S_{ij}\, N_j \\
=& \sum_{S_{ij}<0} |S_{ij}| N_i N_j - \sum_{S_{ij}>0} S_{ij}\, N_i N_j - \sum_i N_i^2 + 1
\end{aligned}
$$

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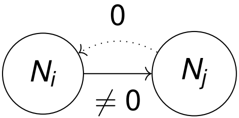
$$= \sum_{S_{ij}<0} |S_{ij}| N_i N_j - \sum_{S_{ij}>0} S_{ij}\, N_i N_j - \sum_i N_i^2 + 1$$

This matches the expected dimension of the quiver moduli space $\mathcal{M}^{Q}_{\vec{N},\vec{\zeta}}$ in the Beilinson branch where $\Phi_{ij\alpha} = 0$ whenever $S_{ij} > 0$.



Coherent sheaves vs. quiver representations

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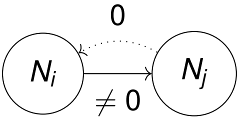
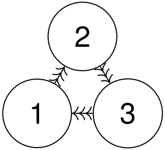


- The Beilinson branch is consistent with $\vec{\zeta}$ only when the slope $\nu_J(E)$ lies in a certain window.

Example: Local $\mathbb{P}^2$



$$W = \sum_{(ijk)\in S_3} \mathrm{sgn}(ijk)\, \Phi^i_{12}\Phi^j_{23}\Phi^k_{31}$$

- The dimension and stability vectors for $\rho = \int_{\mathbb{P}^1} J \gg 1$ are given by

$$\begin{aligned}
\vec{N} &= -\left(\tfrac{3}{2}c_1 + \mathrm{ch}_2 + N, \tfrac{1}{2}c_1 + \mathrm{ch}_2, -\tfrac{1}{2}c_1 + \mathrm{ch}_2\right) \\
\vec{\zeta} &= 3\rho\,(N_2 - N_3, N_3 - N_1, N_1 - N_2) + \left(-\tfrac{N_2+N_3}{2}, \tfrac{N_1+3N_3}{2}, \tfrac{N_1-3N_2}{2}\right)
\end{aligned}$$

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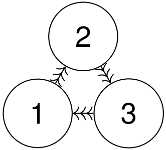

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- In the chamber $\Phi^k_{31} = 0$, the dimensions of $\mathcal{M}^Q$ and $\mathcal{M}^S$ agree,

$$d_{\mathbb{C}} = 3(N_1N_2 + N_2N_3 - N_3N_1) - N_1^2 - N_2^2 - N_3^2 + 1 = c_1^2 - 2N\,\mathrm{ch}_2 - N^2 + 1$$

This requires $\zeta_1 \geq 0, \zeta_3 \leq 0$ hence $-N \leq c_1 \leq 0$.

DT invariants, VW invariants and modularity

- The DT invariants counting semi-stable coherent sheaves on S coincide with the DT invariants counting semi-stable representations of (Q, W). When $J \cdot c_1(S) > 0$, by virtue of vanishing theorems they also coincide with VW invariants.

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- The refined DT/VW invariants are given by the Poincaré polynomial of the moduli space $\mathcal{M} = \mathcal{M}^S_{\gamma,J} = \mathcal{M}^Q_{\vec{N},\vec{\zeta}}$,

$$\Omega(\vec{N}, \vec{\zeta}, y) = \sum_{p=0}^{d_{\mathbb{C}}(\mathcal{M})} (-y)^{2p-d_{\mathbb{C}}(\mathcal{M})} b_p(\mathcal{M})$$

They reduce to $\Omega(\vec{N}, \vec{\zeta}) = (-1)^{d_{\mathbb{C}}(\mathcal{M})} \chi(\mathcal{M})$ as $y \to 1$.

DT invariants, VW invariants and modularity

- When $\vec{N}$ is non-primitive, $\mathcal{M}$ is singular and $b_p(\mathcal{M})$ must be defined using intersection homology. It is useful to introduce the rational DT invariants

$$\bar{\Omega}(\vec{N},\vec{\zeta},y) = \sum_{m|\vec{N}} \frac{y-1/y}{m(y^m-1/y^m)}\,\Omega(\vec{N}/m,\vec{\zeta},y^m),$$

which have simpler behavior under wall-crossing.

