Modularity of Donaldson-Thomas invariants on Calabi-Yau threefolds

Boris Pioline

Physical Mathematics, a celebration of
Albert Schwarz's 70 years in science
IHES, 14/6/2024

Congratulations Albert !

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- Albert's work was quite influential on me and many other researchers working at the juncture between high energy theoretical physics and mathematics. In some ways, he epitomizes the idea of "Physical Mathematics".

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- One of the main physical motivations is to understand the microscopic origin of the Bekenstein-Hawking entropy of black holes. This requires controlling the behavior of these enumerative invariants at large degree. When present, modular symmetries give excellent control on their growth.
- This is loosely connected to Albert's work with Maxim and Vadim Vologodsky (2006), and later with Johannes Walcher (2013-17).

Gromov-Witten invariants

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- Physically, they determine certain protected couplings in the low energy effective action, of the form $F_g(t) R^2 W^{2g-2}$, depending only on the complexified Kähler moduli t and receiving worldsheet instanton corrections: $F_g(t) = \sum_\beta n_\beta^{(g)} e^{2\pi \mathrm{i} t \cdot \beta}$

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- Mirror symmetry allows to compute F_0 and F_1. Holomorphic anomaly equations along with boundary conditions near the discriminant locus and MUM points allow to determine them up to a certain genus $g_{\rm int}$ ($= 51$ for the quintic threefold X_5).

Bershadsky Cecotti Ooguri Vafa'93; Huang Klemm Quackenbush'06

Gopakumar-Vafa invariants

- While GW invariants take rational values, the Gopakumar-Vafa invariants $GV_\beta^{(g)}$ defined by

$$\sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t) = \sum_{g=0}^{\infty} \sum_{k=1}^{\infty} \sum_{\beta} \frac{GV_\beta^{(g)}}{k} \left(2 \sin \tfrac{k\lambda}{2}\right)^{2g-2} e^{2\pi i k t\cdot\beta}$$

take integer values. For $g=0$, $GV_\beta^{(0)} = \sum_{k|\beta} \frac{1}{k^3} n_{\beta/k}^{(0)}$. Moreover, they vanish if g is large enough for fixed β. *[Ionel Parker'13]*

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- $GV_\beta^{(0)}$ counts BPS bound states of D2-branes with charge β, and arbitrary number of D0-branes, while $GV_\beta^{(g\geq 1)}$ keep track of their angular momentum (more on this below).
- The formula above arises by a one-loop computation of the effective action in a constant graviphoton background $W \propto \lambda$ à la Euler-Heisenberg. *[Gopakumar Vafa'98]*

GV invariants and 5D black holes

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- Keeping track of $m = j_L^z$ only, the number of states is

$$\Omega(\beta, m) = \sum_{g=0}^{g_{\max}(\beta)} \binom{2g+2}{g+1+m} GV_\beta^{(g)}$$

Amazingly, it appears that $\Omega(\beta, m) \sim e^{2\pi\sqrt{\beta^3 - m^2}}$ for large β, m keeping m^2/β^3 fixed, in agreement with the Bekenstein-Hawking entropy of 5D black holes ! *[Klemm Marino Tavanfar'07]*.

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- The GV invariants $GV_\beta^{(g)}$ can be defined rigorously using perverse cohomology on the moduli stack of stable sheaves, with some choice of orientation *[Maulik Toda'16]*, but the relation to Gromov-Witten invariants is still mysterious.

GV invariants and D6-brane bound states

- Instead of considering $M/X \times S^1 \times \mathbb{R}^4$, one may take $M/X \times TN \times \mathbb{R}$, where TN is a unit charge Taub-NUT space. This descends to a D6-brane on $X \times \mathbb{R}^{3,1}$.

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- D6-D2-D0 bound states of charge $(1, 0, \beta, n)$ are described mathematically by stable pairs $E : \mathcal{O}_X \stackrel{s}{\to} F$ where F is a pure 1-dimensional sheaf with $\mathrm{ch}_1 F = \beta$ and $\chi(F) = n$ and s has zero-dimensional kernel. The Pandharipande Thomas invariant $PT(\beta, n)$ is the Euler characteristic of the corresponding moduli space (weighted by Behrend's function).

