Unfolding Methods for String Amplitudes

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based on work with C. Angelantonj and I. Florakis (and old work with N. Obers)

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Modular integrals and BPS amplitudes I

- In the low energy effective action of string theory, an interesting class of terms (known as BPS-saturated coupling, topological amplitude or F-term) are given by a modular integral

$$\mathcal{A} = \int_{\mathcal{F}} \mathrm{d}\mu \, \Gamma_{(d+k,d)} \, \Phi(\tau)$$

  - $\mathcal{F} = \Gamma\backslash\mathcal{H}$: fundamental domain of the modular group $\Gamma = SL(2,\mathbb{Z})$ on the Poincaré UHP $\mathcal{H}$;
  - $\mathrm{d}\mu = \mathrm{d}\tau_1 \mathrm{d}\tau_2/\tau_2^2$ is the Γ-invariant measure;
  - $\Gamma_{(d+k,d)} = \tau_2^{d/2} \sum q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2}$: the partition function of an even self-dual (Narain) lattice of signature $(d+k,d)$;
  - $\Phi(\tau)$: an (almost, weak) holomorphic modular form of weight $w = -k/2$, known as the elliptic genus

Modular integrals and BPS amplitudes II

- Such amplitudes arise in a variety of examples:
  - Gauge thresholds, $R^2 F^{2h-2}$ in Het$/K3 \times T^2$ at one-loop

    Dixon Kaplunovsky Louis; Harvey Moore
  - F^4 couplings in Het$/T^d$ at one-loop

    Bachas Fabre Kiritsis Obers Vanhove
  - R^4 couplings in type II/T^d at one-loop ($\Phi = 1$)

    Green Vanhove; Kiritsis BP
  - R^2 couplings in type $II/K3 \times T^2$ at one-loop (")

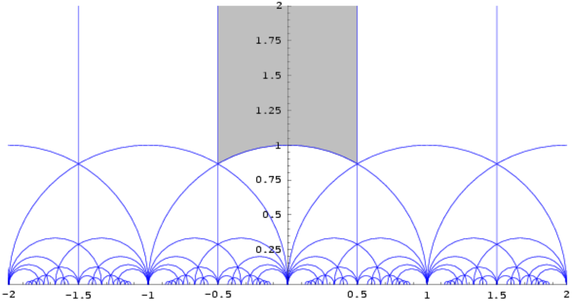
    Harvey Moore; Gregori Kiritsis Kounnas Obers Petropoulos BP
  - F^4 couplings in type $II/T^4/\mathbb{Z}_N$ at tree-level (")

    Obers BP
  - $\nabla^4 R^4$ couplings in M/T^d at two-loops (")

    Green Vanhove Russo
- These amplitudes are strongly constrained by supersymmetry, and offer precise tests of string dualities.

Modular integrals and BPS amplitudes III

- When $\mathcal{A}$ arises at one-loop, and upon choosing $\mathcal{F}$ as the standard 'keyhole' domain, τ_2 can be interpreted as the Schwinger parameter, while τ_1 is a Lagrange multiplier enforcing the level-matching constraint $p_L^2 - p_R^2 = N$.



Theta correspondances

- From the mathematical point of view, modular integrals give a theta correspondence

$$\Phi : \Gamma\backslash\mathcal{H} \to \mathbb{C} \quad \leftrightarrow \quad \mathcal{A} : O(\Gamma_{d+k,d})\backslash G_{d+k,d} \to \mathbb{C}$$

between modular forms on $\mathcal{H}$ and automorphic forms on the Grassmannian $G_{d+k,d}$, or Narain moduli space

$$G_{d+k,d} = \frac{O(d+k,d)}{O(d+k)\times O(d)} \ni (g_{ij}, B_{ij}, Y_i^a)$$

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$$G_{d+k,d} = \frac{O(d+k,d)}{O(d+k)\times O(d)} \ni (g_{ij}, B_{ij}, Y_i^a)$$

- Indeed, $SL(2)\times O(d+k,d)$ forms a dual pair in $Sp(d+k,d)$, and the lattice partition function is invariant under $\Gamma \times O(\Gamma_{d+k,d})$.

Unfolding trick

- In the physics literature, the time-honored way to evaluate such integrals has been the unfolding trick or orbit method where the domain of integration $\mathcal{F}$ is unfolded by grouping the terms in the theta series into orbits.

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- In the physics literature, the time-honored way to evaluate such integrals has been the unfolding trick or orbit method where the domain of integration $\mathcal{F}$ is unfolded by grouping the terms in the theta series into orbits.
- E.g for $d = 1$, representing $\Gamma_{(1,1)} = R \sum_{m,n} e^{-\pi R^2 |m-n\tau|^2/\tau_2}$,

$$\int_{\mathcal{F}} \Gamma_{(1,1)} = R \int_{\mathcal{F}} \mathrm{d}\mu + R \int_{\mathcal{S}} \sum_{m\neq 0} e^{-\pi R^2 m^2/\tau_2}$$
$$= \frac{\pi}{3} R + \frac{\pi}{3} R^{-1}$$

where $\mathcal{S} = \mathcal{H}/\Gamma_\infty$ is the strip $\{-\frac{1}{2} \leq \tau_1 \leq \frac{1}{2}, \tau_2 > 0\}$.

