From attractor indices to single-centered indices

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based on arXiv:2103.03205 with Guillaume Beaujard, Swapnamay Mondal, 2109.nnnnn with Pierre Descombes

and earlier work with Jan Manschot, Ashoke Sen and Sergei Alexandrov.

- Precision counting of BPS black hole microstates is an important challenge, both for physics (probing the consistency of string theory as a model of quantum gravity) and for mathematics (uncovering new topological invariants of Calabi-Yau threefolds).
- The net number of BPS states with fixed electro-magnetic charge γ, called BPS index $\Omega(\gamma, z)$, is known exactly in most string backgrounds with $\mathcal{N} \geq 4$ supersymmetry in $3+1$ dimensions. This is not yet so in $\mathcal{N} = 2$ vacua such as type II on a generic CY3.
- The main difficulty is that $\Omega(\gamma, z)$ depends on the moduli z in an intricate way, due to wall-crossing phenomena associated to BPS bound states with any number of constituents.

Introduction

- The attractor mechanism selects a particular value z_γ of the moduli, known as the attractor or self-stability chamber, where most multi-centered bound states (in particular, all two-centered bound states), have decayed.
- The attractor indices $\Omega_\star(\gamma) = \Omega(\gamma, z_\gamma)$ determine the index $\Omega(\gamma, z)$ for any z through the attractor flow tree formula. For D4-D2-D0 charges at large volume, they possess interesting (mock) modular properties. *[Alexandrov Banerjee Manschot BP, 2016-19]*
- In general, at the attractor point there often exist multi-centered scaling solutions, where the centers can become arbitrarily close to each other, which contribute to the attractor index $\Omega_\star(\gamma)$.

Introduction

- There is a conjectural prescription, known as the Coulomb branch formula, for subtracting the contributions of scaling solutions and extracting the so called single-centered index $\Omega_S(\gamma)$ (aka pure-Higgs indices). However the latter does not have a first principle definition yet.
- After reviewing aspects of multi-centered solutions, I will present some recent progress in proving the Coulomb branch formula in the context of quiver quantum mechanics, using supersymmetric localization.

Single-centered black holes in $\mathcal{N}=2$ supergravity

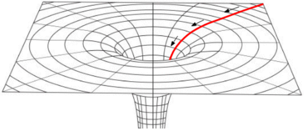
- Recall that $\mathcal{N}=2$ supergravity admits supersymmetric, spherically symmetric solutions corresponding to a BPS black hole of charge γ, with metric

$$ds^2 = -e^{2U(r)}\mathrm{d}t^2 + e^{-2U(r)}(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2)$$

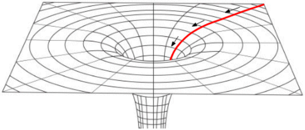
with suitable flux and radial profile for the vector multiplet scalars

$$r^2\frac{\mathrm{d}U}{\mathrm{d}r} = e^U\,|Z_\gamma| \quad , \quad r^2\frac{\mathrm{d}z^a}{\mathrm{d}r} = 2\,e^U g^{a\bar{b}}\partial_{\bar{z}}|Z_\gamma|$$

where $Z_\gamma(z) = e^{\mathcal{K}/2}(q_\Lambda X^\Lambda(z) - p^\Lambda F_\Lambda(z))$ is the central charge.



Single-centered black holes in $\mathcal{N} = 2$ supergravity



- As $r \to 0$, the moduli $z(r)$ are attracted to a critical point z_γ of $|Z_\gamma|$, independent of the moduli z_∞ at spatial infinity. The geometry interpolates from $\mathbb{R}^{3,1}$ at $r = \infty$ to $AdS_2 \times S^2$ at $r = 0$.

Ferrara Kallosh Strominger 1995

- The Bekenstein-Hawking entropy is $S_{BH} = \pi |Z_\gamma(z_\gamma)|^2$, while the mass saturates the BPS bound, $\mathcal{M} = |Z_\gamma(z_\infty)|$.
- Since the solution is static, $\vec{J} = 0$ classically. This remains true quantum mechanically *[Sen 2009]*.

