

Rankin-Selberg methods for String Amplitudes

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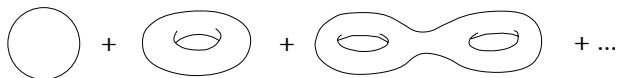
Institut Paul Painlevé, Lille, March 5, 2014

*based on work with C. Angelantonj and I. Florakis,
arXiv:1110.5318,1203.0566,1304.4271,1401.4265 and work in progress*

String theory in a nutshell I

- Scattering amplitudes of n external states in perturbative string theory have a topological expansion

$$\mathcal{A}_n = \sum_{h=0}^{\infty} g_s^{2h-2} \int_{\mathfrak{M}_{h,n}} d\mu_{h,n} F_{h,n}$$



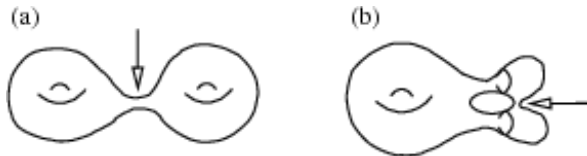
where $F_{h,n}$ is a correlator of n vertex operators (along with ghost insertions) in a certain SCFT on a Riemann surface Σ_h of genus h with n punctures z_i , integrated over the **moduli space of super-Riemann surfaces** $\mathfrak{M}_{h,n}$.

String theory in a nutshell II

- After integrating over the positions of the punctures and fermionic part of supermoduli, one is left with an integral over the (ordinary) **moduli space of Riemann surfaces** \mathcal{M}_h :

$$\mathcal{A}_h = \int_{\mathcal{M}_h} d\mu_h F_h$$

- There is no canonical way of projecting the supermoduli space onto bosonic moduli space. Different projections differ by total derivatives on \mathcal{M}_h , which can in principle be fixed by matching with QFT behavior at the boundaries.

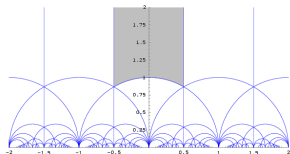


String theory in a nutshell III

- The moduli space $\mathcal{M}_h = \mathcal{T}_h/\Gamma_h$ is the quotient of the **Teichmüller space** \mathcal{T}_h by the **mapping class group** Γ_h . The integrand is naturally a function on \mathcal{T}_h invariant under Γ_h .
- \mathcal{T}_h is the analog of the space of Schwinger/Feynman parameters in QFT, while Γ_h has no analog in QFT. The quotient by Γ_h is largely responsible for the UV finiteness of string theory.
- For genus $h \leq 3$, the Teichmüller space \mathcal{T}_h is isomorphic to (an open set in) the **Siegel-Poincaré upper half plane** \mathcal{H}_h , parametrized by the **period matrix** Ω , a complex $h \times h$ symmetric matrix with positive definite imaginary part. The integrand $F_h(\Omega)$ is a Siegel modular form for $\Gamma_h = Sp(2h, \mathbb{Z})$, acting as $\Omega \mapsto (A\Omega + B) \cdot (C\Omega + D)^{-1}$.

String theory in a nutshell IV

- At genus 1, \mathcal{T}_1 is the Poincaré upper-half plane, parametrized by $\Omega_{11} \equiv \tau = \tau_1 + i\tau_2$ and the integrand F_1 is invariant under $SL(2, \mathbb{Z})$. A convenient choice of fundamental domain is



- τ_2 can be interpreted as a **Schwinger parameter** while τ_1 (for $\tau_2 > 1$) a **Lagrange multiplier** projecting the spectrum on level-matched states

- E.g. the one-loop vacuum amplitude in bosonic closed string theory in $D = 26$ flat space time is proportional to

$$\mathcal{A}_1 = \int_{\mathcal{F}} \frac{d\tau_1 d\tau_2}{\tau_2^{1+D/2}} \frac{1}{|\eta|^{2(D-2)}}$$

where $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ is the Dedekind eta function ($q = e^{2\pi i\tau}$)

- For genus 2, it takes 25 inequalities to define \mathcal{F}_2 !
- For genus $h \geq 4$, \mathcal{T}_h is a codimension $\frac{1}{2}(h-2)(h-3)$ locus inside \mathcal{H}_h known as the **Schottky locus**. It is not clear how to extend F_h to a modular form on \mathcal{H}_h .

Rankin-Selberg method / unfolding trick I

- Our goal is to develop methods to **compute integrals of Siegel modular forms over a fundamental domain of the Siegel upper-half plane analytically**.
- The key idea is to **represent the integrand as a Poincaré series**,

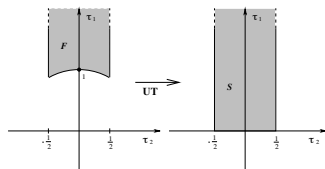
$$F_h(\Omega) = \sum_{\gamma \in \Gamma_{h,\infty} \backslash \Gamma_h} f_h|_{\gamma}(\Omega)$$

where $f_h|_{\gamma}(\Omega) = f_h(\gamma \cdot \Omega)$ and the ‘seed’ $f_h(\Omega)$ is invariant under a subgroup $\Gamma_{h,\infty} \subset \Gamma_h$. Typically, $\Gamma_{h,\infty}$ is the **stabilizer of the cusp at infinity**, acting by integer shifts of Ω_1 .

