Unfolding methods for String Amplitudes

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based on work with C. Angelantonj and I. Florakis,

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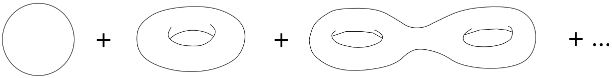
String amplitudes and modular integrals I

- Scattering amplitudes of n external states in perturbative string theory have a topological expansion

$$\mathcal{A}_n = \sum_{h=0}^{\infty} g_s^{2h-2}\, \mathcal{A}_{h,n}\ , \quad \mathcal{A}_{h,n} = \int_{\mathfrak{M}_{h,n}} \mathrm{d}\mu_{h,n}\, F_{h,n}$$

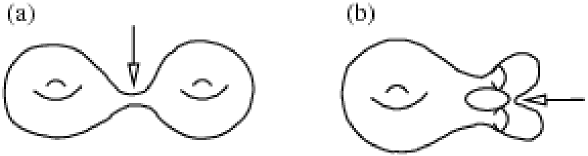


where $F_{h,n}$ is a correlator of n vertex operators (along with ghost insertions) in a certain SCFT on a Riemann surface Σ_h of genus h with n punctures z_i, integrated over the moduli space of super-Riemann surfaces $\mathfrak{M}_{h,n}$.

- Only one topology at each loop order, but the integrand is vastly more complicated than in QFT !

String amplitudes and modular integrals II

- String amplitudes are automatically free of UV divergences: boundaries of $\mathfrak{M}_{h,n}$ correspond to Riemann surfaces with nodes, describing propagation of massless states over long proper time:

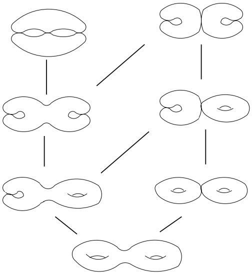


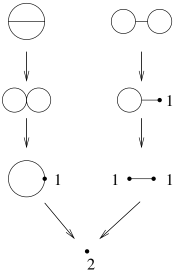
- String amplitudes are expected to have the same infrared divergences as QFT amplitudes. This is tricky to show, since $\mathfrak{M}_{h,n}$ is a non-projected supermanifold. We shall ignore this complication, and assume that $\mathcal{A}_{h,n}$ can be reduced to an integral over the moduli space $\mathcal{M}_h$ of ordinary Riemann surfaces.

Donagi Witten 2013

String amplitudes and modular integrals III

- In a low energy expansion, all Feynman diagrams at h-loop emerge from degenerations of the genus h Riemann surface:





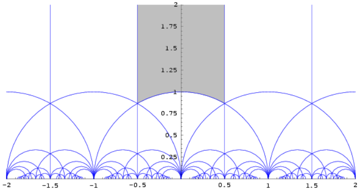
- In order to regulate IR divergences, a cut-off is needed. Dimensional regularization is hardly an option in string theory !

String amplitudes and modular integrals IV

- For genus $h \leq 3$, $\mathcal{M}_h$ is isomorphic to a fundamental domain $\mathcal{F}_h$ in the Siegel-Poincaré upper half plane $\mathcal{H}_h$, parametrized by the period matrix Ω, a complex $h \times h$ symmetric matrix with positive definite imaginary part.
- Entries in $\mathrm{Im}(\Omega)$ correspond to Schwinger/Feynman parameters in QFT, while the entries in $\mathrm{Re}(\Omega)$ are Lagrange multipliers which ensure that only level-matched states propagate.
- The integrand $F_h(\Omega)$ is a Siegel modular form for $\Gamma_h = Sp(2h, \mathbb{Z})$, acting as $\Omega \mapsto (A\Omega + B) \cdot (C\Omega + D)^{-1}$. This includes integer shifts of $\mathrm{Re}(\Omega)$.