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which have simpler behavior under wall-crossing.

- There is yet another notion of 'stack(y) invariants' which I won't need here.

DT invariants and VW invariants

- In a sector with fixed ('t Hooft flux) c_1, the partition function

$$h^S_{N,c_1,J}(\tau, y) = \sum_n \frac{\bar{\Omega}([N, c_1, \frac{1}{2}c_1^2 - n], J, y)}{y - y^{-1}} \, q^{n - \frac{N-1}{2N} c_1^2 - \frac{N\chi(S)}{24}}$$

  is expected to transform as a vector-valued Jacobi form of weight $-\frac{1}{2} b_2(S)$ and index $-\frac{1}{6} K_S^2 (N^3 - N)$.

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$$h^S_{N,c_1,J}(\tau,y) = \sum_n \frac{\bar\Omega([N,c_1,\frac12 c_1^2 - n],J,y)}{y-y^{-1}}\, q^{n-\frac{N-1}{2N}c_1^2-\frac{N\chi(S)}{24}}$$

  is expected to transform as a vector-valued Jacobi form of weight $-\frac12 b_2(S)$ and index $-\frac16 K_S^2(N^3-N)$.
- When $b_2^+(S)=1$, additional non-holomorphic contributions from reducible connections at the boundary of moduli space $\mathcal{M}^S_{\gamma,J}$ are needed to restore modularity. In general $h^S_{N,c_1,J}(\tau,y)$ is a vector-valued *mock* Jacobi form of depth $N-1$, subject to wall-crossing in J.

Vafa Witten 1994; Alexandrov Manschot BP 2019; Dabholkar Putrov Witten 2020

DT invariants and VW invariants

- For $N = 1$, there are no non-holomorphic contributions, nor any dependence on J, and h_1 is truly modular,

$$h_1^S(\tau, y) = \frac{\mathrm{i}}{\theta_1(\tau, y^2)\, \eta(\tau)^{b_2(S)-1}}$$

Göttsche 1990

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Alexandrov 2020 (see previous talk)

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Alexandrov 2020 (see previous talk)

- I shall demonstrate that quivers provide an alternative way of computing these invariants. But first, some more background on wall-crossing and attractor indices is needed.

Wall-crossing and attractor indices

- The DT invariants $\bar{\Omega}(\vec{N}, \vec{\zeta}, y)$ jump on hyperplanes where stable representations become semi-stable. The discontinuity is given by the Konsevitch-Soibelman wall-crossing formula.

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- Physically, the jump can be interpreted as the (dis)appearance of multi-centered black hole bound states.

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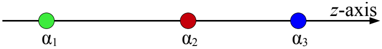
- The KS formula can be derived using localisation in the black hole supersymmetric quantum mechanics. Rational invariants $\bar{\Omega}(\gamma, t)$ arise as effective indices for particles with Boltzmann statistics.



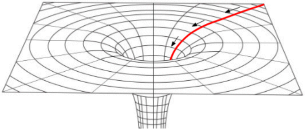
Manschot BP Sen 2010

Wall-crossing and attractor indices

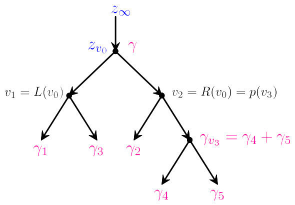
- For fixed $\vec{N}$, there is a particular stability condition

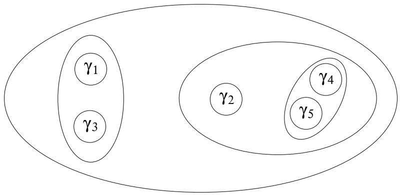
$$\zeta_i^\star(\vec{N}) = -\kappa_{ij} N^j$$

known as 'attractor point' or 'self-stability' where bound states are ruled out. This is analogous to the attractor point for spherically symmetric black holes in $\mathcal{N}=2$ supergravity.