Unfolding trick

- For $d = 2$, a (lengthy) landmark computation shows

$$\int_{\mathcal{F}} \left(\Gamma_{(2,2)}(T,U) - \tau_2\right) \, d\mu = \int_{\mathcal{F}} + \int_{\mathcal{S}} + \int_{\mathcal{H}}$$
$$= (\text{mess})$$
$$= -\log\left(\frac{8\pi e^{1-\gamma}}{3\sqrt{3}} \, T_2 \, U_2 \, |\eta(T)\,\eta(U)|^4\right)$$

Dixon Kaplunovsky Louis

where T, U parametrize the Grassmannian $G_{2,2} = \mathcal{H}_T \times \mathcal{H}_U / \mathbb{Z}_2$.

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- The final result is invariant under T-duality, but intermediate steps do not make T-duality manifest. We shall present a method that preserves T-duality at all steps.

Outline

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Rankin-Selberg method I

- Our method is an extension of the Rankin-Selberg method commonly used in number theory. It relies on the (completed, non-holomorphic) Eisenstein series

$$\begin{aligned} E^\star(\tau;s) \equiv& \zeta^\star(2s) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} [\mathrm{Im}\,(\gamma\cdot\tau)]^s \\ =&\tfrac12\,\zeta^\star(2s) \sum_{(c,d)=1} \frac{\tau_2^s}{|c\,\tau+d|^{2s}} \end{aligned}$$

where $\zeta^\star(s) \equiv \pi^{-s/2}\,\Gamma(s/2)\,\zeta(s) = \zeta^\star(1-s)$ is the completed zeta function with simple poles at $s=1,0$

- $E^\star(\tau;s) = E^\star(\tau;1-s)$ is analytic in s away from $s=0,1$,

$$E^\star(\tau;s) = \frac{1}{2(s-1)} + \tfrac12\left(\gamma - \log(4\pi\,\tau_2\,|\eta(\tau)|^4)\right) + \mathcal{O}(s-1)\,,$$

Rankin-Selberg method (cont.)

- If $F(\tau)$ is a modular function of rapid decay at the cusp, the Rankin-Selberg transform

$$\mathcal{R}^\star(F,s) \equiv \int_{\mathcal{F}} \mathrm{d}\mu\, E^\star(\tau;s)\, F(\tau)$$

can be computed by the same unfolding trick,

$$\begin{aligned}\mathcal{R}^\star(F;s) =& \zeta^\star(2s) \int_{\mathcal{S}} \frac{\mathrm{d}\tau_1\, \mathrm{d}\tau_2}{\tau_2^{2-s}}\, F(\tau) \\ =& \zeta^\star(2s) \int_0^\infty \mathrm{d}\tau_2\, \tau_2^{s-2}\, F_0(\tau_2)\,,\end{aligned}$$

Thus $\mathcal{R}^\star(F,s)$ is proportional to the Mellin transform of the constant term $F_0(\tau_2) = \int_{-1/2}^{1/2} \mathrm{d}\tau_1\, F(\tau)$

Rankin-Selberg method (cont.)

- The analyticity and functional relation for $E^\star(s)$ implies similar properties for $\mathcal{R}^\star(F;s)$. For $F = f.g$ product of two cusp forms, this is used e.g. to show the analyticity and functional relation of the L-function $L(s) = \sum_n a_n b_n n^{-s} \propto \mathcal{R}^\star(F;s)$.
- For us, the main point is that, since the residue of $E^\star$ at $s = 0, 1$ is constant, the residue of $\mathcal{R}^\star(F;s)$ at $s = 0$ is proportional to the modular integral of F,

$$\mathrm{Res}\ \mathcal{R}^\star(F;s)|_{s=1} = \frac{1}{2}\int_{\mathcal{F}} \mathrm{d}\mu\, F = -\mathrm{Res}\ \mathcal{R}^\star(F;s)|_{s=0}\ .$$

Rankin-Selberg-Zagier method I

- This was extended by Zagier to the case where F is of moderate growth $F(\tau) \sim \phi(\tau_2)$ at the cusp ($\phi(\tau_2)$ at most a power).
- To regulate the infrared divergence, one may introduce a hard cut-off $\mathcal{T}$. The unfolding trick generalizes into

$$\int_{\mathcal{F};\tau_2\leq\mathcal{T}} \sum_{\gamma\in\Gamma/\Gamma_\infty} f|_\gamma = \int_{\mathcal{S};\tau_2\leq\mathcal{T}} f + \int_{\mathcal{S};\tau_2>\mathcal{T}} \sum_{\gamma\in\Gamma/\Gamma_\infty,\gamma\neq 1} f|_\gamma$$

where $f|_\gamma(\tau) = f(\gamma\cdot\tau)$.