String amplitudes and modular integrals V

- At genus 1, $\mathcal{M}_1$ is a fundamental domain in the Poincaré upper-half plane, parametrized by $\Omega_{11} \equiv \tau = \tau_1 + \mathrm{i}\tau_2$ and the integrand is invariant under $SL(2,\mathbb{Z})$. A convenient choice is



- τ_2 can be interpreted as the Schwinger time while integral over τ_1 (for $\tau_2 > 1$) projects on level-matched states.
- UV finiteness is manifest. IR divergences can be regulated by introducing a IR cut-off $\tau_2 < \Lambda$.

String amplitudes and modular integrals VI

- E.g. the one-loop vacuum amplitude in bosonic closed string theory in $D = 26$ flat space time is proportional to

$$\mathcal{A}_1 = \int_{\mathcal{F}} \frac{\mathrm{d}\tau_1 \mathrm{d}\tau_2}{\tau_2^{1+D/2}} \frac{e^{\frac{\pi(D-2)}{6}\tau_2}}{|\prod_{n=1}^{\infty}(1-e^{2\pi \mathrm{i} n\tau})|^{2(D-2)}}$$

  This is infrared divergent due to the existence of a tachyon. One-loop amplitudes in superstring theory look similar except the integrand grows only polynomially in τ_2.

String amplitudes and modular integrals VII

- For $h \geq 2$, a standard fundamental domain in $\mathcal{H}_h$ is obtained by choosing $-\frac{1}{2} \leq \mathrm{Re}(\Omega_{\alpha\beta}) \leq \frac{1}{2}$, $\mathrm{Im}(\Omega)$ to be Minkovski-reduced, and further requiring that $\det(C\Omega_2 + D) \geq 1$ for all elements in $Sp(h, \mathbb{Z})$.
- For genus 2, it takes about 25 inequalities to define $\mathcal{F}_2$!
- For genus $h \geq 4$, $\mathcal{M}_h$ sits in a codimension $\frac{1}{2}(h-2)(h-3)$ locus inside $\mathcal{H}_h$ known as the Schottky locus. It is not clear how to extend F_h to a modular form on $\mathcal{H}_h$.

Rankin-Selberg method / unfolding trick I

- Our goal is to develop methods to compute integrals of Siegel modular forms over a fundamental domain of the Siegel upper-half plane analytically.
- The key idea is to represent the integrand as a Poincaré series,

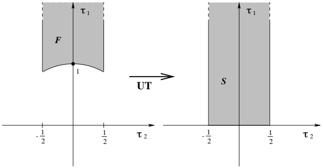
$$F_h(\Omega) = \sum_{\gamma \in \Gamma_{h,\infty} \backslash \Gamma_h} f_h|_\gamma(\Omega)$$

where $f_h|_\gamma(\Omega) = f_h(\gamma \cdot \Omega)$ and the 'seed' $f_h(\Omega)$ is invariant under a subgroup $\Gamma_{h,\infty} \subset \Gamma_h$. Typically, $\Gamma_{h,\infty}$ is the stabilizer of the cusp at infinity, acting by integer shifts of Ω_1.

Rankin-Selberg method / unfolding trick II

- Provided the sum is absolutely convergent, one can exchange the sum and integral and obtain

$$\int_{\Gamma_h \backslash \mathcal{H}_h} \mathrm{d}\mu_h \, F_h(\Omega) = \int_{\Gamma_{\infty,h} \backslash \mathcal{H}_h} \mathrm{d}\mu_h \, f_h(\Omega) \; .$$



- We gain if $\Gamma_{\infty,h} \backslash \mathcal{H}_h$ and f_h are simpler than $\Gamma_h \backslash \mathcal{H}_h$ and F_h !
- If F_h is a cusp form, one can always insert by hand an Eisenstein series $\mathcal{E}(s;\Omega)$ in the integrand, apply the unfolding trick, and then extract the residue at a suitable value of s where $\mathcal{E}(s;\Omega)$ has a pole with constant residue. But F_h is rarely a cusp form !