Rankin-Selberg method / unfolding trick II

- Provided the sum is absolutely convergent, one can exchange the sum and integral and obtain

$$\int_{\Gamma_h \backslash \mathcal{H}_h} d\mu_h F_h(\Omega) = \int_{\Gamma_{\infty, h} \backslash \mathcal{H}_h} d\mu_h f_h(\Omega) .$$



- We gain if $\Gamma_{\infty, h} \backslash \mathcal{H}_h$ and f_h are simpler than $\Gamma_h \backslash \mathcal{H}_h$ and F_h !
- This method is limited by our ability to represent the integrand as a Poincaré series. Not much is known in genus $h > 1$. In genus one, any weakly, almost holomorphic modular form of negative weight can be represented as a Poincaré series.

Rankin-Selberg method / unfolding trick III

- We shall focus on a class of one-loop amplitudes of the form

$$\mathcal{A} = \int_{\mathcal{F}} d\mu \Gamma_{d+k,d} \Phi(\tau), \quad d\mu = \frac{d\tau_1 d\tau_2}{\tau_2^2}$$

where $\Phi(\tau)$ is a weakly, almost holomorphic modular form of weight $w = -k/2$ (the **elliptic genus**) and $\Gamma_{(d+k,d)}$ is a Siegel Theta series (the **Narain lattice partition function**) for an **even self-dual lattice** (Γ, B) of signature $(d+k, d)$,

$$\Gamma_{(d+k,d)} = \tau_2^{d/2} \sum_{p \in \Gamma} e^{-\pi\tau_2 \mathcal{M}^2(p) + 2\pi i \tau_1 \langle p, p \rangle}$$

- The positive definite quadratic form $\mathcal{M}^2(p)$ is parametrized by the orthogonal Grassmannian

$$G_{d+k,d} = \frac{O(d+k, d)}{O(d+k) \times O(d)} \ni (g_{ij}, B_{ij}, Y_i^a),$$

- Such modular integrals arise in certain **BPS-saturated amplitudes**, such as F^2, R^2, F^4, R^4 in type II string theory ($k = 0$) or heterotic string ($k = 8, 16$) compactified on a torus T^d .
- \mathcal{A} is invariant under **T-duality**, i.e. under the automorphisms of the lattice. Mathematically, $\Phi \mapsto \mathcal{A}$ is a **Theta correspondence** between $SL(2, \mathbb{Z})$ and $O(\Gamma_{d+k, d})$ automorphic forms.
- In the physics literature, such integrals are typically computed the **orbit method**, i.e. by applying the unfolding trick to $I_{(d+k, d)}$. Instead, we shall apply the unfolding trick to $\Phi(\tau)$, which has the advantage of keeping T-duality manifest throughout.

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- Consider the completed **non-holomorphic Eisenstein series**

$$E^*(\tau; s) = \zeta^*(2s) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \tau_2^s |\gamma| = \frac{1}{2} \zeta^*(2s) \sum_{(c,d)=1} \frac{\tau_2^s}{|c\tau + d|^{2s}}$$

where $\zeta^*(s) \equiv \pi^{-s/2} \Gamma(s/2) \zeta(s) = \zeta^*(1-s)$.

- $E^*(\tau; s)$ is convergent for $\text{Re}(s) > 1$, and has a meromorphic continuation to all s , invariant under $s \mapsto 1-s$, with simple poles at $s = 0, 1$ with **constant residue**:

$$E^*(\tau; s) = \frac{1}{2(s-1)} + \frac{1}{2} \left(\gamma - \log(4\pi \tau_2 |\eta(\tau)|^4) \right) + \mathcal{O}(s-1),$$

- For any modular function $F(\Omega)$ of rapid decay, consider the **Rankin-Selberg transform**

$$\mathcal{R}^*(F, s) = \int_{\mathcal{F}} d\mu E^*(\tau; s) F(\tau)$$

- By the unfolding trick, $\mathcal{R}^*(F, s)$ is proportional to the **Mellin transform** of the constant term $F_0(\tau_2) = \int_{-1/2}^{1/2} d\tau_1 F(\tau)$,

$$\begin{aligned} \mathcal{R}^*(F; s) &= \zeta^*(2s) \int_{\mathbb{R}^+ \times [-\frac{1}{2}, \frac{1}{2}]} d\mu \tau_2^s F(\tau) \\ &= \zeta^*(2s) \int_0^\infty d\tau_2 \tau_2^{s-2} F_0(\tau_2), \end{aligned}$$

Rankin-Selberg method (cont.)