Rankin-Selberg-Zagier method II

- Defining the renormalized modular integral

$$\text{R.N.} \int_{\mathcal{F}} \mathrm{d}\mu\, F(\tau) \equiv \lim_{\mathcal{T}\to\infty} \left[\int_{\mathcal{F}_{\mathcal{T}}} \mathrm{d}\mu\, F(\tau) - \hat{\varphi}(\mathcal{T}) \right]$$

where $\hat{\varphi}$ is the anti-derivative of φ (i.e. $d\hat{\varphi}/d\mathcal{T} = \varphi(\mathcal{T})/\mathcal{T}^2$), one finds that it is again related to the (regularized) Mellin transform of the constant term

$$\mathcal{R}^\star(F;s) = \zeta^\star(2s) \int_0^\infty \mathrm{d}\tau_2\, \tau_2^{s-2}\, (F_0 - \varphi) \ ,$$

via

$$\text{R.N.} \int_{\mathcal{F}} \mathrm{d}\mu\, F(\tau) = 2\,\mathrm{Res}\left[\mathcal{R}^\star(F;s)\right]_{s=1} + \delta$$

Rankin-Selberg-Zagier method III

- The correction δ depends only of the leading behavior $\phi(\tau_2)$, and is given by

$$\delta = 2\,\mathrm{Res}\left[\zeta^\star(2s)\,h_\mathcal{T}(s) + \zeta^\star(2s-1)\,h_\mathcal{T}(1-s)\right]_{s=1} - \hat{\varphi}(\mathcal{T})\,,$$

where

$$h_\mathcal{T}(s) = \int_0^\mathcal{T} \mathrm{d}\tau_2\,\varphi(\tau_2)\,\tau_2^{s-2}\,, \qquad \hat{\phi}(\mathcal{T}) = \mathrm{Res}\left[\frac{h_\mathcal{T}(s)}{s-1}\right]_{s=1}$$

- Other renormalization schemes may give a different constant δ

Epstein series from modular integrals

- The RSZ method applies immediately to modular integrals with unit elliptic genus $\Phi = 1$:

$$\mathcal{R}^\star(\Gamma_{(d,d)}; s) = \text{R.N.} \int_{\mathcal{F}} \mathrm{d}\mu\, \tau_2^{d/2} \sum_{m_i, n^i} q^{\frac{1}{2}p_L^2}\, \bar{q}^{\frac{1}{2}p_R^2}\, E^\star(s,\tau)\,,$$

$$= \zeta^\star(2s) \int_0^\infty \mathrm{d}\tau_2\, \tau_2^{s+d/2-2} \sum_{m_i n^i = 0} e^{-\pi\tau_2 \mathcal{M}^2}$$

$$= \zeta^\star(2s)\, \frac{\Gamma(s+\frac{d}{2}-1)}{\pi^{s+\frac{d}{2}-1}}\, \mathcal{E}^d_V(g,B; s+\tfrac{d}{2}-1)$$

where $\mathcal{E}^d_V(g,B;s)$ is the constrained Epstein Zeta series

$$\mathcal{E}^d_V(g,B;s) \equiv \sum_{\substack{(m_i,n^i)\in\mathbb{Z}^{2d}\backslash(0,0) \\ m_i n^i = 0}} \mathcal{M}^{-2s}\,, \qquad \mathcal{M}^2 = p_L^2 + p_R^2$$

Epstein series from modular integrals

- The constrained Epstein Zeta series $\mathcal{E}^{d}_{V}(g, B; s)$ converges absolutely for $s + \frac{d}{2} - 1 > 1$. The RSZ method shows that it admits a meromorphic continuation in the s-plane satisfying

$$\mathcal{E}^{d\star}_{V}(g, B; s) = \mathcal{E}^{d\star}_{V}(g, B; d - 1 - s)\,,$$

where

$$\mathcal{E}^{d\star}_{V}(g, B; s) = \pi^{-s}\Gamma(s)\zeta^{\star}(2s - d + 2)\,\mathcal{E}^{d}_{V}(g, B; s)$$

- Moreover $\mathcal{E}^{d\star}_{V}(s)$ has a simple pole at $s = 0, \frac{d}{2} - 1, \frac{d}{2}, 1$ (for $d > 2$) and is an eigenmode of the Laplace-Beltrami operator on the Grassmannian $G_{d,d}$ with eigenvalue $s(s - d + 1)$, as a result of

$$0 = \left[\Delta_{\mathrm{SO}(d,d)} - 2\,\Delta_{\mathrm{SL}(2)} + \tfrac{1}{4}\,d(d - 2)\right]\,\mathit{\Gamma}_{(d,d)}(g, B)$$
$$0 = \left[\Delta_{\mathrm{SL}(2)} - \tfrac{1}{2}\,s(s - 1)\right]\,E^{\star}(\tau; s)\,,$$