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- Since TN is locally $\mathbb{R}^4$, one expects the same low energy effective action as in flat space. This suggests a relation of the form

$$\sum_{\beta,n} PT(\beta, n) e^{2\pi \mathrm{i} t\cdot\beta} q^n \simeq \exp\left(\sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t)\right)$$

Dijkgraaf Vafa Verlinde'06

GV invariants and D6-brane bound states

- More precisely, PT invariants are related to GV invariants by *[Maulik Nekrasov Okounkov Pandharipande'06]*

$$\sum_{\beta,n} PT(\beta,n)\, e^{2\pi i t\cdot\beta} q^n = \prod_{\beta,g,\ell} \left(1-(-q)^{g-\ell-1}\, e^{2\pi i t\cdot\beta}\right)^{(-1)^{g+\ell}\binom{2g-2}{\ell} N_\beta^{(g)}}$$

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- For n close to the Castelnuovo bound $n \geq 1-g_{\max}(\beta)$, this reduces to $PT(\beta,n) = \sum_{g=1}^{g_{\max}(\beta)} \binom{2g-2}{g-1-n} GV_\beta^{(g)}$

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- The Donaldson-Thomas invariant $DT(\beta,n)$ is a variant of PT invariant treating D0-branes differently,

$$\sum_{\beta,n} DT(\beta,n) e^{2\pi i t\cdot\beta} q^n = M(-q)^{\chi_X} \sum_{\beta,n} PT(\beta,n) e^{2\pi i t\cdot\beta} q^n$$

where $M(q) = \prod_k (1-q^k)^{-k}$ is the Mac-Mahon function.

Generalized Donaldson-Thomas invariants

- More generally, D6-D4-D2-D0 bound states are described by stable objects in the bounded derived category of coherent sheaves $D^b\mathrm{Coh}(X)$ *[Kontsevich'95, Douglas'01]*. Objects $E \in \mathcal{C}$ are bounded complexes $E = (\dots \stackrel{d^{-2}}{\to} \mathcal{E}^{-1} \stackrel{d^{-1}}{\to} \mathcal{E}^{0} \stackrel{d^{0}}{\to} \mathcal{E}^{1} \stackrel{d^{1}}{\to} \dots)$

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- Stable objects are counted by the generalized Donaldson-Thomas invariant $\bar\Omega_\sigma(\gamma)$, where $\gamma \in K_{\mathrm{num}}(X) \sim \mathbb{Z}^{2b_2(X)+2}$ and $\sigma = (Z, \mathcal{A})$ is a Bridgeland stability condition. In particular, $\forall E \in \mathcal{A}$, $(i)\,\mathrm{Im}Z(E) \geq 0$ and $(ii)\,\mathrm{Im}Z(E) = 0 \Rightarrow \mathrm{Re}Z(E) < 0$.

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- The space of stability conditions $\mathrm{Stab}\,\mathcal{C}$ is a complex manifold of dimension $\dim K_{\mathrm{num}}(X) = 2b_2(X) + 2$, *unless it is empty*.
- For X a a projective CY3, stability conditions are only known to exist only for the quintic threefold X_5 and a couple of other examples *[Li'18, Koseki'20, Liu'21]*

Generalized Donaldson-Thomas invariants

- $\bar{\Omega}_\sigma(\gamma)$ is roughly the weighted Euler number of the moduli stack of semi-stable objects $M_\sigma(\gamma)$, where semi-stability means $\arg Z(E') \leq \arg Z(E)$ for any subobject $E' \subset E$.

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- For $\gamma = (0, 0, \beta, n)$ and $\gamma = (1, 0, \beta, n)$, $\Omega_\sigma(\gamma)$ coincides with $GV_\beta^{(0)}$ and $PT(\beta, n)$ OR $GV(\beta, n)$ at large volume, respectively.

D4-D2-D0 indices as rank 0 DT invariants

- The main interest in this talk will be rank 0 DT invariants $\Omega(0, p, \beta, n)$ counting D4-D2-D0 brane bound states supported on a divisor $\mathcal{D}$ with class $[\mathcal{D}] = p \in H_4(X, \mathbb{Z})$.