Multi-centered black holes in $\mathcal{N} = 2$ supergravity

- In addition, there may also exist multi-centered supersymmetric solutions. Near each center they reduce to the previous solution with charge γ_i. Near ∞ they look like a black hole of charge $\gamma = \sum_{i=1}^{n} \gamma_i$ and angular momentum $\vec{J} = \frac{1}{2} \sum_{i<j} \kappa_{ij} \frac{\vec{r}_{ij}}{r_{ij}}$, where $\kappa_{ij} = \langle \gamma_i, \gamma_j \rangle$ is the Dirac pairing.
- The distances $r_{ij} = |\vec{r}_i - \vec{r}_j|$ are constrained by Denef's equations

$$\forall i = 1 \dots n, \qquad \sum_{j \neq i} \frac{\kappa_{ij}}{r_{ij}} = \zeta_i$$

where $\zeta_i = R\, \mathrm{Im}[e^{-\mathrm{i}\varphi} Z_{\gamma_i}(z_\infty)]$ where $\varphi = \arg Z_\gamma(z_\infty)$ and $R > 0$ (hence $\sum_i \zeta_i = 0$). *One should also check the absence of closed timelike curves.*

- At the attractor point $z_\infty = z_\gamma$, $\zeta_i = -R \sum_j \kappa_{ij}$.

Multi-centered black holes in $\mathcal{N} = 2$ supergravity

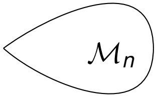
- If not empty, the space $\mathcal{M}_n(\{\gamma_i, \zeta_i\})$ of solutions mod translations has dimension $3n - (n-1) - 3 = 2n - 2$. It carries a symplectic two-form $\omega = \frac{1}{2}\sum_{i<j} \kappa_{ij} \sin\theta_{ij} \mathrm{d}\theta_{ij} \mathrm{d}\phi_{ij}$ such that $O(3)$ rotations are generated by the moment map $\vec{J}$.

de Boer El Showk Messamah van den Bleeken 2008

- For example, $\mathcal{M}_2$ is empty when $\kappa_{12}\zeta_1 < 0$ (in particular at the attractor point $\zeta_1 = -R\kappa_{12}$). If $\kappa_{12}\zeta_1 > 0$, $\mathcal{M}_2$ is a two-sphere with κ_{12} units of flux, corresponding to the dipole orientation. *Wall-crossing takes place when Z_{γ_1} and Z_{γ_2} become aligned.*
- Easy fact: If one can split the centers into two sets $S \cup \bar{S}$ such that $\kappa_{ij} > 0$ for all $i \in S, j \in \bar{S}$, then $\mathcal{M}_n$ is compact away from walls of marginal stability, and empty at the attractor point $\zeta_i = -R\sum_j \kappa_{ij}$.

Scaling solutions

In general, $\mathcal{M}_n$ can be non-compact due to some scaling regions where the centers become arbitrarily close to each other.



- The simplest example occurs for $n = 3$ and $\kappa_{12}, \kappa_{23}, \kappa_{31}$ of same sign (say positive) and satisfy the triangular inequalities

$$\kappa_{12} \leq \kappa_{23} + \kappa_{31}, \quad \kappa_{23} \leq \kappa_{31} + \kappa_{12}, \quad \kappa_{31} \leq \kappa_{12} + \kappa_{23}$$

  There is a one-parameter family of solutions such that $r_{ij} \sim \lambda \kappa_{ij} + \mathcal{O}(\lambda)$ as $\lambda \to 0$, irrespective of the parameters ζ_i. In particular, such solutions continue to exist at the attractor point.
- Since $\vec{J} = \frac{1}{2} \sum_i \zeta_i \vec{r}_i$ and $\sum_i \zeta_i = 0$, such solutions have $\vec{J} \simeq 0$ and become undistinguishable from single-centered black holes.

Existence of scaling solutions

- The existence of scaling solutions can be analyzed by setting $\zeta_i = 0$ in Denef's equations. By exploiting the geometric inequalities on the edges of n-gons, one finds necessary conditions generalizing the ones above to any $n \geq 4$. *[Beaujard Mondal BP 2021; Descombes and BP, to appear]*
- In order to state the conditions, let us introduce a quiver Q with vertices $Q_0 = \{v_i, i = 1 \dots n\}$ and with one arrow $v_i \to v_j$ whenever $\kappa_{ij} > 0$. Let Q_1 be the set of arrows and Q_2 the set of simple oriented cycles. We define a cut as a subset $I \subset Q_1$ such that each cycle $C \in Q_2$ contains one and only one arrow.
- A necessary condition for existence of a scaling region where all n centers coalesce is that Q is strongly connected, and for any cut I,

$$\sum_{(i \to j) \in I} \kappa_{ij} \leq \sum_{(i \to j) \notin I} \kappa_{ij}$$