Rankin-Selberg method / unfolding trick III

- We shall focus on a class of one-loop amplitudes of the form

$$\mathcal{A} = \int_{\mathcal{F}_1} \mathrm{d}\mu\, \Gamma_{d+k,d}\, \Phi(\tau) \ , \quad \mathrm{d}\mu = \frac{\mathrm{d}\tau_1 \mathrm{d}\tau_2}{\tau_2^2}$$

  where $\Phi(\tau)$ is a weakly, almost holomorphic modular form of weight $w = -k/2$ (the elliptic genus) and $\Gamma_{(d+k,d)}$ is a Siegel Theta series (the Narain lattice partition function) for an even self-dual lattice Λ of signature $(d+k, d)$,

$$\Gamma_{d+k,d} = \tau_2^{d/2} \sum_{p \in \Lambda} e^{-\pi\tau_2 \mathcal{M}^2(p) + \pi \mathrm{i} \tau_1 \langle p,p \rangle}$$

- The positive definite quadratic form $\mathcal{M}^2(p)$ is parametrized by the orthogonal Grassmannian

$$G_{d+k,d} = \frac{O(d+k,d)}{O(d+k) \times O(d)} \ni (g_{ij}, B_{ij}, Y_i^a) \ ,$$

Rankin-Selberg method / unfolding trick IV

- Such modular integrals arise in certain BPS-saturated amplitudes, such as $F^2, \mathcal{R}^2, F^4, \mathcal{R}^4$ in type II string theory ($k=0$) or heterotic string ($k=8,16$) compactified on a torus T^d.
- $\mathcal{A}$ is invariant under T-duality, i.e. under the automorphisms of the lattice. Mathematically, $\Phi \mapsto \mathcal{A}$ is a Theta correspondence between $SL(2,\mathbb{Z})$ and $O(\Gamma_{d+k,d})$ automorphic forms.

Borcherds; Kudla Rallis; Harvey Moore

- We shall also consider higher loop integrals of the form $\int_{\mathcal{F}_h} \mathrm{d}\mu_h \, \Gamma_{d,d,h}$, which arise e.g. in $D^4\mathcal{R}^4$ and $D^6\mathcal{R}^4$ amplitudes in type II string theory compactified on T^d.

Outline

1. String amplitudes and modular integrals
2. The Rankin-Selberg method at one-loop
3. Rankin-Selberg method at higher genus

Outline

Rankin-Selberg method I

- The simplest Poincaré series is the (completed) non-holomorphic Eisenstein series

$$\mathcal{E}^\star(s;\tau) = \tfrac{1}{2}\,\zeta^\star(2s) \sum_{(c,d)=1} \frac{\tau_2^s}{|c\tau+d|^{2s}}$$

where $\zeta^\star(s) \equiv \pi^{-s/2}\,\Gamma(s/2)\,\zeta(s) = \zeta^\star(1-s)$.

- $\mathcal{E}^\star(s;\tau)$ is convergent for $\mathrm{Re}(s)>1$, and has a meromorphic continuation to all s, invariant under $s \mapsto 1-s$, with simple poles at $s=0,1$ with constant residue:

$$\mathcal{E}^\star(s) = \frac{1}{2(s-1)} + \tfrac{1}{2}\left(\gamma - \log(4\pi\,\tau_2\,|\eta(\tau)|^4)\right) + \mathcal{O}(s-1)\,,$$

Rankin-Selberg method (cont.)

- For any modular function $F(\Omega)$ of rapid decay, consider the Rankin-Selberg transform

$$\mathcal{R}^\star(F,s) = \int_{\mathcal{F}} \mathrm{d}\mu\, \mathcal{E}^\star(s;\tau)\, F(\tau)$$

- By the unfolding trick, $\mathcal{R}^\star(F,s)$ is proportional to the Mellin transform of the constant term $F_0(\tau_2) = \int_{-1/2}^{1/2} \mathrm{d}\tau_1\, F(\tau)$,

$$\begin{aligned}\mathcal{R}^\star(F;s) =& \zeta^\star(2s) \int_{\mathbb{R}^+\times[-\frac12,\frac12]} \mathrm{d}\mu\, \tau_2^s\, F(\tau) \\ =& \zeta^\star(2s) \int_0^\infty \mathrm{d}\tau_2\, \tau_2^{s-2}\, F_0(\tau_2)\,,\end{aligned}$$