- The full spectrum can be constructed as bound states of these attractor BPS states, labelled by attractor flow trees:





Denef '00; Denef Green Raugas '01; Denef Moore'07

Wall-crossing and attractor indices

- The 'flow tree formula' allows to express $\bar\Omega(\vec N,\vec\zeta,y)$ in terms of the attractor indices $\bar\Omega^\star(\vec N_i,y) := \bar\Omega(\vec N_i,\vec\zeta^\star(\vec N_i),y)$:

$$\boxed{\bar\Omega(\vec N,\vec\zeta,y) = \sum_{\vec N=\sum_{i=1}^n \vec N_i} \frac{g_{\rm tr}(\{\vec N_i,\vec\zeta_i\},y)}{|{\rm Aut}\{\vec N_i\}|} \prod_{i=1}^n \bar\Omega_*(\vec N_i,y,t)}$$

  where $g_{\rm tr}$ is a sum over all possible stable flow trees ending on the leaves $\gamma_1,\dots,\gamma_n$.

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  where $g_{\rm tr}$ is a sum over all possible stable flow trees ending on the leaves $\gamma_1, \dots, \gamma_n$.
- The flow tree formula is purely combinatoric, and does not require integrating the attractor flow !

Alexandrov BP 2018

Wall-crossing and attractor indices

- Remarkably, attractor indices for quivers coming from Fano surfaces have a special property:

  $\boxed{\text{$\Omega_\star(\vec{N},y)=0$ unless $\vec{N}$ is supported on a single node with height 1 (in which case $\Omega_\star=1$) or $\vec{N}\propto\vec{N}_{D0}$ (for a pure D0-brane)}}$

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- To see this, we exhibit a positive quadratic form $\mathcal{Q}(\vec{N})$ and rational coefficients $\lambda_i \in \mathbb{Q}$ such that the expected dimension of the moduli space $\mathcal{M}^Q_{\vec{N},\vec{\zeta}^\star(\vec{N})}$ in the attractor chamber can be written as

$$d^*_{\mathbb{C}} = 1 - \mathcal{Q}(\vec{N}) - \sum_i \lambda_i N_i \zeta_i^\star$$

where $\lambda_i = 0$ or $\mathrm{sgn}(\lambda_i) = \mathrm{sgn}(\zeta_i^\star)$ for all i. The quadratic form is degenerate along $\vec{N}_{D0}$. $\mathcal{Q}(\vec{N})$ is found case-by-case.

Beaujard Manschot BP 2020

Wall-crossing and attractor indices

- Since $\langle \vec{N}_{D0}, \vec{N} \rangle = 0$ for any $\vec{N}$, the flow tree formula does not involve the unknown indices $\Omega_\star(p\vec{N}_{D0})$. Thus it can be used to compute $\bar{\Omega}(\vec{N}, \vec{\zeta}, y)$ for any $(\vec{N}, \vec{\zeta})$!

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- Presumably this micro-structure is revealed as one travels from large volume to the genuine (finite volume) attractor point.

Example: Local $\mathbb{P}^2$

- In the attractor chamber $\vec{\zeta} = \rho \vec{\zeta^\star}$, the expected dimension can be written as

$$d^*_{\mathbb{C}} = 1 - \mathcal{Q}(\vec{N}) + \begin{cases} \frac23 N_3 \zeta^\star_3 - \frac23 N_1 \zeta^\star_1 & \zeta^\star_1 \geq 0, \zeta^\star_3 \leq 0 \\ \frac23 N_1 \zeta^\star_1 - \frac23 N_2 \zeta^\star_2 & \zeta^\star_2 \geq 0, \zeta^\star_1 \leq 0 \\ \frac23 N_2 \zeta^\star_2 - \frac23 N_3 \zeta^\star_3 & \zeta^\star_3 \geq 0, \zeta^\star_2 \leq 0 \end{cases}$$

$$\mathcal{Q}(\vec{N}) = \frac12 (N_1 - N_2)^2 + \frac12 (N_2 - N_3)^2 + \frac12 (N_3 - N_1)^2$$

hence $d^*_{\mathbb{C}} < 0$ unless $\vec{N} \in \{(1,0,0), (0,1,0), (0,0,1), (p,p,p)\}$. Hence $\Omega_\star(\vec{N}) = 0$ except in those cases.