- The residue at $s = \frac{d}{2}$ produces the modular integral of interest:

$$\mathrm{R.N.} \int_{\mathcal{F}} \mathrm{d}\mu \, \Gamma_{(d,d)}(g,B) = \frac{\pi}{3} \frac{\Gamma(d/2)}{\pi^{d/2}} \, \mathrm{Res} \, \mathcal{E}^{d}_{V}(g,B;s)\Big|_{s=d/2}$$
$$= \frac{\Gamma(d/2-1)}{\pi^{d/2-1}} \, \mathcal{E}^{d}_{V}\left(g,B;\tfrac{1}{2}d-1\right)$$

rigorously proving an old conjecture of Obers and myself.

- This is identified as a sum over all BPS states of momentum m_i and winding n^i along the torus, $O(\Gamma_{d,d})$-invariant mass

$$\mathcal{M}^2 = (m_i + B_{ik} n^k) g^{ij} (m_j + B_{jl} n^l) + n^i g_{ij} n^j$$

subject to the $O(\Gamma_{d,d})$-invariant BPS condition $m_i n^i = 0$.

Epstein series and BPS state sums II

- For $d=1$ or $d=2$:

$$\mathcal{E}_V^{1,\star}(g,B;s-\tfrac{1}{2}) = 2\,\zeta^\star(2s)\,\zeta^\star(2s-1)\,\left(R^{1-2s}+R^{2s-1}\right)$$
$$\mathcal{E}_V^{2\star}(T,U;s) = 2\,E^\star(T;s)\,E^\star(U;s)$$

leading immediately to advertized results.

- By the Siegel-Weil formula, $\mathcal{E}_V^{d\star}(g,B;s)$ is also equal to the Langlands-Eisenstein series of $G=O(d,d)$ with character $\lambda=-2s\alpha_1$,

$$\mathcal{E}_V^{d\star}(g;s) = \sum_{G(\mathbb{Z})/(P\cap G(\mathbb{Z}))} e^{\langle\rho+\lambda,a(g)\rangle}|_\gamma\,, \quad g=k\cdot a\cdot n$$

- The residue at $s=\frac{d}{2}$ is the minimal theta series, attached to the minimal representation of $SO(d,d)$ (functional dimension $2d-3$).

Ginzburg Rallis Soudry; Kazhdan BP Waldron; Green Vanhove Miller

Higher genus I

- Similar techniques can be used to evaluate modular integrals at genus $h > 1$, at least for $h = 2, 3$ where the Schottky problem does not arise. Consider the completed Eisenstein series

$$\mathcal{E}^{\star}(\Omega, s) = \zeta^{\star}(2s) \prod_{j=1}^{[h/2]} \zeta^{\star}(4s - 2j) \sum_{(C,D)} \left[\frac{\det \mathrm{Im}(\Omega)}{|\det(C\Omega + D)|^2} \right]^{-s} ,$$

It converges for $\mathrm{Re}(s) > (h+1)/2$, can be meromorphically continued to the full s plane and satisfies the functional relation

$$\mathcal{E}^{\star}(\Omega, s) = \mathcal{E}^{\star}(\Omega, \tfrac{h+1}{2} - s)$$

and has a simple pole at $s = \frac{h+1}{2}$ (and $s = 0$) with residue

$$\frac{1}{2} \prod_{j=1}^{[h/2]} \zeta^{\star}(2j+1)$$

- The (suitably renormalized) modular integral can be computed by unfolding

$$I_h(s) = \mathrm{R.N.} \int_{\mathcal{F}_h} \mathrm{d}\mu \, \mathcal{E}(\Omega, s) \, \Gamma^{(h)}_{d,d}(g, B; \Omega)$$
$$= \int_{\mathcal{S}_h} \frac{\mathrm{d}\Omega}{[\det(\mathrm{Im}\Omega)]^{m+1-s}} \left(\Gamma^{(h)}_{d,d}(g, B; \Omega) - [\det(\mathrm{Im}\Omega)]^{d/2} \right)$$

The integral over $\mathrm{Re}(\Omega)$ imposes the BPS constraints

$$P_L \, P_L^t = P_R \, P_R^t$$

Higher genus III

- The integral over $\omega \equiv \mathrm{Im}(\Omega) \in GL(h)/SO(h)$ leads to

$$I_h(s) = \prod_{k=1}^{h} \Gamma\left(\tfrac{s-h-1}{2} - \tfrac{k-1}{2} + \tfrac{d}{4}\right) \sum_{\mathrm{BPS}} \left[\det(P_L P_L^t + P_R P_R^t)\right]^{-\frac{s-h-1}{2}-\frac{d}{4}}$$