Rankin-Selberg method (cont.) I

- It inherits the meromorphicity and functional relations of $\mathcal{E}^\star$, e.g. $\mathcal{R}^\star(F;s) = \mathcal{R}^\star(F;1-s)$. *In analytic number theory, this is useful for analyzing properties of L-functions.*
- Since the residue of $\mathcal{E}^\star(s)$ at $s=0,1$ is constant, the residue of $\mathcal{R}^\star(F;s)$ at $s=1$ is proportional to the modular integral of F,

$$\mathrm{Res}_{s=1}\mathcal{R}^\star(F;s) = \frac{1}{2}\int_{\mathcal{F}} \mathrm{d}\mu\, F$$

- For this it was important that F was of rapid decay near the cusp. This is rarely so for string theory applications !

Rankin-Selberg method (cont.) II

- Zagier extended the RS method to the case where $F^{(0)}$ is of power-like growth $F(\tau) \sim \varphi(\tau_2) = \sum_\alpha c_\alpha \tau_2^\alpha$ at the cusp: the renormalized integral defined by

$$\mathrm{R.N.} \int_{\mathcal{F}} \mathrm{d}\mu\, F(\tau) \equiv \lim_{\Lambda\to\infty} \left[\int_{\mathcal{F}_\Lambda} \mathrm{d}\mu\, F(\tau) - \hat\varphi(\Lambda) \right]$$

where $\hat\varphi(\Lambda) = \sum_{\alpha\neq 1} c_\alpha \frac{\tau_2^{\alpha-1}}{\alpha-1} + \sum_{\alpha=1} c_\alpha \log\tau_2$, is related to the renormalized Rankin-Selberg transform

$$\mathcal{R}^\star(F;s) \equiv \zeta^\star(2s) \int_0^\infty \mathrm{d}\tau_2\, \tau_2^{s-2} \left(F^{(0)} - \varphi \right) ,$$

via $$\boxed{\mathrm{R.N.} \int_{\mathcal{F}} \mathrm{d}\mu\, F(\tau) = 2\,\mathrm{Res}_{s=1} \mathcal{R}^\star(F;s) + \delta}$$

Rankin-Selberg method (cont.) III

- Here δ is a scheme-dependent correction which depends only on the leading behavior $\varphi(\tau_2)$:

$$\delta = 2\,\mathrm{Res}_{s=1}\left[\zeta^\star(2s)\,h_\Lambda(s) + \zeta^\star(2s-1)\,h_\Lambda(1-s)\right] - \hat{\varphi}(\Lambda)\,,$$

where $h_\Lambda(s) = \int_0^\Lambda \mathrm{d}\tau_2\,\varphi(\tau_2)\,\tau_2^{s-2}$.

- With these definitions, the Rankin-Selberg transform $\mathcal{R}^\star(F;s)$ is equal to the renormalized integral

$$\mathcal{R}^\star(F;s) = \mathrm{R.N.}\int_{\mathcal{F}} \mathrm{d}\mu\,F(\tau)\,\mathcal{E}^\star(s;\tau)$$

moreover $\mathrm{R.N.}\int_{\mathcal{F}} \mathrm{d}\mu\,\mathcal{E}^\star(\tau;s) = \mathrm{R.N.}\int_{\mathcal{F}} \mathrm{d}\mu\,\mathcal{E}^\star(\tau;s_1)\,\mathcal{E}^\star(\tau;s_2) = 0$!

Epstein series from modular integrals

- The RSZ method applies immediately to $\mathcal{A} = \int_{\mathcal{F}} \mathrm{d}\mu\, \Gamma_{d,d}$:

$$\mathcal{R}^\star(\Gamma_{d,d}; s) = \zeta^\star(2s) \int_0^\infty \mathrm{d}\tau_2\, \tau_2^{s+d/2-2} \sum_{\langle p,p\rangle=0}{}' e^{-\pi\tau_2\, \mathcal{M}^2(p)}$$

$$= \zeta^\star(2s)\, \frac{\Gamma(s+\frac{d}{2}-1)}{\pi^{s+\frac{d}{2}-1}}\, \mathcal{E}_V^d(g,B; s+\tfrac{d}{2}-1)$$

where $\mathcal{E}_V^d(g,B;s)$ is the constrained Epstein series

$$\mathcal{E}_V^d(g,B;s) \equiv \sum_{\langle p,p\rangle=0}{}' \left[\mathcal{M}^2(p)\right]^{-s} ,$$