Example: Local $\mathbb{P}^2$

- Using the flow tree formula with $\Omega_\star = 0$, we get results consistent with other approaches:

$[N; c_1; c_2]$	$\vec{N}$	$\Omega_c(\vec{N})$
$[1;0;2]$	$(1,2,2)$	$y^4+2y^2+3+\dots$
$[1;0;3]$	$(2,3,3)$	$y^6+2y^4+5y^2+6+\dots$
$[2;0;3]$	$(1,3,3)$	$-y^9-2y^7-4y^5-6y^3-6y-\dots$
$[2;-1;2]$	$(1,2,1)$	$y^4+2y^2+3+\dots$
$[2;-1;3]$	$(2,3,2)$	$y^8+2y^6+6y^4+9y^2+12+\dots$
$[3;-1;3]$	$(1,3,2)$	$y^8+2y^6+5y^4+8y^2+10+\dots$
$[4;-2;4]$	$(1,3,1)$	$y^5+y^3+y+\dots$
$[4;-2;5]$	$(2,4,2)$	$-y^{13}-2y^{11}-6y^9-10y^7-17y^5-21y^3-24y-.$

Attractor indices vs. single-centered indices

- While there are no genuine bound states at the attractor point $\vec{\zeta} = \vec{\zeta}^{\star}(\vec{N})$, from the Coulomb branch prospective there can still be contributions from 'scaling solutions', where several centers approach at arbitrary small distance.

Bena Wang Warner 2007; de Boer El-Showk Messamah Den Bleeken 2008

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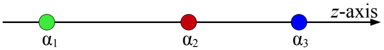
- The Coulomb branch formula gives a (conjectural) general prescription for removing these scaling contributions. It expresses $\bar{\Omega}(\vec{N}, \vec{\zeta}, y)$ in terms of 'single-centered' or 'pure-Higgs' indices :

$$\boxed{\bar{\Omega}(\vec{N}, \vec{\zeta}, y) = \sum_{\vec{N} = \sum_{i=1}^{n} \vec{N}_i} \frac{g_{\rm tr}(\{\vec{N}_i, \vec{\zeta}_i\}, y)}{|\mathrm{Aut}\{\vec{N}_i\}|} \prod_{i=1}^{n} \bar{\Omega}_S(\vec{N}_i, y, t)}$$

Denef Moore 2007, Manschot BP Sen 2011, Lee Yang Yi 2012

Attractor indices and pure Higgs indices

- The Coulomb index $g_{\rm tr}(\{\vec{N}_i, \vec{\zeta}_i\}, y)$ is computed by localization as in the KS formula, with an ad hoc prescription for incorporating contributions from cusps in phase space, which ensures that the result is a symmetric Laurent polynomial in y.



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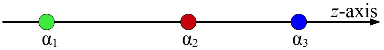

- The indices $\Omega_{\rm S}(\vec{N}_i)$ do not depend on $\vec{\zeta}$, and are conjectured to count harmonic forms supported on the middle cohomology of the quiver moduli space.

Attractor indices vs. single-centered indices

- Applying this formula, one finds evidence that, similar to $\Omega_\star$,

  $\Omega_S(\vec{N}, y) = 0$ unless $\vec{N}$ is supported on a single node with height 1 (in which case $\Omega_S = 1$) or $\vec{N} \propto \vec{N}_{D0}$ (for a pure D0-brane)

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- In particular, $\Omega_S(\vec{N}, y) = \Omega_\star(\vec{N}, y)$ unless $\vec{N} \propto \vec{N}_{D0}$. This is surprising since scaling solutions do exist classically. However, for quivers associated to Fano surfaces they appear to be removed by quantum effects, under the 'minimal modification hypothesis'.