- The modular integral of interest is proportional to the residue at $s = (h+1)/2$,

$$\int_{\mathcal{F}_h} \mathrm{d}\mu\, \Gamma_{d,d}^{(h)}(g,B;\Omega) = \mathrm{Res}_{s=(h+1)/2} \frac{2\, I_h(s)}{\prod_{j=1}^{[h/2]} \zeta^\star(2j+1)}$$

$$\propto \sum_{\mathrm{BPS}} \left[\det(P_L P_L^t + P_R P_R^t)\right]^{-\frac{d}{4}}$$

Higher genus IV

- This should be compared with the Obers-BP conjecture

$$\int_{\mathcal{F}_h} \mathrm{d}\mu \, \Gamma^{(h)}_{d,d}(g, B; \Omega) \propto \mathcal{E}^{d\star}_{S}(h) + \mathcal{E}^{d\star}_{C}(h)$$

  where $\mathcal{E}^{d\star}_{S,C}(s)$ are constrained lattice sums $\sum \mathcal{M}^{-2s}$ in spinor representations of $SO(d, d)$.

Outline

Modular integrals with unphysical tachyons I

- For many cases of interest, the integrand is NOT of moderate growth at the cusp, rather it grows exponentially, due to the heterotic unphysical tachyon.
- In mathematical terms, $\Phi(\tau) \in \mathbb{C}[\hat{E}_2, E_4, E_6, 1/\Delta]$ is a weak almost holmorphic modular form with weight $w = -k/2 \leq 0$.
- The RSZ method fails, however the unfolding trick could still work provided $\Phi(\tau)$ had a uniformly convergent Poincaré representation

$$\Phi(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f(\tau)|_w \gamma$$

where the seed $f(\tau)$ is invariant under $\tau \to \tau + 1$ and

$$(f|_w\gamma)(\tau) = (c\tau + d)^{-w} f(\gamma \cdot \tau) , \qquad \gamma \cdot \tau = \frac{a\tau + b}{c\tau + d} .$$

Various Poincaré series representations I

- Naively, one requires $f(\tau) = 1/q^\kappa$ ($\kappa = 1$ for physics applications), however convergence requires $f(\tau) \ll \tau_2^{1-\frac{w}{2}}$ as $\tau_2 \to 0$. This is OK for $w > 2$ but fails for $w \leq 0$. We need to regularize.
- Any weak holomorphic modular form can be represented as a linear combination of regularized holomorphic Poincaré series

$$P(\kappa, w) = \frac{1}{2} \sum_{(c,d)=1}^{!} (c\tau + d)^{-w} \, e^{-2\pi i\kappa \, \frac{a\tau+b}{c\tau+d}} \, R_w\left(\frac{2\pi i\kappa}{c(c\tau + d)} \right) ,$$

where $R_w(x) \sim x^{1-w}/\Gamma(2 - w)$ as $x \to 0$ and approaches 1 as $x \to \infty$. However this is only conditionally convergent, and $P(\kappa, w)$ in general has modular anomalies.

Niebur; Knopp; Manschot Moore

Various Poincaré series representations II

- Another option is to insert a non-holomorphic convergence factor and consider the Selberg-Poincaré series with $f(\tau) = \tau_2^{s-\frac{w}{2}}\, q^{-\kappa}$,

$$E(s,\kappa,w) \equiv \frac{1}{2} \sum_{(c,d)=1} \frac{\tau_2^{s-\frac{w}{2}}}{|c\tau+d|^{2s-w}}\,(c\tau+d)^{-w}\, e^{-2\pi\mathrm{i}\kappa\,\frac{a\tau+b}{c\tau+d}}$$

Selberg;Goldfeld Sarnak; Pribitkin

This converges absolutely for $\mathrm{Re}(s) > 1$, but the analytic continuation to $s = \frac{w}{2}$ is tricky (no modular anomaly, but in general holomorphic anomalies).

- Moreover, $E(s,\kappa,w)$ is not an eigenmode of the Laplacian, rather

$$\left[\Delta_w + \tfrac{1}{2}\, s(1-s) + \tfrac{1}{8}\, w(w+2)\right]\, E(s,\kappa,w) = 2\pi\kappa\,(s-\tfrac{w}{2})\, E(s+1,\kappa,w)$$

Niebur-Poincaré series I

- There exist yet another regularization which does not require analytic continuation and is still an eigenmode of the Laplacian: the Niebur-Poincaré series

$$\mathcal{F}(s,\kappa,w) = \frac{1}{2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathcal{M}_{s,w}(-\kappa\tau_2)\, e^{-2\pi i\kappa\tau_1} \,|_w\, \gamma$$

Niebur; Hejhal; Bruinier Ono Bringmann...