Existence of scaling solutions

- For example, for a cyclic quiver $v_1 \to v_2 \to \cdots \to v_n \to v_1$, this requires

$$\kappa_{n1} \leq \kappa_{12} + \kappa_{23} + \cdots + \kappa_{n-1,n}$$

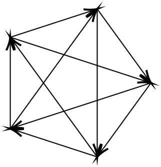
  and cyclic permutations thereof. In that case, this follows trivially from $r_{i,i+1} = \lambda \kappa_{i,i+1} + \mathcal{O}(\lambda^2)$.

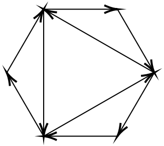
- In cases where the quiver admits a cut and a simple oriented cycle $v_1 \to v_2 \to \cdots \to v_n \to v_1$ running through all centers, the condition agrees with the conjecture in our work with Guillaume and Swapno:

$$\sum_{i<j} \kappa_{ij} \geq 0 \qquad \text{and cyclic perm.}$$

Existence of scaling solutions

- In cases where no cuts exist (which happens when Q admits no R-charge), one has similar necessary conditions using a notion of weak cut.





- Remarkably, the same necessary conditions apply for the existence of multi-centered solutions at the attractor point, and for the existence of stable representations on the Higgs branch at the attractor point ! *[P. Descombes and BP, to appear]*

Quantizing the space of multi-centered solutions

- In the absence of scaling regions, the centers become far separated as $R \to 0$, so we expect that the internal degrees of freedom of each black hole decouple from the configurational degrees of freedom.
- The latter are described classically by the BPS phase space $(\mathcal{M}_n, \omega)$. Quantum mechanically, they correspond to zero-modes of the Dirac operator for a charged particle on $\mathcal{M}_n$ with flux $F = \omega$.
- Hence we expect that the total index can be written

$$\Omega(\gamma, z) \stackrel{?}{=} \sum_{\gamma=\sum \gamma_i} g(\{\gamma_i, \zeta_i\}) \prod_i \Omega_\star(\gamma_i)$$

where $\Omega_\star(\gamma_i)$ are the attractor indices and $g(\{\gamma_i, \zeta_i\})$ is the index of the Dirac operator on $(\mathcal{M}_n, \omega)$. This is broadly correct but naive...

Quantizing the space of multi-centered solutions

- First, when some of the charges γ_i coincide, we must enforce Bose-Fermi statistics. As argued in *[Manschot BP Sen 2010]*, one can use the simpler Boltzmann statistics, provided one replaces the BPS index by the rational index $\bar{\Omega}(\gamma, z) = \sum_{m|\gamma} \frac{1}{m^2} \Omega(\gamma/m, z)$. Hence a better guess is

$$\bar{\Omega}(\gamma, z) \stackrel{?}{=} \sum_{\gamma=\sum \gamma_i} \frac{g(\{\gamma_i, \zeta_i\})}{|\mathrm{Aut}(\{\gamma_i\})|} \prod_i \bar{\Omega}_\star(\gamma_i)$$

- Second, in the presence of scaling regions, the phase space $\mathcal{M}_n$ is non-compact and the Dirac operator is not self-adjoint (unless one specifies appropriate conditions at the boundary). Moreover it is unclear whether each center contributes $\bar{\Omega}_\star(\gamma_i)$ or $\bar{\Omega}_S(\gamma_i)$.