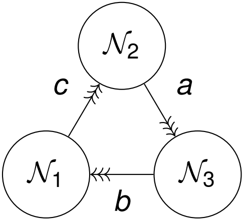
Example 2: Three-block collections

- For $\mathbb{F}_0$ and all del Pezzo surfaces dP_k with $k \neq 1, 2$, Karpov and Nogin have constructed strong cyclic exceptional collections with three-blocks structure with $\alpha + \beta + \gamma = \chi(S)$

$$S = \left(\begin{array}{c|c|c} 1_\alpha & -c & b \\ \hline & 1_\beta & -a \\ \hline & & 1_\gamma \end{array} \right) , \quad \kappa = \left(\begin{array}{c|c|c} 0_\alpha & c & -b \\ -c & 0_\beta & a \\ b & -a & 0_\gamma \end{array} \right) ,$$

where $\alpha x^2 + \beta y^2 + \gamma z^2 = xyz \sqrt{K_S^2 \, \alpha\beta\gamma}$

$$\begin{aligned} a &= \alpha x \, K' \\ b &= \beta y \, K' \\ c &= \gamma z \, K' \\ K' &= \sqrt{K_S^2/(\alpha\beta\gamma)} \end{aligned}$$



$$\mathcal{A} = a^2 \beta\gamma \, , \quad \mathcal{B} = b^2 \alpha\gamma \, , \quad \mathcal{C} = c^2 \alpha\beta , \quad \mathcal{A} + \mathcal{B} + \mathcal{C} = \sqrt{\mathcal{ABC}}$$

Example 2: Three-block collections

S	K_S^2	$\#$	(α,β,γ)	(x,y,z)	(a,b,c)	$(\mathcal{A},\mathcal{B},\mathcal{C})$
$\mathbb{P}_2$	9	(1)	$(1,1,1)$	$(1,1,1)$	$(3,3,3)$	$(9,9,9)$
$\mathbb{P}_1\times\mathbb{P}_1$	8	(2)	$(1,2,1)$	$(1,1,1)$	$(2,4,2)$	$(8,16,8)$
dP_3	6	(3)	$(1,2,3)$	$(1,1,1)$	$(1,2,3)$	$(6,12,18)$
dP_4	5	(4)	$(1,1,5)$	$(1,2,1)$	$(1,2,5)$	$(5,20,25)$
dP_5	4	(5)	$(2,2,4)$	$(1,1,1)$	$(1,1,2)$	$(8,8,16)$
dP_6	3	(6.1)	$(3,3,3)$	$(1,1,1)$	$(1,1,1)$	$(9,9,9)$
		(6.2)	$(1,2,6)$	$(2,1,1)$	$(1,1,3)$	$(12,6,18)$
dP_7	2	(7.1)	$(1,1,8)$	$(2,2,1)$	$(1,1,4)$	$(8,8,16)$
		(7.2)	$(2,4,4)$	$(2,1,1)$	$(1,1,1)$	$(16,8,8)$
		(7.3)	$(1,3,6)$	$(3,1,1)$	$(1,1,2)$	$(18,6,12)$
dP_8	1	(8.1)	$(1,1,9)$	$(3,3,1)$	$(1,1,3)$	$(9,9,9)$
		(8.2)	$(1,2,8)$	$(4,2,1)$	$(1,1,2)$	$(16,8,8)$
		(8.3)	$(2,3,6)$	$(3,2,1)$	$(1,1,1)$	$(6,12,18)$
		(8.4)	$(1,5,5)$	$(5,2,1)$	$(1,2,1)$	$(25,20,5)$

Example 2: Three-block collections

- In the Beilinson chamber where $\Phi_{31,\alpha} = 0$, the expected dimension of $\mathcal{M}^Q$ agrees with that of $\mathcal{M}^S$,

$$d_{\mathbb{C}} = c\mathcal{N}_1\mathcal{N}_2 + a\mathcal{N}_2\mathcal{N}_3 - b\mathcal{N}_1\mathcal{N}_3 - \sum_i N_i^2 + 1$$

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- In the attractor chamber when $\varsigma_3^\star \le 0, \varsigma_1^\star \ge 0$, one has instead

$$\begin{aligned} d_{\mathbb{C}}^* &= 1 - \mathcal{Q}(\vec{N}) + \tfrac{2\mathcal{A}}{\mathcal{A}+\mathcal{B}+\mathcal{C}}\mathcal{N}_3\varsigma_3^\star - \tfrac{2\mathcal{C}}{\mathcal{A}+\mathcal{B}+\mathcal{C}}\mathcal{N}_1\varsigma_1^\star \\ \mathcal{Q} &= \sum_{i=1}^r N_i^2 - \tfrac{\mathcal{A}+\mathcal{B}-\mathcal{C}}{\mathcal{A}+\mathcal{B}+\mathcal{C}}c\,\mathcal{N}_1\mathcal{N}_2 - \tfrac{\mathcal{B}+\mathcal{C}-\mathcal{A}}{\mathcal{A}+\mathcal{B}+\mathcal{C}}a\,\mathcal{N}_2\mathcal{N}_3 - \tfrac{\mathcal{C}+\mathcal{A}-\mathcal{B}}{\mathcal{A}+\mathcal{B}+\mathcal{C}}b\,\mathcal{N}_3\mathcal{N}_1 \end{aligned}$$