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- Viewing IIA=M/S^1, they arise from M5-branes wrapped on $\mathcal{D} \times S^1$. In the limit where S^1 is much larger than X, they are described by a two-dimensional superconformal field theory with $(0, 4)$ SUSY. DT invariants $\Omega(0, p, \beta, n)$ (in suitable chamber) arise as Fourier coefficients of the elliptic genus.

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- When $\mathcal{D}$ is very ample, the central charges are

$$c_L = p^3 + c_2(X) \cdot p = \chi(\mathcal{D}) \,, \quad c_R = p^3 + \frac{1}{2} c_2(X) \cdot p$$

Cardy's formula predicts a growth $\Omega(0, p, \beta, n \to \infty) \sim e^{2\pi\sqrt{p^3\, n}}$ in perfect agreement with Bekenstein-Hawking formula *[Maldacena Strominger Witten'97]*.

Mock modularity of rank 0 DT invariants

- When p is primitive, there are no walls extending to large volume, so the choice of chamber is moot. The generating series

$$h_{p^a,\mu_a}(\tau) := \sum_n \Omega(0, p^a, \mu_a, n)\, \mathrm{q}^{n+\frac12 \mu_a \kappa^{ab} \mu_b - \frac12 p^a \mu_a - \frac{\chi(\mathcal{D})}{24}}$$

  should be a vector-valued, weakly holomorphic modular form of weight $w = -\frac12 b_2(X) - 1$ in the Weil representation of the lattice $\Lambda^* = H_4(X,\mathbb{Z})$ with quadratic form $\kappa_{abc}p^c$. Note that $\mu \in \Lambda/\Lambda^*$, and n is bounded from below by the Bogomolov-Gieseker inequality

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- In general, we predict that the generating series of DT invariants $\Omega_*(0,p,\beta,n)$ at the large volume attractor point $t^a=\kappa^{ab}\mu_b+\mathrm{i}\lambda p^a$, $\lambda\to\infty$ is a weakly holomorphic mock modular form of depth $k-1$, where k is the largest integer such that p/k is primitive.

  [Alexandrov BP Manschot'16-20]

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- Specifically, there exists explicit non-holomorphic theta series $\Theta_n(\{p_i\},\tau,\bar\tau)$ such that

$$\widehat{h}_p(\tau,\bar\tau) = h_p(\tau) + \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \Theta_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^{n} h_{p_i}(\tau)$$

transforms as a modular form of weight $-\frac{1}{2}b_2(\mathfrak{Y})-1$. Moreover the completion satisfies an explicit holomorphic anomaly equation,

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- The derivation relies on the study of instanton corrections to the low energy effective action after compactifying on a circle, and implementing $SL(2,\mathbb{Z})$ symmetry manifest from $IIA/S^1 = M/T^2$.

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- When X is K3-fibered, modularity is known to hold for vertical D4-brane charge, using the relation to Noether-Lefschetz invariants. In that case, no modular anomaly due to $\kappa_{abc}p^a p^b p^c = 0$. *[Bouchard Creutzig Diaconescu Doran Quigley Sheshmani'16]*

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- For non-compact CY threefolds of the form $X = K_S$ where S is a Fano surface, rank 0 DT invariants reduce to Vafa-Witten invariants. They coincide with DT invariants for the moduli space of certain quivers with potential. Modularity holds for rank $r = 1$ by Goettsche's formula. Mock modularity holds for $S = \mathbb{P}^2$, $r = 2, 3$ by results of *[Klyachko'91, Yoshioka'94, Manschot]*

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- In general however, the origin of this (mock) modularity is completely obscure.

- Our aim is to test this prediction for CY threefolds with Picard rank 1, by computing the first few coefficients in the q expansion and determine the putative (mock) modular form.

Mock modularity of rank 0 DT invariants

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- We shall compute many terms rigorously, obtaining high precision tests of modularity, and generalize to two units of D4-brane charge for some models.