Flow tree formula

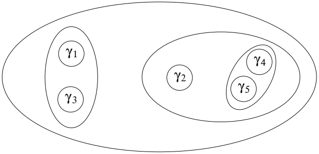
- One way to proceed is to insist that each center contributes $\bar{\Omega}_\star(\gamma_i)$. The prefactor $g(\{\gamma_i, \zeta_i\})$ then arises as a sequence of wall-crossings, leading to the attractor flow tree formula

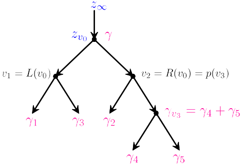
$$\bar{\Omega}(\gamma, z) = \sum_{\gamma=\sum \gamma_i} \frac{g_{\rm tr}(\{\gamma_i, \zeta_i\})}{|{\rm Aut}(\{\gamma_i\})|} \prod_i \bar{\Omega}_\star(\gamma_i)$$

where $g_{\rm tr}(\{\gamma_i, \zeta_i\})$ is given by a sum over stable flow trees:

$$g_{\rm tr}(\{\gamma_i, \zeta_i\}) = \sum_T \prod_{v \in V_T} \langle \gamma_{L(v)}, \gamma_{R(v)} \rangle$$

corresponding to multi-centered solutions with nested structure:





Flow tree formula

- Each vertex v carries different ζ_v, obtained from the value at the parent vertex $\zeta_{p(v)}$ by evolving the attractor flow equations until it crosses the wall of marginal stability for $\gamma_{p(v)} \to \gamma_{L(p(v))} + \gamma_{R(p(v))}$. Up to rescaling, one has

$$\zeta_v = \zeta_{p(v)} + \frac{\langle \gamma_v, - \rangle}{\langle \gamma_{L(v)}, \gamma_{R(v)} \rangle} \zeta_{p(v)}(\gamma_{L(v)})$$

  The flow tree contributes only if $\langle \gamma_{L(v)}, \gamma_{R(v)} \rangle \times \zeta_v(\gamma_{L(v)}) > 0$ for all vertices *[Manschot'10, Alexandrov BP '18;* `CoulombHiggs` *Mathematica package]*

- *For this we need to assume that the flow never crosses fake walls where the central charges Z_{γ_R} and Z_{γ_L} are anti-aligned; moreover in case some of the γ_i's coincide one needs to perturb ζ_∞ at the root vertex to avoid non-primitive wall-crossings.*

- This formula is now a mathematical theorem in the context of DT invariants for quivers. *[Argüz Bousseau '21]*

Coulomb branch formula

- The other way is to include only the bulk contribution to Dirac index $g(\{\gamma_i, \zeta_i\})$, but assign the boundary contributions to the indices carried by each center:

$$\bar{\Omega}(\gamma, z) = \sum_{\gamma=\sum \gamma_i} \frac{g_{\rm C}(\{\gamma_i, \zeta_i\})}{|{\rm Aut}(\{\gamma_i\})|} \prod_i \bar{\Omega}_{\rm tot}(\gamma_i)$$

Manschot BP Sen 2011; `CoulombHiggs` *Mathematica package*

where $\bar{\Omega}_{\rm tot}(\gamma_i)$ includes contributions from single-centered black holes and scaling configurations thereof:

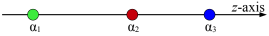
$$\Omega_{\rm tot}(\alpha) = \Omega_{\rm S}(\alpha) + \sum_{m\geq 3} \sum_{\alpha=\sum_{i=1}^m \beta_i} H(\{\beta_i\}) \prod_{i=1}^m \Omega_{\rm S}(\beta_i)$$

- The coefficient $H(\{\beta_i\})$ comes from the boundary contribution to the Dirac index on the non-compact phase space $\mathcal{M}_m(\{\beta_i, \zeta_i\})$
- Evaluating the Coulomb branch formula at z_γ allows to express $\Omega_\star(\gamma)$ in terms of $\Omega_{\rm S}(\gamma)$.

Coulomb index

- The bulk contribution to the Dirac index can be computed by localization with respect to rotations J_3. The fixed points correspond to collinear configurations satisfying a 1D version of Denef's equations *[Manschot BP Sen 2010]*:

$$\forall i = 1 \dots n, \qquad \sum_{j \neq i} \frac{\kappa_{ij}}{|x_i - x_j|} = \zeta_i$$



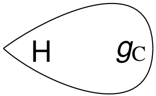
- Solutions (when they exist) are isolated, and labelled by the order σ in which centers appear along the axis. Each such solution contributes $\pm y^{2J_3}$ to the equivariant Dirac index:

$$g_{\rm C}(\{\gamma_i, \zeta_i\}, y) = \frac{(-1)^{n-1+\sum_{i<j}\kappa_{ij}}}{(y-1/y)^{n-1}} \sum_{\sigma \in S_n} \epsilon(\sigma)\, y^{\sum_{i<j}\kappa_{\sigma(i)\sigma(j)}}$$

- When $\mathcal{M}_n$ is compact, this produces a symmetric Laurent coefficient with integer coefficients, with a smooth limit as $y \to 1$.