- It inherits the meromorphicity and functional relations of E^* , e.g. $\mathcal{R}^*(F; s) = \mathcal{R}^*(F; 1 - s)$.
- Since the residue of $E^*(\tau; s)$ at $s = 0, 1$ is constant, the residue of $\mathcal{R}^*(F; s)$ at $s = 1$ is proportional to the modular integral of F ,

$$\text{Res}_{s=1} \mathcal{R}^*(F; s) = \frac{1}{2} \int_{\mathcal{F}} d\mu F$$

- This was extended by Zagier to the case where $F^{(0)}$ is of power-like growth $F^{(0)}(\tau) \sim \varphi(\tau_2)$ at the cusp:

$$\text{R.N.} \int_{\mathcal{F}} d\mu F(\tau) = 2 \text{Res}_{s=1} \mathcal{R}^*(F; s) + \delta$$

where δ is a scheme-dependent correction which depends only on the leading behavior $\varphi(\tau_2)$.

Epstein series from modular integrals

- The RSZ method applies immediately to $\mathcal{A} = \int_{\mathcal{F}} d\mu \Gamma_{d,d}(g, B)$:

$$\begin{aligned}\mathcal{R}^*(\Gamma_{d,d}; s) &= \zeta^*(2s) \int_0^\infty d\tau_2 \tau_2^{s+d/2-2} \sum_{p_L^2 - p_R^2 = 0} e^{-\pi\tau_2(p_L^2 + p_R^2)} \\ &= \zeta^*(2s) \frac{\Gamma(s + \frac{d}{2} - 1)}{\pi^{s + \frac{d}{2} - 1}} \mathcal{E}_V^d(g, B; s + \frac{d}{2} - 1)\end{aligned}$$

where $\mathcal{E}_V^d(g, B; s)$ is the **constrained Epstein series**

$$\mathcal{E}_V^d(g, B; s) \equiv \sum_{\substack{p=(m_i, n^i) \in \mathbb{Z}^{2d} \setminus (0,0) \\ m_i n^i = 0}} [\mathcal{M}^2(p)]^{-s},$$

a.k.a. **degenerate Langlands-Eisenstein series with infinitesimal character $\rho - 2s\alpha_1$**

Epstein series and BPS state sums I

- This is identified as a **sum over all BPS states** of momentum m_i and winding n^j , with mass

$$\mathcal{M}^2(p) = (m_i + B_{ik} n^k) g^{ij} (m_j + B_{jl} n^l) + n^i g_{ij} n^j$$

subject to the **BPS condition** $\langle p, p \rangle = m_i n^i = 0$. Invariance under $O(\Gamma_{d,d})$ is manifest.

- The constrained Epstein Zeta series $\mathcal{E}_V^d(g, B; s)$ converges absolutely for $\text{Re}(s) > d$. The RSZ method shows that it admits a meromorphic continuation in the s -plane satisfying

$$\mathcal{E}_V^{d*}(s) = \pi^{-s} \Gamma(s) \zeta^*(2s - d + 2) \mathcal{E}_V^d(s) = \mathcal{E}_V^{d*}(d - 1 - s),$$

with a simple pole at $s = 0, \frac{d}{2} - 1, \frac{d}{2}, d - 1$ (double poles if $d = 2$).

Epstein series and BPS state sums II

- The residue at $s = \frac{d}{2}$ produces the modular integral of interest:

$$\text{R.N.} \int_{\mathcal{F}} d\mu \Gamma_{d,d}(g, B) = \frac{\Gamma(\frac{d}{2} - 1)}{\pi^{\frac{d}{2}-1}} \mathcal{E}_V^d(g, B; \frac{d}{2} - 1)$$

rigorously proving an old conjecture of Obers and myself (1999).

- For $d = 2$, the BPS constraint $m_i n^i = 0$ can be solved, leading to

$$\mathcal{E}_V^{2*}(T, U; s) = 2 E^*(T; s) E^*(U; s)$$

hence to Dixon-Kaplunovsky-Louis famous result (1989)

$$\int_{\mathcal{F}} (\Gamma_{2,2}(T, U) - \tau_2) d\mu = -\log \left(T_2 U_2 |\eta(T) \eta(U)|^4 \right) + \text{cte}$$

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Modular integrals with unphysical tachyons I

- For many cases of interest, the integrand is NOT of moderate growth at the cusp, rather it grows exponentially, due to the heterotic unphysical tachyon, $\Phi(\tau) \sim 1/q^\kappa + \mathcal{O}(1)$ with $\kappa = 1$.
- In mathematical terms, $\Phi(\tau) \in \mathbb{C}[\hat{E}_2, E_4, E_6, 1/\Delta]$ is an **almost, weakly holomorphic modular** form with weight $w = -k/2 \leq 0$.
- The RSZ method fails, however the unfolding trick could still work provided $\Phi(\tau)$ can be represented as a **uniformly convergent Poincaré series** with seed $f(\tau)$ is invariant under $\Gamma_\infty : \tau \rightarrow \tau + n$,

$$\Phi(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f(\tau)|_w \gamma$$

- Convergence requires $f(\tau) \ll \tau_2^{1-\frac{w}{2}}$ as $\tau_2 \rightarrow 0$. The choice $f(\tau) = 1/q^\kappa$ works for $w > 2$ but fails for $w \leq 2$.