Alexandrov, Feyzbakhsh, Klemm, BP, Schimannek'23

From rank 1 to rank 0 DT invariants

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- The key idea is to use wall-crossing in a family of weak stability conditions parametrized by $b+\mathrm{i}t \in \mathbb{H}$, with central charge

$$Z_{b,t}(E) = \frac{\mathrm{i}}{6} t^3 \, \mathrm{ch}_0^b(E) - \frac{1}{2} t^2 \, \mathrm{ch}_1^b(E) - \mathrm{i} t \, \mathrm{ch}_2^b(E) + 0 \, \mathrm{ch}_3^b(E)$$

with $\mathrm{ch}_k^b(E) = \int_{\mathfrak{Y}} H^{3-k} e^{-bH} \mathrm{ch}(E)$. The heart $\mathcal{A}_b$ is generated by length-two complexes $\mathcal{E} \xrightarrow{d} \mathcal{F}$ with $\mathrm{ch}_1^b(\mathcal{E}) > 0, \mathrm{ch}_1^b(\mathcal{F}) \leq 0$.

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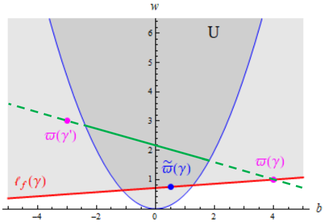
- The JS wall-crossing formula holds for this family, even though they are not genuine stability conditions. In fact, tilt-stability provide the first step in constructing genuine stability conditions near the large volume point *[Bayer Macri Toda'11]*

Rank 0 DT invariants from GV invariants

- Tilt stability agrees with Gieseker stability at large volume, but the chamber structure is much simpler: walls are straight lines in the plane spanned by $(b, w = \frac{1}{2}b^2 + \frac{1}{6}t^2)$, with $w > \frac{1}{2}b^2$.



$$\nu_{b,w}(E) = \frac{\mathrm{ch}_2 .H - w\, \mathrm{ch}_0 .H^3}{\mathrm{ch}_1 .H^2 - b\, \mathrm{ch}_0 .H^3}$$

$$\varpi(E) = \left(\frac{\mathrm{ch}_1 .H^2}{\mathrm{ch}_0 .H^3}, \frac{\mathrm{ch}_2 .H}{\mathrm{ch}_0 .H^3} \right)$$

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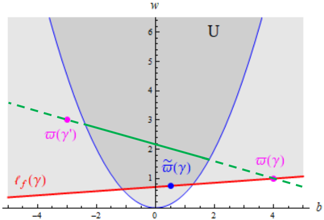

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- Importantly, for any $\nu_{b,w}$-semistable object E there is a conjectural inequality on Chern classes $C_i := \int_{\mathfrak{Y}} \mathrm{ch}_i(E).H^{3-i}$ *[Bayer Macri Toda'11; Bayer Macri Stellari'16]*

$$(C_1^2 - 2C_0C_2)w + (3C_0C_3 - C_1C_2)b + (2C_2^2 - 3C_1C_3) \geq 0$$

Rank 0 DT invariants from GV invariants

- By studying wall-crossing between the empty chamber provided by BMT bound and large volume, *[Feyzbakhsh Thomas'20-22]* show that D4-D2-D0 indices can be computed from rank 1 DT or PT invariants, which are in turn related to GV invariants.

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- By studying wall-crossing between the empty chamber provided by BMT bound and large volume, *[Feyzbakhsh Thomas'20-22]* show that D4-D2-D0 indices can be computed from rank 1 DT or PT invariants, which are in turn related to GV invariants.
- In particular for (β, n) large enough, the PT invariant counting tilt-stable objects of class $(-1, 0, \beta, n)$ is given by *[Feyzbakhsh'22]*

$$PT(\beta, n) = (-1)^{\langle \overline{D6(1)}, \gamma \rangle + 1} \langle \overline{D6(1)}, \gamma \rangle \, \Omega(\gamma)$$

with $\overline{D6(1)} := \mathcal{O}_{\mathfrak{Y}}(H)[1]$ and $\gamma = (0, H, \beta, n)$. By tensoring with $\mathcal{O}_X(mH)$ for $m \geq m_0(\beta, n)$ large enough,

$$\boxed{\Omega(\gamma) = \frac{(-1)^{\langle \overline{D6(1-m)}, \gamma \rangle + 1}}{\langle \overline{D6(1-m)}, \gamma \rangle} PT(\beta', n')} \qquad \begin{cases} \beta' = \beta + mH \\ n' = n - m\beta.H - \frac{H^3}{2} m(m+1) \end{cases}$$