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$$\mathcal{E}_V^d(g,B;s) \equiv \sum_{\langle p,p\rangle=0}' \left[\mathcal{M}^2(p)\right]^{-s} ,$$

- Mathematically, $\mathcal{E}_V^d$ is recognized as a degenerate Langlands-Eisenstein series with infinitesimal character $\rho - 2s\alpha_1$

Epstein series and BPS state sums I

- Physically, $\mathcal{E}^d_V$ is identified as a sum over all BPS states of momentum m_i and winding n^i, with mass

$$\mathcal{M}^2(p) = (m_i + B_{ik} n^k) g^{ij} (m_j + B_{jl} n^l) + n^i g_{ij} n^j$$

subject to the BPS condition $\langle p, p \rangle = m_i n^i = 0$. Invariance under $O(\Gamma_{d,d})$ is manifest.

- $\mathcal{E}^d_V(g, B; s)$ converges absolutely for $\mathrm{Re}(s) > d$. The RSZ method shows that it admits a meromorphic continuation in the s-plane satisfying

$$\mathcal{E}^{d\star}_V(s) \equiv \pi^{-s}\,\Gamma(s)\,\zeta^\star(2s - d + 2)\,\mathcal{E}^d_V(s) = \mathcal{E}^{d\star}_V(d - 1 - s)\,,$$

with a simple pole at $s = 0, \frac{d}{2} - 1, \frac{d}{2}, d - 1$ (double poles if $d = 2$).

Epstein series and BPS state sums II

- The residue at $s=\frac{d}{2}$ produces the modular integral of interest:

$$\text{R.N.} \int_{\mathcal{F}} \mathrm{d}\mu\, \Gamma_{d,d}(g,B) = \frac{\Gamma(\frac{d}{2}-1)}{\pi^{\frac{d}{2}-1}}\, \mathcal{E}^{d}_{V}\left(g,B;\tfrac{d}{2}-1\right)$$

  rigorously proving an old conjecture of Obers and myself (1999).
- For $d=2$, the BPS constraint $m_i n^i = 0$ can be solved, leading to

$$\mathcal{E}^{2\star}_{V}(s;T,U) = 2\,\mathcal{E}^{\star}(s;T)\,\mathcal{E}^{\star}(s;U)$$

  hence to Dixon-Kaplunovsky-Louis famous result (1989)

$$\text{R.N.} \int_{\mathcal{F}} \mathrm{d}\mu\, \Gamma_{2,2}(T,U) = -\log\left(T_2\, U_2\, |\eta(T)\,\eta(U)|^4\right) + \text{cte}$$

  up to a scheme-dependent additive constant.

Modular integrals with unphysical tachyons I

- For many cases of interest, the integrand is NOT of moderate growth at the cusp, rather it grows exponentially, due to the heterotic unphysical tachyon, $\Phi(\tau) \sim 1/q + \mathcal{O}(1)$.
- The RSZ method fails, however the unfolding trick can still be used if $\Phi(\tau)$ can be represented as a uniformly convergent Poincaré series.
- If $\Phi(\tau)$ is an almost, weakly holomorphic modular form of negative weight, as is usually the case for BPS amplitudes, it can be decomposed as a sum of Niebur-Poincaré series, for which the unfolding trick applies.
- This provides a new method for evaluating one-loop threshold corrections, alternative to the one of Harvey-Moore, which keeps T-duality manifest throughout. *A topic for another talk.*

Angelantonj Florakis BP, Bruinier, Bringmann Kane

Outline

Rankin-Selberg method at higher genus I

- String amplitudes at genus $h \leq 3$ can be written as

$$\mathcal{A}_h = \int_{\mathcal{F}_h} \mathrm{d}\mu_h \, F_h(\Omega) \ , \quad \mathrm{d}\mu_h = \frac{\mathrm{d}\Omega_1 \mathrm{d}\Omega_2}{|\Omega_2|^{h+1}}$$

where $\mathcal{F}_h$ is a fundamental domain of the action of $\Gamma = Sp(2h, \mathbb{Z})$ on $\mathcal{H}_h$, and F_h is a Siegel modular function.