where $\mathcal{M}_{s,w}(y)$ is proportional to a Whittaker function, so that

$$\left[\Delta_w + \frac{1}{2}\, s(1-s) + \frac{1}{8}\, w(w+2)\right] \mathcal{F}(s,\kappa,w) = 0$$

- The seed $f(\tau) = \mathcal{M}_{s,w}(-\kappa\tau_2)\, e^{-2\pi i\kappa\tau_1}$ satisfies

$$f(\tau) \sim_{\tau_2\to 0} \tau_2^{\mathrm{Re}(s)-\frac{w}{2}}\, e^{-2\pi i\kappa\tau_1} \qquad f(\tau) \sim_{\tau_2\to\infty} \frac{\Gamma(2s)}{\Gamma(s+\frac{w}{2})}\, q^{-\kappa}$$

hence $\mathcal{F}(s,\kappa,w)$ converges absolutely for $\mathrm{Re}(s) > 1$.

Niebur-Poincaré series II

- For $s = 1 - \frac{w}{2}$ the eigenvalue coincides with that of a holomorphic modular form, and the seed simplifies to

$$f(\tau) = \Gamma(2-w)\left(q^{-\kappa} - \bar{q}^{\kappa} \sum_{\ell=0}^{-w} \frac{(4\pi\kappa\tau_2)^{\ell}}{\ell!}\right)$$

- For $w < 0$, the value $s = 1 - \frac{w}{2}$ lies in the convergence domain. $\mathcal{F}(1-\frac{w}{2},\kappa,w)$ is in general NOT holomorphic, but rather a weak harmonic Maass form.
- For $s = \frac{w'}{2}$ and $w' > 0$, $\mathcal{F}(\frac{w'}{2},\kappa,w')$ IS weakly holomorphic. For $w' = 2 - w$, it is the Farey transform (or the 'ghost') of the weak harmonic Maass form $\mathcal{F}(1-\frac{w}{2},\kappa,w)$.

Niebur-Poincaré series III

w	$\mathcal{F}(1-\frac{w}{2},1,w)$	$\mathcal{F}(1-\frac{w}{2},1,2-w)$
0	$j+24$	$E_4^2 E_6\, \Delta^{-1}$
-2	$3!\, E_4 E_6\, \Delta^{-1}$	$E_4(j-240)$
-4	$5!\, E_4^2\, \Delta^{-1}$	$E_6(j+204)$
-6	$7!\, E_6\, \Delta^{-1}$	$E_4^2(j-480)$
-8	$9!\, E_4\, \Delta^{-1}$	$E_4 E_6(j+264)$
-10	$11!\, \Phi_{-10}$	$(mess)$
-12	$13!\, \Delta^{-1}$	$E_4^2 E_6(j+24)$
-14	$15!\, \Phi_{-14}$	$(mess)$

Niebur-Poincaré series IV

- Indeed, for $w = -10$, there does not exist any weak holomorphic modular form with a simple pole at the cusp. Rather, there exist a weak harmonic Maass form

$$\begin{aligned}\Phi_{-10} =& q^{-1} - \tfrac{65520}{691} - 1842.89\, q - 23274.08\, q^2 + \dots \\ &+ \sum_{m=1}^{\infty} m^{-11}\, \bar{b}_m\, \Gamma(11, 4\pi m \tau_2)\, q^{-m}\end{aligned}$$

Ono

with shadow $\sum b_m q^m$ proportional to the cusp form Δ.

- Theorem (Bruinier) : any weak holomorphic modular form of weight $w \le 0$ with polar part $\Phi = \sum_{-\kappa \le m < 0} a_m\, q^m + \mathcal{O}(1)$ can be represented as a linear combination of Niebur-Poincaré series

$$\Phi = \frac{1}{\Gamma(2-w)} \sum_{-\kappa \le m < 0} a_m\, \mathcal{F}(1 - \tfrac{w}{2}, m, w) + a_0'\, \delta_{w,0}$$

Niebur-Poincaré series V

- Almost weak holomorphic modular forms can be reached by raising and lowering operators

$$D_w = \frac{\mathrm{i}}{\pi}\left(\partial_\tau - \frac{\mathrm{i}w}{2\tau_2}\right)\,, \qquad \bar{D}_w = -\mathrm{i}\pi\,\tau_2^2\partial_{\bar\tau}\,,$$