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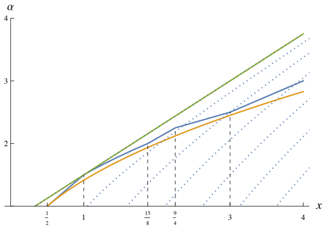
- Let $(\mathfrak{Y}, H)$ be a smooth polarised CY threefold with $\mathrm{Pic}(\mathfrak{Y}) = \mathbb{Z}.H$ satisfying the BMT conjecture.
- Fix $m \in \mathbb{Z}, \beta \in H_2(\mathfrak{Y}, \mathbb{Z})$ and define $x = \frac{\beta.H}{H^3}, \quad \alpha = -\frac{3m}{2\beta.H}$

$$f(x) := \begin{cases} x + \frac{1}{2} & \text{if } 0 < x < 1 \\ \sqrt{2x + \frac{1}{4}} & \text{if } 1 < x < \frac{15}{8} \\ \frac{2}{3}x + \frac{3}{4} & \text{if } \frac{15}{8} \leq x < \frac{9}{4} \\ \frac{1}{3}x + \frac{3}{2} & \text{if } \frac{9}{4} \leq x < 3 \\ \frac{1}{2}x + 1 & \text{if } 3 \leq x \end{cases}$$



A new explicit formula (S. Feyzbakhsh'23)

Theorem (wall-crossing for class $(-1, 0, \beta, -m)$):

- If $f(x) < \alpha$ then the stable pair invariant $PT(\beta, m)$ equals

$$\sum_{(m', \beta')} (-1)^{\chi_{m',\beta'}} \chi_{m',\beta'} PT(\beta', m')\, \Omega\left(0,\ H,\ \frac{H^2}{2} - \beta' + \beta\ ,\ \frac{H^3}{6} + m' - m - \beta'.H\right)$$

where $\chi_{m',\beta'} = \beta.H + \beta'.H + m - m' - \frac{H^3}{6} - \frac{1}{12} c_2(\mathfrak{Y}).H$.

Corollary (Castelnuovo bound): $PT(\beta, m) = 0$ unless

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where $\chi_{m',\beta'} = \beta.H + \beta'.H + m - m' - \frac{H^3}{6} - \frac{1}{12} c_2(\mathfrak{Y}).H$.

- The sum runs over $(\beta', m') \in H_2(\mathfrak{Y}, \mathbb{Z}) \oplus H_0(\mathfrak{Y}, \mathbb{Z})$ such that

$$0 \leq \beta'.H \leq \frac{H^3}{2} + \frac{3mH^3}{2\beta.H} + \beta.H$$

$$-\frac{(\beta'.H)^2}{2H^3} - \frac{\beta'.H}{2} \leq\ m'\ \leq \frac{(\beta.H - \beta'.H)^2}{2H^3} + \frac{\beta.H + \beta'.H}{2} + m$$

In particular, $\beta'.H < \beta.H$.

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Modularity for one-modulus compact CY

- Using Soheyla's formula and known GV invariants, we could compute a large number of coefficients in the generating series of Abelian (=unit D4-brane charge) rank 0 DT invariants in one-parameter hypergeometric threefolds, including the quintic X_5.

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- In all cases (except $X_{4,2}, X_{3,2,2}, X_{2,2,2,2}$ where current knowledge of GV invariants is insufficient), we could find a linear combination of the following vv modular forms matching all computed coeffs:

$$\frac{E_4^a E_6^b}{\eta^{4\kappa+c_2}} D^\ell(\vartheta_\mu^{(\kappa)}) \quad \text{with} \quad \vartheta_\mu^{(\kappa)} = \sum_{k\in\mathbb{Z}+\frac{\mu}{\kappa}+\frac{1}{2}} \mathrm{q}^{\frac{1}{2}\kappa k^2}$$

where $D = \mathrm{q}\partial_\mathrm{q} - \frac{w}{12}E_2$, and $4a + 6b + 2\ell - 2\kappa - \frac{1}{2}c_2 = -2$.