Coulomb index (continued)

- In presence of scaling regions, $g_{\rm C}(\{\gamma_i, \zeta_i\}, y)$ is singular in the limit $y \to 1$. We can repair this by adjusting $H(\{\beta_i\}, y)$. Since scaling solutions have $\vec{J} \sim 0$, we postulate that $H(\{\beta_i\}, y)$ introduces contributions y^{2J_3} with smallest possible J_3.



- Specifically, $H(\{\beta_i\}, y)$ is determined recursively by requiring
  1. i) $H(\{\beta_i\}, y)$ is invariant under $y \to 1/y$
  2. ii) $H(\{\beta_i\}, y) \to 0$ as $y \to \infty$
  3. iii) in the expression for $\Omega(\gamma, z)$ in terms of single-centered invariants, the coefficient of the monomial $\prod_i \Omega_{\rm S}(\beta_i)$ – denoted by $\hat{g}(\{\beta_i, \zeta_i\}, y)$ – is a symmetric Laurent polynomial in y.
- Since $H(\{\beta_i\}, y)$ does not depend on ζ, it can be evaluated at any point, e.g. at the attractor point.

Coulomb index (continued)

- As an example, for $n=3$ centers satisfying triangular inequalities, the Coulomb branch formula for $\gamma=\gamma_1+\gamma_2+\gamma_3$ at the attractor point gives

$$\Omega_\star(\gamma) = \Omega_{\rm S}(\gamma) + [g_{\rm C}^\star + H]\, \Omega_{\rm S}(\gamma_1)\Omega_{\rm S}(\gamma_2)\Omega_{\rm S}(\gamma_3)$$

  where (assuming that $\kappa_{12} > \kappa_{23}, \kappa_{31}$ and $\kappa_{12}+\kappa_{23}+\kappa_{31}$ is even)

$$g_{\rm C}^\star = \frac{y^{\kappa_{23}+\kappa_{31}-\kappa_{12}} + y^{-\kappa_{23}-\kappa_{31}+\kappa_{12}}}{(y-1/y)^2}, \quad H = \frac{-2}{(y-1/y)^2}$$

  such that $g_{\rm C}^\star + H$ is a symmetric Laurent polynomial, with a finite, integer limit $\frac14(\kappa_{23}+\kappa_{31}-\kappa_{12})^2$ as $y\to 1$.
- Our aim will be to derive this apparently ad hoc prescription in the framework of quiver quantum mechanics.

Quiver quantum mechanics

- Consider a SUSY quantum mechanics in $0+1$ dimensions, obtained by reducing $\mathcal{N}=1$ gauge theory in $3+1$ dimension, with matter content encoded in a quiver: each node $\ell = 1...K$ represents a $U(N_\ell)$ vector multiplet (VM), each arrow $k \to \ell$ represents a chiral multiplet (CM) $\Phi^\alpha_{k,\ell}$ in $(N_\ell, \bar{N}_k)$ representation of $U(N_\ell) \times U(N_k)$.
- In addition, one must specify Fayet-Iliopoulos terms $\zeta_\ell \in \mathbb{R}$ and (in presence of closed oriented loops) a superpotential $W(\Phi)$. We assume that one can assign R-charge $R_{k\ell}$ to $\Phi^\alpha_{k,\ell}$ such that W has R-charge 2.
- The quantum mechanics admits two branches: the Higgs branch, where the gauge group $G = \prod_{\ell=1}^K U(N_\ell)$ is completely broken by vevs of chiral multiplet scalars, and the Coulomb branch where G is broken to its Cartan torus by vevs of vector multiplet scalars.