Niebur-Poincaré series I

- A very convenient basis is provided by the **Niebur-Poincaré series**

$$\mathcal{F}(s, \kappa, w) = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} f(\tau)|_w \gamma$$

where the seed $f(\tau) = |4\pi\kappa\tau_2|^{-\frac{w}{2}} M_{-\frac{w}{2} \operatorname{sgn}(\kappa), s - \frac{1}{2}}(4\pi|\kappa|\tau_2) e^{-2\pi i \kappa \tau_1}$
is chosen so that

$$f(\tau) \sim_{\tau_2 \rightarrow 0} \tau_2^{s - \frac{w}{2}} e^{-2\pi i \kappa \tau_1} \quad f(\tau) \sim_{\tau_2 \rightarrow \infty} \frac{\Gamma(2s)}{\Gamma(s + \frac{w}{2})} q^{-\kappa}$$

- $\mathcal{F}(s, \kappa, w)$ converges absolutely for $\operatorname{Re}(s) > 1$ and satisfies

$$\left[\Delta_w + \frac{1}{2} \left(s - \frac{w}{2} \right) \left(1 - s - \frac{w}{2} \right) \right] \mathcal{F}(s, \kappa, w) = 0$$

Niebur; Hejhal; Bruinier Ono Bringmann...

Niebur-Poincaré series II

- Under raising and lowering operators,

$$D_w = \frac{i}{\pi} \left(\partial_\tau - \frac{iW}{2\tau_2} \right), \quad \bar{D}_w = -i\pi \tau_2^2 \partial_{\bar{\tau}},$$

the NP series transforms as

$$D_w \cdot \mathcal{F}(s, \kappa, w) = 2\kappa \left(s + \frac{w}{2} \right) \mathcal{F}(s, \kappa, w + 2),$$

$$\bar{D}_w \cdot \mathcal{F}(s, \kappa, w) = \frac{1}{8\kappa} \left(s - \frac{w}{2} \right) \mathcal{F}(s, \kappa, w - 2).$$

- Under Hecke operators,

$$H_{\kappa'} \cdot \mathcal{F}(s, \kappa, w) = \sum_{d|(\kappa, \kappa')} d^{1-w} \mathcal{F}(s, \kappa \kappa' / d^2, w).$$

- For congruence subgroups of $SL(2, \mathbb{Z})$, one can similarly define NP series $\mathcal{F}_\alpha(s, \kappa, w)$ for each cusp.

Niebur-Poincaré series III

- For $s = 1 - \frac{w}{2}$, the value relevant for weakly holomorphic modular forms, the seed simplifies to

$$f(\tau) = \Gamma(2 - w) \left(q^{-\kappa} - \bar{q}^{\kappa} \sum_{\ell=0}^{-w} \frac{(4\pi\kappa\tau_2)^\ell}{\ell!} \right)$$

- For $w < 0$, the value $s = 1 - \frac{w}{2}$ lies in the convergence domain, but $\mathcal{F}(1 - \frac{w}{2}, \kappa, w)$ is in general NOT holomorphic, but rather a **weakly harmonic Maass form**,

$$\Phi = \sum_{m=-\kappa}^{\infty} a_m q^m + \sum_{m=1}^{\infty} m^{w-1} \bar{b}_m \Gamma(1 - w, 4\pi m\tau_2) q^{-m}$$

- For any such form, $\bar{D}\Phi = \tau_2^{2-w} \bar{\Psi}$ where $\Psi = \sum_{m \geq 1} b_m q^m$ is a holomorphic cusp form of weight $2 - w$, the **shadow** of the Mock modular form $\Phi^- = \sum_{m=-\kappa}^{\infty} a_m q^m$.

Niebur-Poincaré series IV

- If $|w|$ is small enough, the negative frequency coefficients b_m vanish and Φ is in fact a weakly holomorphic modular form:

w	$\mathcal{F}(1 - \frac{w}{2}, 1, w)$
0	$j + 24$
-2	$3! E_4 E_6 / \Delta$
-4	$5! E_4^2 / \Delta$
-6	$7! E_6 / \Delta$
-8	$9! E_4 / \Delta$
-10	$11! \Phi_{-10}$
-12	$13! / \Delta$
-14	$15! \Phi_{-14}$

Here Φ_{-10} and Φ_{-14} are genuine harmonic Maass forms with shadow $2.8402... \times \Delta$ and $1.3061... \times E_4 \Delta$.