Modularity for one-modulus compact CY

$\mathfrak{Y}$	$\chi_{\mathfrak{Y}}$	κ	$c_2(T\mathfrak{Y})$	$\chi(\mathcal{O}_{\mathcal{D}})$	n_1	C_1
$X_5(1^5)$	-200	5	50	5	7	0
$X_6(1^4,2)$	-204	3	42	4	4	0
$X_8(1^4,4)$	-296	2	44	4	4	0
$X_{10}(1^3,2,5)$	-288	1	34	3	2	0
$X_{4,3}(1^5,2)$	-156	6	48	5	9	0
$X_{4,4}(1^4,2^2)$	-144	4	40	4	6	1
$X_{6,2}(1^5,3)$	-256	4	52	5	7	0
$X_{6,4}(1^3,2^2,3)$	-156	2	32	3	3	0
$X_{6,6}(1^2,2^2,3^2)$	-120	1	22	2	1	0
$X_{3,3}(1^6)$	-144	9	54	6	14	1
$X_{4,2}(1^6)$	-176	8	56	6	15	1
$X_{3,2,2}(1^7)$	-144	12	60	7	21	1
$X_{2,2,2,2}(1^8)$	-128	16	64	8	33	3

Modular predictions for the quintic threefold

- Using Soheyla's formula we can compute many terms

$$h_{1,0} = q^{-\frac{55}{24}} \Big(\underline{5 - 800q + 58500q^2} + 5817125q^3 + 75474060100q^4 + 28096675153255q^5 + 3756542229485475q^6 + 277591744202815875q^7 + 13610985014709888750q^8 + \dots \Big),$$

$$h_{1,\pm 1} = q^{-\frac{55}{24}+\frac{3}{5}} \Big(\underline{0 + 8625q} - 1138500q^2 + 3777474000q^3 + 3102750380125q^4 + 577727215123000q^5 + \dots \Big)$$

$$h_{1,\pm 2} = q^{-\frac{55}{24}+\frac{2}{5}} \Big(\underline{0 + 0q} - 1218500q^2 + 441969250q^3 + 953712511250q^4 + 217571250023750q^5 + 22258695264509625q^6 + \dots \Big)$$

Modular predictions for the quintic threefold

- The space of vv modular forms has dimension 7. Remarkably, all terms above are reproduced by *[Gaiotto Strominger Yin'06]*

$$h_\mu = \frac{1}{\eta^{70}}\left[-\frac{222887 E_4^8 + 1093010 E_4^5 E_6^2 + 177095 E_4^2 E_6^4}{35831808} + \frac{25\left(458287 E_4^6 E_6 + 967810 E_4^3 E_6^3 + 66895 E_6^5\right)}{53747712} D + \frac{25\left(155587 E_4^7 + 1054810 E_4^4 E_6^2 + 282595 E_4 E_6^4\right)}{8957952} D^2 \right] \vartheta^{(5)}_\mu ,$$

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- For other models, Gaiotto et al were not so lucky, e.g. for X_{10} they predicted

$$h_{1,0} \stackrel{?}{=} \mathrm{q}^{-\frac{35}{24}} \left(\underline{3 - 576\mathrm{q}} + 271704\mathrm{q}^2 + 206401533\mathrm{q}^3 + \cdots \right)$$

whereas the correct result turns out to be

$$h_{1,0} \stackrel{!}{=} \frac{203 E_4^4 + 445 E_4 E_6^2}{216\, \eta^{35}} = \mathrm{q}^{-\frac{35}{24}} \left(\underline{3 - 575\mathrm{q}} + 271955\mathrm{q}^2 + \cdots \right)$$