Quiver quantum mechanics: Higgs branch

- On the Higgs branch, VM are massive and can be integrated out. Classically, SUSY vacua $\mathcal{M}_H(\gamma,\zeta)$ correspond to solutions of the F-term and D-term equations modulo the action of G,

$$\forall \ell : \sum_{\gamma_{\ell k}>0} \Phi^*_{\ell k}\, T^a\, \Phi_{\ell k} - \sum_{\gamma_{k\ell}>0} \Phi^*_{k\ell}\, T^a\, \Phi_{k\ell} = \zeta_\ell\, \mathrm{Tr}(T^a)$$

$$\forall k,\ell,\alpha : \ \partial_{\Phi_{k\ell,\alpha}} W = 0$$

- Mathematically, $\mathcal{M}_H$ is the moduli space $\mathcal{M}_H(\gamma,\zeta)$ of stable quiver representations with dimension vector $\gamma = (N_1,\dots,N_K)$ and stability condition ζ.

Quiver quantum mechanics: Higgs branch

- BPS states correspond to Dolbeault cohomology classes of degree (p,q) on in $\mathcal{M}_H(\gamma,\zeta)$, counted by the Hodge polynomial

$$\Omega(\gamma,y,t,\zeta) = \sum_{p,q=0}^{2d} h_{p,q}(\mathcal{M}_H(\gamma,\zeta))\,(-y)^{p+q-d}\,t^{p-q}$$

  The fugacity y keeps track of angular momentum J_3^L, while t is conjugate to J_3^R inside R-symmetry group $SU(2)_L \times SU(2)_R$.

- The refined BPS index is the special value at $t = 1/y$, known as χ_{y^2}-genus. When Dolbeault cohomology is supported in degree $p = q$, it coincides with the Poincaré polynomial. In either case, it reduces to the Euler number in the unrefined limit $y \to 1$.

Quiver quantum mechanics: Coulomb branch

- On the Coulomb branch, after integrating out CM and off-diagonal VM, the remaining VM scalars $\vec{r}_i$ for the Cartan torus satisfy Denef's equations for $n = \sum_{\ell=1}^{K} N_\ell$ centers,

$$\forall i = 1 \dots n, \qquad \sum_{j \neq i} \frac{\kappa_{ij}}{r_{ij}} = \zeta_i$$

  where κ_{ij} is the (signed) number of arrows $k \to \ell$ whenever $i \in U(k), j \in U(\ell)$ (or zero when $k = \ell$).
- When the phase space has no scaling regions, the distances r_{ij} are bounded from below and it is legitimate to integrate out the CM and off-diagonal VM. The equivariant Dirac index of $(\mathcal{M}_n, \omega)$ is then expected to agree with the χ_{y^2}-genus of the Higgs branch.
- In the presence of scaling solutions, one may hope to restore agreement for appropriate values of single-centered indices $\Omega_{\rm S}(\gamma)$.

Manschot BP Sen 2012, Lee Wang Yi 2012

Witten index from localization

- Using supersymmetric localization, one may reduce the functional integral computing the Witten index (i.e. the χ_{y^2}-genus of the Higgs branch) to a finite dimensional integral. For Abelian quivers,

$$\Omega(\gamma,\zeta) = \int \left(\frac{\beta^2}{8\pi^3} \mathrm{d}u \mathrm{d}\bar{u} \mathrm{d}D \right)^{n-1} g(u,D)\, \det h(u,D)\, e^{-\beta S(D,\zeta)}$$

Benini Eager Hori Tachikawa 2013; Hori Kim Yi 2014

where $u_i = \frac{\beta}{2\pi}(A_i - \mathrm{i} x_i)$ are the complexified gauge fields (subject to $\sum_i u_i = 0$), D_i are the auxiliary fields with action

$$S(D) = \frac{1}{2e^2} \sum_{i=1}^{n} D_i^2 - \mathrm{i} \sum_{i=1}^{n} \zeta_i\, D_i\ ,$$

and $\det h(u,D)$ comes from saturating fermionic zero-modes.

Witten index from localization

- $g(u, D)$ is a one-loop fluctuation determinant (with $y = e^{\mathrm{i}\pi z}$):

$$g(u,D) = (\sin \pi z)^{n-1} \prod_{i\to j} \left[\prod_{m\in\mathbb{Z}} \frac{\left(m+\bar{u}_i-\bar{u}'_j+\frac{1}{2}R_{ij}\,\bar{z}\right)\left(m+u_i-u'_j+(\frac{1}{2}R_{ij}-1)z\right)}{|m+u_i-u'_j+\frac{1}{2}R_{ij}\,z|^2-\frac{\mathrm{i}\beta^2}{4\pi^2}(D_i-D'_j)} \right]^{\kappa_{ij}}$$

- Upon using the key identity

$$\partial_{\bar{u}_i} g(u,D) = -\frac{\mathrm{i}\beta^2}{4\pi^2}\, h_{ij}(u,D)\, D^j\, g(u,D)$$

the integral over $u, \bar{u}$ can be cast into a contour integral in the u-plane, and the integral over D evaluated by computing the residue at $D = 0$. This leads to the Jeffrey-Kirwan residue formula for the index. *[Hori Kim Yi 2014]*

- Instead, we shall perform the integral using saddle point methods, which are exact as $e \to 0, \beta \to \infty$.