- Theorem (Bruinier) : any **weakly holomorphic** modular form of weight $w \leq 0$ with polar part $\Phi = \sum_{0 < m \leq \kappa} a_{-m} q^{-m} + \mathcal{O}(1)$ is a linear combination of Niebur-Poincaré series

$$\Phi = \frac{1}{\Gamma(2-w)} \sum_{0 < m \leq \kappa} a_{-m} \mathcal{F}\left(1 - \frac{w}{2}, m, w\right) + a'_0 \delta_{w,0}$$

(The same holds for congruence subgroups of $SL(2, \mathbb{Z})$, including contributions from all cusps)

- **Weakly almost holomorphic** modular forms of weight $w \leq 0$ can similarly be represented as linear combinations of $\mathcal{F}\left(1 - \frac{w}{2} + n, m, w\right)$ with $0 < m \leq \kappa, 0 \leq n \leq p$ where p is the depth. This fails for positive weight, as such forms are not necessarily harmonic !

Unfolding the modular integral

- By Bruinier's thm, any modular integral is a linear combination of

$$\mathcal{I}_{d+k,d}(s, \kappa) = \text{R.N.} \int_{\mathcal{F}} d\mu \Gamma_{d+k,d}(G, B, Y) \mathcal{F}(s, \kappa, -\frac{k}{2})$$

- Using the unfolding trick, one arrives at the **BPS state sum**

$$\begin{aligned} \mathcal{I}_{d+k,d}(s, \kappa) &= (4\pi\kappa)^{1-\frac{d}{2}} \Gamma(s + \frac{2d+k}{4} - 1) \\ &\times \sum_{\substack{p \in \Gamma \\ \langle p, p \rangle = \kappa}} {}_2F_1\left(s - \frac{k}{4}, s + \frac{2d+k}{4} - 1; 2s; \frac{4\kappa}{p_L^2}\right) \left(\frac{p_L^2}{4\kappa}\right)^{1-s-\frac{2d+k}{4}} \end{aligned}$$

Bruinier; Angelantonj Florakis BP

where $p_L^2 = \mathcal{M}^2(p) + 4\langle p, p \rangle$. This converges absolutely for $\text{Re}(s) > \frac{2d+k}{4}$ and can be analytically continued to $\text{Re}(s) > 1$ with a simple pole at $s = \frac{2d+k}{4}$.

Unfolding the modular integral

- For $s = 1 - \frac{w}{2} + n$, the values relevant for almost holomorphic modular forms, the summand can be written using elementary functions, e.g.

$$\mathcal{I}_{2+k,2}(1 + \frac{k}{4}, \kappa) = -\Gamma(2 + \frac{k}{2}) \sum_{\text{BPS}} \left[\log \left(\frac{p_R^2}{p_L^2} \right) + \sum_{\ell=1}^{k/2} \frac{1}{\ell} \left(\frac{p_L^2}{4\kappa} \right)^{-\ell} \right]$$

- The result is manifestly $O(\Gamma_{d+k,d})$ invariant, and requires no choice of chamber in Narain modular space. Singularities on $G_{d+k,d}$ arise when $p_L^2 = 0$ for some lattice vector.

Fourier-Jacobi expansion I

- For $d = 2, k = 0$, the Fourier expansion in T_1 (or U_1) can be obtained by solving the BPS constraint. E.g. for $\kappa = 1$, all solutions to $m_1 n^1 + m_2 n^2 = 1$ are

$$\begin{cases} m_1 = b + dM, & n^1 = -c \\ m_2 = a + cM, & n^2 = d \end{cases} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash SL(2, \mathbb{Z}), \quad M \in \mathbb{Z}$$

- After Poisson resummation over M , the sum over γ neatly produces Niebur-Poincaré series in U ,

$$\begin{aligned} \mathcal{I}(s, 1) &= 2^{2s} \sqrt{4\pi} \Gamma(s - \frac{1}{2}) T_2^{1-s} \mathcal{E}(U; s) \\ &+ 4 \sum_{N>0} \sqrt{\frac{T_2}{N}} K_{s-\frac{1}{2}}(2\pi N T_2) \left[e^{2\pi i N T_1} \underbrace{\mathcal{F}(s, N, 0; U)}_{=H_N \cdot \mathcal{F}(s, 1, 0; U)} + \text{cc} \right] \end{aligned}$$

Fourier-Jacobi expansion II

- The same result is obtained by the usual orbit method. In fact, both methods end up computing the same integral,

$$\int_{\mathcal{H}} d\mu e^{-\pi T_2 \frac{|U-\tau|^2}{\tau_2 U_2}} \mathcal{F}(\tau) = 2 T_2^{-1/2} e^{2\pi T_2} K_{s-\frac{1}{2}}(2\pi T_2) \mathcal{F}(U),$$

where $\mathcal{F}(\tau)$ is the seed of the NP series in the unfolding method, or the full NP series $\mathcal{F}(s, \kappa, 0; \tau)$ in the old orbit method.

- This formula works for any solution of $[\Delta + \frac{1}{2}s(1-s)]\mathcal{F}(\tau) = 0$, irrespective of modular invariance. It generalizes the **average value property** of harmonic functions.