Mock modularity for non-Abelian D4-D2-D0 indices

- Let us consider D4-D2-D0 indices with $N = 2$ units of D4-brane charge. In that case, $\{h_{2,\mu}, \mu \in \mathbb{Z}/(2\kappa\mathbb{Z})\}$ should transform as a vv mock modular form with modular completion

$$\widehat{h}_{2,\mu}(\tau,\bar{\tau}) = h_{2,\mu}(\tau) + \sum_{\mu_1,\mu_2=0}^{\kappa-1} \delta^{(\kappa)}_{\mu_1+\mu_2-\mu}\, \Theta^{(\kappa)}_{\mu_2-\mu_1+\kappa}\, h_{1,\mu_1}\, h_{1,\mu_2}$$

where

$$\Theta^{(\kappa)}_{\mu} = \frac{(-1)^\mu}{8\pi} \sum_{k\in 2\kappa\mathbb{Z}+\mu} |k|\, \beta\left(\frac{\tau_2 k^2}{\kappa}\right) e^{-\frac{\pi i \tau}{2\kappa}\, k^2},$$

and $\beta(x^2) = 2|x|^{-1} e^{-\pi x^2} - 2\pi \mathrm{Erfc}(\sqrt{\pi}|x|)$ such that

$$\partial_{\bar{\tau}} \Theta^{(\kappa)}_{\mu} = \frac{(-1)^\mu \sqrt{\kappa}}{16\pi i \tau_2^{3/2}} \sum_{k\in 2\kappa\mathbb{Z}+\mu} e^{-\frac{\pi i \bar{\tau}}{2\kappa}\, k^2}$$

Mock modularity for non-Abelian D4-D2-D0 indices

- Suppose there exists a holomorphic function $g_\mu^{(\kappa)}$ such that $\Theta_\mu^{(\kappa)} + g_\mu^{(\kappa)}$ transforms as a vv modular form. Then

$$\widetilde{h}_{2,\mu}(\tau,\bar\tau) = h_{2,\mu}(\tau) - \sum_{\mu_1,\mu_2=0}^{\kappa-1} \delta^{(\kappa)}_{\mu_1+\mu_2-\mu}\, g^{(\kappa)}_{\mu_2-\mu_1+\kappa}\, h_{1,\mu_1}\, h_{1,\mu_2}$$

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  will be an ordinary weak holomorphic vv modular form, hence uniquely determined by its polar part.
- For $\kappa = 1$, the series $\Theta_\mu^{(1)}$ is the same one appearing in the modular completion of the generating series of Hurwitz class numbers *[Hirzebruch Zagier 1973]*, or rank 2 Vafa-Witten invariants on $\mathbb{P}^2$ *[Yoshioka'93; Bringmann Manschot'10]*

$$H_0(\tau) = -\frac{1}{12} + \frac{1}{2}\mathrm{q} + \mathrm{q}^2 + \frac{4}{3}\mathrm{q}^3 + \frac{3}{2}\mathrm{q}^4 + \dots$$
$$H_1(\tau) = \mathrm{q}^{\frac{3}{4}}\left(\frac{1}{3} + \mathrm{q} + \mathrm{q}^2 + 2\mathrm{q}^3 + \mathrm{q}^4 + \dots\right)$$

  Thus we can choose $g_\mu^{(1)} = H_\mu(\tau)$.

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$\mathfrak{Y}$	$\chi_{\mathfrak{Y}}$	κ	c_2	$\chi(\mathcal{O}_{2\mathcal{D}})$	n_2	C_2
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Mock modularity for non-Abelian D4-D2-D0 indices

- For X_{10}, we computed the 7 polar terms + 4 non-polar terms and found a unique mock modular form reproducing this data:

$$h_{2,\mu} = \frac{5397523 E_4^{12} + 70149738 E_4^9 E_6^2 - 12112656 E_4^6 E_6^4 - 61127530 E_4^3 E_6^6 - 2307075 E_6^8}{46438023168 \eta^{100}} \vartheta_\mu^{(1,2)} + \frac{-10826123 E_4^{10} E_6 - 14574207 E_4^7 E_6^3 + 20196255 E_4^4 E_6^5 + 5204075 E_4 E_6^7}{1934917632 \eta^{100}} D\vartheta_\mu^{(1,2)} + (-1)^{\mu+1} H_{\mu+1}(\tau)\, h_1(\tau)^2$$