Witten index from localization

- The infinite product can be evaluated explicitly, leading to

$$\frac{g(u,D)}{g(u,0)} = \prod_{i\to j} \left[\frac{\cosh\beta\Sigma_{ij} - \cos\beta V_{ij}}{\cosh(\beta\sqrt{\Sigma_{ij}^2 - \mathrm{i}D_{ij}}) - \cos\beta V_{ij}} \right]^{\kappa_{ij}}$$

$$g(u,0) = \prod_{i\to j} \left[\frac{\sin\pi(u_j - u_i)}{\sin\pi(u_i - u_j - z)} \right]^{\kappa_{ij}}$$

where $u_i - u_j + \frac{R_{ij}}{2} z = \frac{\beta}{2\pi}(V_{ij} - \mathrm{i}\Sigma_{ij})$ and $D_{ij} = D_i - D_j$.

- In the limit where $\beta|\Sigma| \gg 1$ and $|D| \leq |\Sigma|^2$, the ratio simplifies to

$$\frac{g(u,D)}{g(u,0)} \sim \prod_{i\to j} \left[e^{-\beta\sqrt{\Sigma_{ij}^2 - \mathrm{i}D_{ij}} + \beta|\Sigma_{ij}|} \right]^{\kappa_{ij}} \sim e^{\frac{\mathrm{i}}{2}\sum_{i\to j}\kappa_{ij}\frac{D_{ij}}{|\Sigma_{ij}|}}$$

Witten index from localization

- Plugging back into the integral yields

$$\Omega(\gamma,\zeta) = \int \left(\frac{\beta^2}{8\pi^3} \mathrm{d}u \mathrm{d}\bar{u} \mathrm{d}D \right)^{n-1} g(u,0) \det h(u,D)\, e^{-\beta S(D,\Sigma,\zeta)}$$

with a Σ-dependent action,

$$S(D,\Sigma,\zeta) = \frac{1}{2e^2} \sum_{i=1}^{n} D_i^2 - \mathrm{i} \sum_{i=1}^{n} \zeta_i D_i - \frac{\mathrm{i}}{2} \sum_{i\to j} \kappa_{ij} \frac{D_{ij}}{|\Sigma_{ij}|} ,$$

- The integral over D is Gaussian, dominated by a saddle point at

$$D_i^\star = -\mathrm{i}\, e^2 \left(\zeta_i - \sum_{j\neq i} \frac{\kappa_{ij}}{2|\Sigma_{ij}|} \right)$$

where $\Sigma_{ij} = \Sigma_i - \Sigma_j - \frac{\pi \mathrm{Im} z}{\beta} R_{ij}$ with $\Sigma_i = -\frac{2\pi}{\beta} \mathrm{Im} u_i$.

Witten index from localization

- After integrating over D, the integral over $\Sigma_i \sim \mathrm{Im} u_i$ is dominated by configurations such that $D_i^\star(\Sigma) = 0$. This produces a deformation of the 1D Denef equations:

$$\forall i = 1 \dots n, \qquad \sum_{j \neq i} \frac{\kappa_{ij}}{|\Sigma_i - \Sigma_j - \frac{\pi \mathrm{Im} z}{\beta} R_{ij}|} = \zeta_i \qquad (*)$$