Fourier-Jacobi expansion III

- For $s = 1$, using $\mathcal{F}(1, 1, 0; U) = j(U) + 24$ one finds

$$\begin{aligned}\mathcal{A} &= 8\pi \operatorname{Res}_{s=1} \left[T_2^{1-s} \mathcal{E}(s; U) \right] + 2 \sum_{N>0} \left[\frac{q_T^N}{N} H_N^{(U)} \cdot [j(U) + 24] + \text{cc} \right] \\ &= -24 \log \left[T_2 U_2 |\eta(T)\eta(U)|^4 \right] - \log |j(T) - j(U)|^4\end{aligned}$$

consistently with Borcherds product

$$q_T [j(T) - j(U)] = \prod_{M>0, N \in \mathbb{Z}} (1 - q_T^M q_U^N)^{c(MN)}, \quad j = \sum c(M) q^M$$

Borcherds; Harvey Moore

Fourier-Jacobi expansion IV

- For $s = 1 + n$, relevant for almost holomorphic modular forms of depth $p \geq n$, one can express $\mathcal{I}_{2,2}(n+1, 1)$ as the iterated derivative of a generalized prepotential,

$$\mathcal{I}_{2,2}(n+1, 1) = 4 \operatorname{Re} \left[\frac{(-D_T D_U)^n}{n!} f_n(T, U) \right]$$

where f_n is holomorphic in T but harmonic in U ,

$$f_n(T, U) = 2 (2\pi)^{2n+1} \mathcal{E}(n+1, -2n; U) \\ + \sum_{N>0} \frac{2q_T^N}{(2N)^{2n+1}} \mathcal{F}(n+1, N, -2n; U)$$

Fourier-Jacobi expansion V

- One can turn f_n into a holomorphic function $\tilde{f}_n(T, U)$ by replacing the Eisenstein series $\mathcal{E}(n+1, -2n; U)$ by its analytic part

$$\tilde{E}(n+1, -2n; \tau) = \frac{\zeta(2n+2)(2\pi i\tau)^{2n+1}}{(-4\pi^2)^{n+1}} + \frac{1}{2}\zeta(2n+1) + \sum_{N \geq 1} \sigma_{-1-2n}(N) q^N$$

without affecting the real part of its iterated derivative.

- The generalized holomorphic prepotential becomes

$$\begin{aligned} \tilde{f}_n(T, U) = & \sum_{N, M} c_n(NM) \operatorname{Li}_{2n+1}(q_T^M q_U^N) + \Gamma(2n+2) \operatorname{Li}_{2n+1}\left(\frac{q_T}{q_U}\right) \\ & + \frac{(-1)^n (2\pi)^{2n+2}}{2\zeta(2n+2)} \left[\zeta(2n+1) + \frac{\zeta(-2n-1)}{\Gamma(2n+2)} (2\pi i U)^{2n+1} \right] \end{aligned}$$

where $\mathcal{F}(n+1, 1, -2n) = \sum_{M \geq -1} c_n(M) q^M$.

Fourier-Jacobi expansion VI

- $\tilde{f}_n(T, U)$ now transforms as an **Eichler integral** of weight $(-2n, -2n)$ under $SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U \ltimes (T \leftrightarrow U)$,

$$(cU + d)^{2n} \tilde{f}_n \left(T, \frac{aU + b}{cU + d} \right) = \tilde{f}_n(T, U) + P_\gamma(U),$$

where $P_\gamma(U)$ is a computable polynomial of degree $\leq 2n$.

- The case $n = 1$ describes the standard prepotential appearing in string vacua with $\mathcal{N} = 2$ supersymmetry. Its modular anomaly was discussed by Antoniadis, Ferrara, Gava, Narain, Taylor in 1995, which is **the first occurrence of Eichler integrals in string theory !**
- The case $n = 2$ has appeared in the context of 1/4-BPS amplitudes in Het/K_3 . Higher n has not come up in physics yet, but is suggestive of CY_{2n+1} -fold.

Lerche Stieberger Warner

Outline

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Rankin-Selberg method at higher genus I

- String amplitudes at genus $h \leq 3$ take the form

$$\mathcal{A}_h = \int_{\mathcal{F}_h} d\mu_h \Gamma_{d+k,d,h}(G, B, Y; \Omega) \Phi(\Omega), \quad d\mu_h = \frac{d\Omega_1 d\Omega_2}{[\det \Omega_2]^{h+1}}$$

- \mathcal{F}_h is a fundamental domain of the action of $\Gamma = Sp(2h, \mathbb{Z})$ on Siegel's upper half plane $\{\Omega = \Omega^t \in \mathbb{C}^{h \times h}, \Omega_2 > 0\}$
- $\Gamma_{d+k,d,h}$ a Siegel-Narain theta series of signature $(d+k, d)$

$$\Gamma_{d+k,d,h} = [\det \Omega_2]^{d/2} \sum_{p_\alpha \in \Gamma_{d+k,d}, \alpha=1 \dots h} e^{-\pi \Omega_2^{\alpha\beta} \mathcal{M}^2(p_\alpha, p_\beta) + 2\pi i \Omega_1^{\alpha\beta} \langle p_\alpha, p_\beta \rangle}$$

- $\Phi(\Omega)$ a Siegel modular form of weight $-k/2$.
- We would like to generalize the previous methods to the case where $\Phi(\Omega)$ is an almost holomorphic modular form with poles inside \mathcal{F}_h , such as $1/\chi_{10}$. As a first step, take $k=0$, $\Phi=1$.