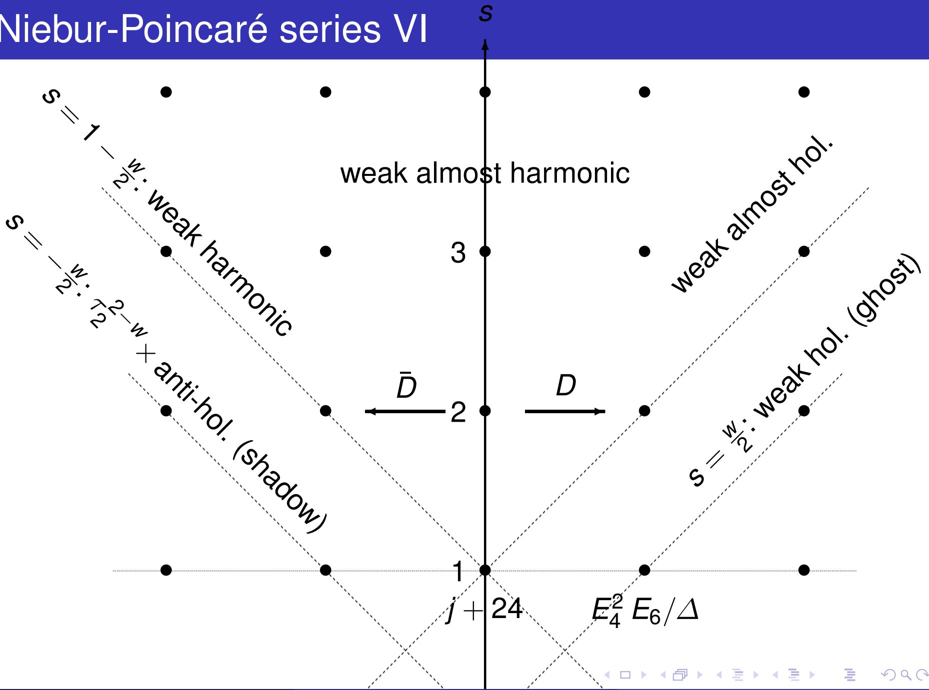
under which

$$D_w\cdot\mathcal{F}(s,\kappa,w) = 2\kappa\,(s+\tfrac{w}{2})\,\mathcal{F}(s,\kappa,w+2)\,,$$
$$\bar{D}_w\cdot\mathcal{F}(s,\kappa,w) = \frac{1}{8\kappa}(s-\tfrac{w}{2})\,\mathcal{F}(s,\kappa,w-2)\,.$$

The relevant values of s are $s = 1-\frac{w}{2}+n$ with $n\geq 0$. E.g.

$$\frac{\hat{E}_2 E_4 E_6}{\Delta} = \mathcal{F}(2,1,0) - 5\,\mathcal{F}(1,1,0) - 144$$

Niebur-Poincaré series VI



Unfolding the modular integral

- Using Bruinier's thm, any modular integral can be expressed as a linear combination of

$$\mathcal{I}_{d+k,d}(s,\kappa;\,) = \text{R.N.} \int_{\mathcal{F}} \mathrm{d}\mu \, \Gamma_{d+k,d}(G,B,Y)\, \mathcal{F}(s,\kappa,-\tfrac{k}{2})$$

Unfolding the modular integral

- Using Bruinier's thm, any modular integral can be expressed as a linear combination of

$$\mathcal{I}_{d+k,d}(s,\kappa;\,) = \mathrm{R.N.} \int_{\mathcal{F}} \mathrm{d}\mu\, \Gamma_{d+k,d}(G,B,Y)\, \mathcal{F}(s,\kappa,-\tfrac{k}{2})$$

- Using the unfolding trick, one arrives at the BPS state sum

$$\begin{aligned}\mathcal{I}_{d+k,d}(s,\kappa) =& (4\pi\kappa)^{1-\frac{d}{2}}\, \Gamma(s+\tfrac{2d+k}{4}-1) \\ & \times \sum_{\mathrm{BPS}} {}_2F_1\left(s-\tfrac{k}{4}\,,\, s+\tfrac{2d+k}{4}-1\,;\, 2s\,;\, \tfrac{4\kappa}{p_{\mathrm{L}}^2}\right) \left(\frac{p_{\mathrm{L}}^2}{4\kappa}\right)^{1-s-\frac{2d+k}{4}}\end{aligned}$$

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where $\sum_{\mathrm{BPS}} \equiv \sum_{p_{\mathrm{L}},\, p_{\mathrm{R}}} \delta(p_{\mathrm{L}}^2 - p_{\mathrm{R}}^2 - 4\kappa)$. This converges absolutely for $\mathrm{Re}(s) > \frac{2d+k}{4}$ and can be analytically continued to $\mathrm{Re}(s) > 1$ with a simple pole at $s = \frac{2d+k}{4}$.