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- $\mathcal{Q}$ is the positive quadratic form, degenerate along the direction $\vec{N}_{D0} = (x, \dots; y, \dots; z, \dots)$. Hence $\Omega_\star(\vec{N}) = 0$ except for simple representations or for D0-branes. Using flow tree formula we get agreement with computations by other methods.

Example 3: Local toric surfaces

- Smooth toric surfaces are described by a toric fan spanned by vectors $v_1, \dots, v_r \in \mathbb{Z}^2$ forming a convex polygon. Each vector corresponds to a toric divisor D_i, subject to linear equivalences

$$\sum_i (u, v_i)\, D_i = 0$$

The intersection $D_i \cdot D_j$ vanishes unless $i - j \in \{-1, 0, 1\} \pmod r$, and $D_i \cdot D_{i+1} = 1$, $D_i \cdot D_i = a_i$ where a_i are determined by

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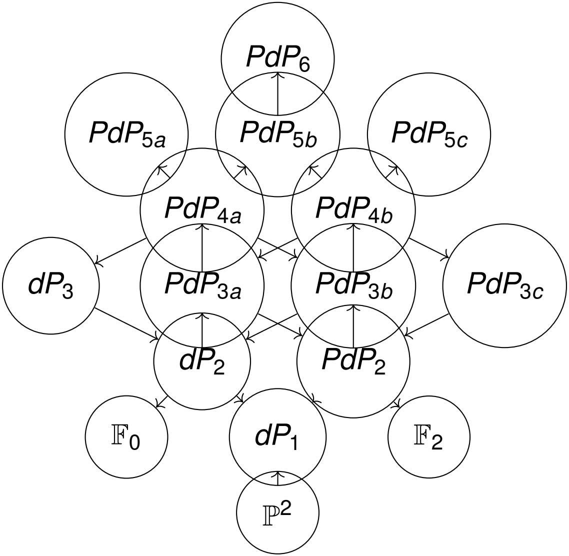
- Fano surfaces have $a_i \geq -1$ for all i, weak Fano surfaces have $a_i \geq -2$. There are 5 smooth toric Fano surfaces, and 11 weak Fano, related by blow-up/down.

Example 3: Local toric surfaces



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- Toric Fano surfaces admit strongly cyclic exceptional collections.

$$\mathcal{O}(0), \mathcal{O}(D_1), \mathcal{O}(D_1 + D_2), \dots, \mathcal{O}(D_1 + \dots + D_{r-1})$$

For weak Fano surfaces, this is not strongly exceptional but there is an alternative choice $D_i \to \tilde{D}_i$.

Hille Perling 2011

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- In all these examples, the BPS indices computed using the attractor flow formula are in agreement with the result form the blow-up and wall-crossing formulae.

Outline

Summary and Outlook

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- For $S = \mathbb{P}^2$, BPS indices can also be computed using scattering diagrams. Are those equivalent to the attractor flow trees ?

Gross Pandharipande Siebert 2010; Bridgeland 2017; Bousseau (2019)

Summary and Outlook

- An important consequence is that generating functions of quiver indices $Z_{\vec{N}_0}(\tau) \sim \sum_n \Omega(\vec{N}_0 + n\vec{N}_{D0})q^n$ should have (mock) modular properties.

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- Modularity should follow from some vertex operator algebra acting on the cohomology of quiver varieties, generalizing Nakajima's action of $\hat{A}_1$ on the Hilbert scheme of points. When S is toric, a natural candidate is the Quiver Yangian.

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- It would be interesting to compute BPS indices in compact CY threefolds, where non-trivial single-centered black holes are expected to occur !

Thank you for your attention, and mind the wall !