Rankin-Selberg method at higher genus II

- The genus h analog of $\mathcal{E}^*(s; \tau)$ is the non-holomorphic Siegel-Eisenstein series

$$\mathcal{E}_h^*(s; \Omega) = \zeta^*(2s) \prod_{j=1}^{[h/2]} \zeta^*(4s - 2j) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} [\det \Omega_2]^s |\gamma|$$

where $\Gamma_\infty = \left\{ \begin{pmatrix} A & B \\ 0 & A^{-t} \end{pmatrix} \right\} \subset \Gamma$.

- The sum converges absolutely for $\operatorname{Re}(s) > \frac{h+1}{2}$ and can be meromorphically continued to the full s plane. The analytic continuation is invariant under $s \mapsto \frac{h+1}{2} - s$, and has a simple pole at $s = \frac{h+1}{2}$ with constant residue $r_h = \frac{1}{2} \prod_{j=1}^{[h/2]} \zeta^*(2j + 1)$

Rankin-Selberg method at higher genus III

- For any modular function $F(\Omega)$ of rapid decay, the Rankin-Selberg transform can be computed by the unfolding trick,

$$\begin{aligned}\mathcal{R}_h^*(F; s) &= \int_{\mathcal{F}_h} d\mu_h F(\Omega) \mathcal{E}_h^*(\Omega, s) \\ &= \zeta^*(2s) \prod_{j=1}^{[h/2]} \zeta^*(4s - 2j) \int_{GL(h, \mathbb{Z}) \backslash \mathcal{P}_h} d\Omega_2 |\Omega_2|^{s-h-1} F_0(\Omega_2)\end{aligned}$$

where \mathcal{P}_h is the space of positive definite real matrices,
 $|\Omega_2| = \det \Omega_2$ and $F_0(\Omega_2) = \int_0^1 d\Omega_1 F(\Omega)$ is the constant term of F .

- The residue at $s = \frac{h+1}{2}$ is proportional to the average of F ,

$$\text{Res}_{s=\frac{h+1}{2}} \mathcal{R}_h^*(F; s) = r_h \int_{\mathcal{F}_h} F.$$

Rankin-Selberg method at higher genus IV

- The Siegel-Narain theta series is not a cusp form, instead its zero-th Fourier mode is

$$\Gamma_{d,d,h}^{(0)}(g, B; \Omega) = |\Omega_2|^{d/2} \sum_{(m_i^\alpha, n^{i\alpha}) \in \mathbb{Z}^{2d \times h}, m_i^{(\alpha} n^{i\beta)} = 0} e^{-\pi \Omega_{2\alpha\beta} \mathcal{M}^{2;\alpha\beta}}$$

where

$$\mathcal{M}^{2;\alpha\beta} = (m_i^\alpha + B_{ik} n^{k\alpha}) g^{ij} (m_j^\beta + B_{jl} n^{l\beta}) + n^{i\alpha} g_{ij} n^{j\beta}$$

Terms with $\text{Rk}(m_i^\alpha, n^{i\alpha}) < h$ do not decay rapidly at $\Omega_2 \rightarrow \infty$. For $d < h$, this is always the case.

- The Siegel-Eisenstein series $\mathcal{E}_h^*(\Omega, s)$ similarly has non-decaying constant term of the form $\sum_T e^{-\text{Tr}(T\Omega_2)}$ with $\text{Rk}(T) < h$.

Rankin-Selberg method at higher genus V

- The regularized Rankin-Selberg transform is obtained by subtracting non-suppressed terms, and yields a field theory-type amplitude, with BPS states running in the loops,

$$\begin{aligned}\mathcal{R}_h(\Gamma_{d,d,h}; \mathbf{s}) &= \int_{GL(h,\mathbb{Z}) \setminus \mathcal{P}_h} \frac{d\Omega_2}{|\Omega_2|^{h+1-s-\frac{d}{2}}} \sum_{\text{BPS}} e^{-\pi \text{Tr}(\mathcal{M}^2 \Omega_2)} \\ &= \Gamma_h\left(s - \frac{h+1-d}{2}\right) \sum_{\text{BPS}} \left[\det \mathcal{M}^2 \right]^{\frac{h+1-d}{2} - s}\end{aligned}$$

where

$$\sum_{\text{BPS}} = \sum_{\substack{(m_j^\alpha, n^{i\alpha}) \in \mathbb{Z}^{2d \times h}, \\ m_j^{(\alpha} n^{j\beta)} = 0, \det \mathcal{M}^2 \neq 0}}, \quad \Gamma_h(s) = \pi^{\frac{1}{4} h(h-1)} \prod_{k=0}^{h-1} \Gamma\left(s - \frac{k}{2}\right)$$