with $h_1 = \frac{203 E_4^4 + 445 E_4 E_6^2}{216\, \eta^{35}} = \mathrm{q}^{-\frac{35}{24}} (\underline{3 - 575\mathrm{q}} + \dots)$, leading to integer DT invariants

$$h_{2,0}^{(\mathrm{int})} = \mathrm{q}^{-\frac{19}{6}} \left(\underline{7 - 1728\mathrm{q} + 203778\mathrm{q}^2 - 13717632\mathrm{q}^3} - 23922034036\mathrm{q}^4 + \dots\right.$$

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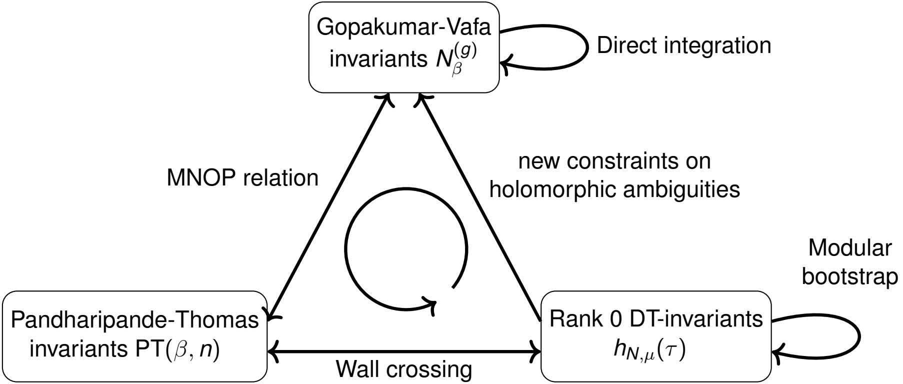
- Similar results for X_8. For other models including the quintic threefold, the current knowledge of GV invariants insufficient.

Quantum geometry from stability and modularity

Conversely, we can use our knowledge of Abelian D4-D2-D0 invariants to compute GV invariants and push the direct integration method to higher genus !



Alexandrov Feyzbakhsh Klemm BP Schimannek'23

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$X_{6,6}(1^2,2^2,3^2)$	-120	1	K	18	22	22
$X_{4,4}(1^4,2^2)$	-144	4	K	26	34	34
$X_{3,3}(1^6)$	-144	9	K	29	33	33
$X_{4,2}(1^6)$	-176	8	C	50	66	50
$X_{6,2}(1^5,3)$	-256	4	C	63	78	49

A remark on the BMT inequality

- Requiring the existence of empty chamber, the discriminant at $w = \frac{1}{2}x^2$ must be positive:

$$8C_0C_2^3 + 6C_1^3C_3 + 9C_0^2C_3^2 - 3C_1^2C_2^2 - 18C_0C_1C_2C_3 \geq 0$$

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- In terms of the electric and magnetic charges

$$p^0=C_0/\kappa,\quad p^1=C_1/\kappa,\quad q_1=-C_2-\frac{c_2}{24\kappa}C_0,\quad q_0=C_3+\frac{c_2}{24\kappa}C_1$$

and ignoring the c_2-dependent terms this becomes

$$\tfrac{8}{9\kappa}p^0q_1^3-\tfrac{2}{3}\kappa q_0(p^1)^3-(p^0q_0)^2+\tfrac{1}{3}(p^1q_1)^2-2p^0p^1q_0q_1\leq 0$$

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  hence an empty chamber arises when single centered black hole solutions are ruled out !

- Can one understand the fulll BMT inequality physically, perhaps on the B-model side ? Is there an improved version of BMT incorporating c_2-dependent corrections ?

Some open questions

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- Higher rank DT invariants can also be computed in terms of GV invariants. Do they define some higher rank version of topological string theory ?
- Modularity constraints were derived by thinking about Euclidean D-brane instanton corrections to hypermultiplet moduli space near infinite volume. Can one also include NS5-brane instantons ?

Congratulations Albert Solomonivich !