- Denoting by S the set of solutions for $\mathrm{Im} u$, the Gaussian integral around S cancels (up to a crucial sign) the factor of $\det h$ in the measure, and one is left with the integral over $\mathrm{Re} u_i \in [0,1]$,

$$\Omega(\gamma,\zeta) = \sum_{s \in S} \int_{[0,1]^\ell} \mathrm{sgn}(\det \partial_i \partial_j \mathcal{W})\, g(u_i(s),0)\, \frac{\mathrm{d}^{n-1} \mathrm{Re}(u)}{(1/y - y)^{n-1}}$$

where $\mathcal{W}$ has critical points at solutions of $(*)$,

$$\mathcal{W} = -\frac{1}{2} \sum_{i<j} \kappa_{ij}\, \mathrm{sgn}(\Sigma_j - \Sigma_i - \tfrac{\pi \mathrm{Im} z}{\beta} R_{ji}) \log |\Sigma_i - \Sigma_j - \tfrac{\pi \mathrm{Im} z}{\beta} R_{ij}| - \sum_i \zeta_i \Sigma_i$$

Witten index from localization

- For quivers without oriented loops, the R-charge R_{ij} can be reabsorbed in Σ_i, and one recovers the standard 1D equations.
- In the limit $\beta \to \infty$, the prefactor $g(u_i(s), 0)$ becomes independent of $\mathrm{Re}\,u$ and reduces to the standard angular momentum $y^{2J_3(s)}$,

$$\prod_{i\to j} \left[\frac{\sin \pi(u_j - u_i)}{\sin \pi(u_i - u_j - z)} \right]^{\kappa_{ij}} \longrightarrow (-1)^{\sum_{i<j} \kappa_{ij}} \, e^{\mathrm{i}\pi z \sum_{i<j} \kappa_{ij} \,\mathrm{sgn}(\Sigma_i - \Sigma_j)}$$

This reproduces the MPS prescription for the Coulomb index !

Ohta Sasai 2015; Beaujard Mondal BP 2021

Witten index from localization

- For quivers with loops, there are two classes of solutions to (*), differing by their behavior as $\beta \to \infty$.
  1. In the first class, the solution reduces to the usual solution of undeformed Denef equations and the same result applies. This produces the bulk part g_C of the equivariant Dirac index.
  2. In the second class, $\beta|\Sigma|$ stays of order $\mathrm{Im} z$ and one cannot neglect the deformation, but one can set $\zeta_i = 0$. This produces the sum $\Omega_{\mathrm{tot}} = \Omega_S + H \prod \Omega_S$ of the single-centered index and the boundary part H of the equivariant Dirac index.
- Unfortunately, it does not appear to be possible to disentangle the two separate contributions in $\Omega_{\mathrm{tot}} = \Omega_S + H \prod \Omega_S$.

Single-centered invariants vs. stacky invariants

- For cyclic quivers, we can instead split $\Omega_{\rm tot} = \Omega_{\rm equal} + \Omega_{\rm unequal}$ depending whether the signs $\sigma_\ell = {\rm sgn}(\Sigma_\ell - \Sigma_{\ell+1} - \frac{\pi {\rm Im} z}{\beta} R_\ell)$ for scaling collinear solutions are all equal or distinct.
- Curiously, $\Omega_{\rm equal}$ agrees with the stacky invariant $\mathcal{I}(\gamma)$ of the moduli space of quiver representations for trivial stability, $\zeta_i = 0$! When triangular inequalities are violated, $\Omega_{\rm equal}$ and $\Omega_{\rm unequal}$ are non-zero but cancel in the sum.
- Since both H and the 'unequal' contributions grow polynomially as $\kappa_{ij} \to \infty$, while $\Omega_S(\gamma)$ grows exponentially, one has $\Omega_S(\gamma) \sim \mathcal{I}(\gamma)$ to exponential accuracy.

Conclusion

- We have outlined a derivation of the Coulomb branch formula in the context of quiver quantum mechanics. Some subtleties remain to be understood for non-Abelian quivers with loops.
- A first principle definition of the single-center/pure Higgs invariant $\Omega_S(\gamma)$ remains an outstanding problem. With Ashoke and Jan, we conjectured that $\Omega_S(\gamma, y, t)$ is independent of y (i.e. supported in middle cohomology), which is a powerful prediction on the structure of the cohomology of quiver moduli spaces.
- The Coulomb branch formula and attractor flow formulae should hold more generally for DT invariants on compact Calabi-Yau threefolds. Can one compute $\Omega_S(\gamma)$ and $\Omega_\star(\gamma)$ for some class of charges, and perform precision tests of holography ?
- Tune in for Jan's talk tomorrow for more on this.

Conclusion

Thank you for your attention !