Rankin-Selberg method at higher genus VI

- For $d > h$, this is recognized as the Langlands-Eisenstein series of $SO(d, d, \mathbb{Z})$ with infinitesimal character $\rho - 2(s - \frac{h+1-d}{2})\lambda_h$, associated to $\Lambda^h V$ where V is the defining representation,

$$\mathcal{R}_h(\Gamma_{d,d}; s) \propto \mathcal{E}_{\Lambda^h V}^{SO(d,d)}(s - \frac{h+1-d}{2}) \quad (h > d)$$

- The modular integral of $\Gamma_{d,d,h}$ is proportional to the residue of $\mathcal{R}_h(\Gamma_{d,d,h}; s)$ at $s = \frac{h+1}{2}$, up to a scheme dependent term δ which remains to be computed. For $d < h$, the entire result ought to come from δ .

Rankin-Selberg method at higher genus VII

- For $d = 1$, any h ,

$$\mathcal{A}_h = \mathcal{V}_h(R^h + R^{-h}), \quad \mathcal{V}_h = \int_{\mathcal{F}_h} d\mu_h = 2 \prod_{j=1}^h \zeta^*(2j)$$

- For $h = d = 2$, either by computing the BPS sum, or by unfolding the Siegel-Narain theta series, one finds

$$\begin{aligned} \mathcal{R}_2^*(\Gamma_{2,2}, s) &= 2\zeta^*(2s)\zeta^*(2s-1)\zeta^*(2s-2) \\ &\quad \times [\mathcal{E}_1^*(T; 2s-1) + \mathcal{E}_1^*(U; 2s-1)] \end{aligned}$$

hence

$$\mathcal{A}_2 = 2\zeta^*(2) [\mathcal{E}_1^*(T; 2) + \mathcal{E}_1^*(U; 2)]$$

proving the conjecture by Obers and BP (1999).

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Black holes in $D = 4$ and instantons in $D = 3$ I

- 1/4-BPS black holes in $\mathcal{N} = 4$ CHL-type vacua are counted by a Siegel modular form of genus 2 and weight k , where $r = 2k + 8$ is the rank of the lattice of electric charges ($k = 10$ in Het/ T^6). Invariance under $G_4 = SL(2, \mathbb{Z}) \times SO(6, r - 6, \mathbb{Z})$ is manifest.

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- Suitable BPS couplings in $D = 3$ (e.g. $\nabla^2 R^2$) should receive instanton corrections from 1/4-BPS black holes in $D = 4$, along with KK monopoles. Yet they should be invariant under the 3D U-duality group $G_3 = SO(8, r - 4, \mathbb{Z})$.
- 1/4-BPS black holes in $M/K3 \times T^4$ can be represented by M5-branes wrapping around genus 2 curve $\Sigma \subset T^4$. On the heterotic side, one should include all genus 2 worksheet instantons in T^7 , plus NS5 and KK monopoles.

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Black holes in $D = 4$ and instantons in $D = 3$ II

- Thus, it is natural to conjecture

$$f_{\nabla^2 R^2}^{D=3} = \int_{\mathcal{F}_2} \frac{d^3\Omega d^3\bar{\Omega}}{(\det \text{Im}\Omega)^3} \frac{Z_{8,r-4}(\Omega) \hat{E}_2(\Omega) (\det \text{Im}\Omega)^4}{\Phi_k}$$

where $Z_{8,r-4}$ is the partition function of the non-perturbative lattice, and \hat{E}_2 is the almost holomorphic Eisenstein series of weight 2.

- At weak heterotic coupling, one should recover the two-loop amplitude, invariant under $SO(7, r-5, \mathbb{Z})$, plus other perturbative corrections.
- At large radius, one should recover a sum over 1/4-BPS states, weighted by their entropy, along with KK monopoles

$$f_{\nabla^2 R^2}^{D=3} \stackrel{?}{=} f_{\nabla^2 R^2}^{D=4} + \sum \Omega(\gamma) e^{-R\mathcal{M}(\gamma)} + \sum_k e^{-R^2 k} + \dots$$

Since $\Omega(l\gamma) \sim e^{\ell^2}$ while $\mathcal{M}(l\gamma) \sim l$, the series is at best asymptotic...

- Modular integrals can be efficiently computed using Rankin-Selberg type methods. The result is expressed as a field theory amplitude with BPS states running in the loop.
- T-duality and singularities from enhanced gauge symmetry are manifest. Fourier-Jacobi expansions can be obtained in some cases by solving the BPS constraint.
- The RSZ method also works at higher genus, at least for $h = 2, 3$. For computing modular integrals with $\Phi \neq 1$ it will be important to develop Poincaré series representations for Siegel modular forms with poles at Humbert divisors, such as $1/\Phi_{10}$.
- Non-BPS amplitudes where Φ is not almost weakly holomorphic are challenging ! So are amplitudes with $h \geq 4$!