Unfolding the modular integral

- The result is manifestly T-duality invariant, and requires no choice of chamber in Narain modular space. Singularities on $G_{d+k,d}$ arise when $p_L^2 = 0$ for some lattice vector.
- For the relevant values $s = 1 - \frac{w}{2} + n$, the result can be written using elementary functions, e.g.

$$\mathcal{I}_{1,1}(1+n,\kappa) = \frac{1}{2}\sqrt{\pi}\,(16\,\kappa)^{1+n}\,\Gamma(n+\tfrac{1}{2}) \times \sum_{\substack{p,q\in\mathbb{Z}\\ pq=\kappa}} \left(\left|p\,R + q\,R^{-1}\right| + \left|p\,R - q\,R^{-1}\right|\right)^{-1-2n}$$

$$\mathcal{I}_{2+k,2}(1+\tfrac{k}{4},\kappa) = -\,\Gamma(2+\tfrac{k}{2}) \sum_{\mathrm{BPS}} \left[\log\left(\frac{p_{\mathrm{R}}^2}{p_{\mathrm{L}}^2}\right) + \sum_{\ell=1}^{k/2} \frac{1}{\ell}\left(\frac{p_{\mathrm{L}}^2}{4\kappa}\right)^{-\ell}\right]$$

One example

- Consider $\mathrm{Het}/T^2 \times K3$ at $\mathbb{Z}_2$ orbifold point with gauge group broken to $E_8 \times E_7 \times SU(2)$. The gauge threshold for E_7 is

$$\Delta_{\mathrm{E}_7} = -\frac{1}{12} \int_{\mathcal{F}} \mathrm{d}\mu\, \Gamma_{2,2}\, \frac{\hat{E}_2\, E_4\, E_6 - E_4^3}{\Delta}$$

Expressing the elliptic genus as a linear combination

$$\frac{\hat{E}_2\, E_4\, E_6 - E_4^3}{\Delta} = \mathcal{F}(2,1,0) - 6\, \mathcal{F}(1,1,0) - 864$$

one arrives at

$$\Delta_{\mathrm{E}_7} = \sum_{\mathrm{BPS}} \left[1 + \frac{p_{\mathrm{R}}^2}{4} \log \left(\frac{p_{\mathrm{R}}^2}{p_{\mathrm{L}}^2} \right) \right] - 72 \log \left(4\pi\, e^{-\gamma}\, T_2\, U_2\, |\eta(T)\, \eta(U)|^4 \right)$$

Outline

Black holes in $D=4$ and instantons in $D=3$ I

- 1/4-BPS black holes in $\mathcal{N}=4$ CHL-type vacua are counted by a Siegel modular form of genus 2 and weight k, where $r=2k+8$ is the rank of the lattice of electric charges ($k=10$ in Het/T^6). Invariance under $G_4 = SL(2,\mathbb{Z}) \times SO(6, r-6, \mathbb{Z})$ is manifest.

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- Suitable BPS couplings in $D=3$ (e.g. $\nabla^2 R^2$) should receive instanton corrections from 1/4-BPS black holes in $D=4$, along with KK monopoles. Yet they should be invariant under the 3D U-duality group $G_3 = SO(8, r-4, \mathbb{Z})$.
- 1/4-BPS black holes in $M/K3 \times T^4$ can be represented by M5-branes wrapping around genus 2 curve $\Sigma \subset T^4$. On the heterotic side, one should include all genus 2 wordsheet instantons in T^7, plus NS5 and KK monopoles.

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Black holes in $D = 4$ and instantons in $D = 3$ II

- Thus, it is natural to conjecture

$$f^{D=3}_{\nabla^2 R^2} = \int_{\mathcal{F}_2} \frac{\mathrm{d}^3\Omega \mathrm{d}^3\bar{\Omega}}{(\det \mathrm{Im}\Omega)^3} \; \frac{Z_{8,r-4}(\Omega)\; \hat{E}_2(\Omega)\,(\det \mathrm{Im}\Omega)^4}{\Phi_k}$$

  where $Z_{8,r-4}$ is the partition function of the non-perturbative lattice, and $\hat{E}_2$ is the almost holomorphic Eisenstein series of weight 2.
- At weak heterotic coupling, one should recover the two-loop amplitude, invariant under $SO(7, r-5, \mathbb{Z})$, plus other perturbative corrections.
- At large radius, one should recover a sum over 1/4-BPS states, weighted by their entropy, along with KK monopoles

$$f^{D=3}_{\nabla^2 R^2} \stackrel{?}{=} f^{D=4}_{\nabla^2 R^2} + \sum \Omega(\gamma)\, e^{-R\mathcal{M}(\gamma)} + \sum_k e^{-R^2 k} + \dots$$

  Since $\Omega(\ell\gamma) \sim e^{\ell^2}$ while $\mathcal{M}(\ell\gamma) \sim \ell$, the series is at best asymptotic...

Conclusion - Outlook

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- Modular integrals can be efficiently computed using Rankin-Selberg type methods. While the time-honored 'orbit method' is still useful for studying large volume limits, our method makes T-duality and singularities manifest.
- For application to orbifold models, it would be useful to extend this method to congruence subgroups of $SL(2,\mathbb{Z})$.
- More interestingly, it would be useful to find Poincaré series representations for Siegel modular forms of higher genus.