- For example, in type II compactified on T^d, $D^4\mathcal{R}^4$ and $D^6\mathcal{R}^4$ amplitudes at two-loop and three-loop are given by above where F_h is the Narain partition function at genus h

$$\Gamma_{d,d,h} = |\Omega_2|^{d/2} \sum_{p \in \Gamma_{d,d}^{\otimes h}} e^{-\pi \Omega_2^{\alpha\beta} \mathcal{M}^2(p_\alpha, p_\beta) + 2\pi \mathrm{i}\, \Omega_1^{\alpha\beta} \langle p_\alpha, p_\beta \rangle}$$

For $D^6\mathcal{R}^4$ at two-loop, $F_2 = \varphi(\Omega)\,\Gamma_{d,d,2}$ where $\varphi(\Omega)$ is the Kawazumi-Zhang invariant.

D'Hoker Gutperle Phong; D'Hoker Green; Gomez Mafra; D'Hoker Green BP Russo

Rankin-Selberg method at higher genus II

- The genus h analog of $\mathcal{E}^\star(s;\tau)$ is the non-holomorphic Siegel-Eisenstein series

$$\mathcal{E}_h^\star(s;\Omega) = \mathcal{N}_h(s) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} |\Omega_2|^s|\gamma$$

where $\Gamma_\infty = \{\begin{pmatrix} A & B \\ 0 & A^{-t}\end{pmatrix}\} \subset \Gamma$, $\mathcal{N}_h(s) = \zeta^\star(2s) \prod_{j=1}^{[h/2]} \zeta^\star(4s-2j)$.

- The sum converges absolutely for $\mathrm{Re}(s) > \frac{h+1}{2}$ and can be meromorphically continued to the full s plane. The analytic continuation is invariant under $s \mapsto \frac{h+1}{2} - s$, and has a simple pole at $s = \frac{h+1}{2}$ with constant residue $r_h = \frac{1}{2} \prod_{j=1}^{[h/2]} \zeta^\star(2j+1)$

Rankin-Selberg method at higher genus III

- For any modular function $F(\Omega)$ of rapid decay, the Rankin-Selberg transform can be computed by the unfolding trick,

$$\mathcal{R}^\star_h(F;s) = \int_{\mathcal{F}_h} \mathrm{d}\mu_h \, F(\Omega)\, \mathcal{E}^\star_h(\Omega,s)$$
$$= \mathcal{N}_h(s) \int_{GL(h,\mathbb{Z})\backslash\mathcal{P}_h} \mathrm{d}\Omega_2 \, |\Omega_2|^{s-h-1} \, F_0(\Omega_2)$$

  where $\mathcal{P}_h$ is the space of positive definite real matrices, and $F_0(\Omega_2) = \int_{[0,1]^{h(h+1)/2}} \mathrm{d}\Omega_1 \, F(\Omega)$ is the constant term of F wrt. Γ_∞.

- The residue at $s = \frac{h+1}{2}$ is proportional to the average of F,

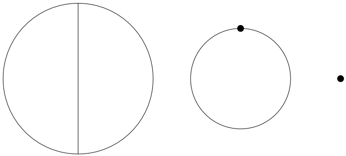
$$\mathrm{Res}_{s=\frac{h+1}{2}} \mathcal{R}^\star_h(F;s) = r_h \int_{\mathcal{F}_h} \mathrm{d}\mu_h \, F \,.$$

Rankin-Selberg method at higher genus IV

- This procedure cannot be directly applied to $F = \Gamma_{d,d,h}$: it is not exponentially suppressed near all cusps, due to contributions of momenta with $\mathrm{Rk}(p^\alpha) < h$.
- Boundaries of $\mathcal{F}_h$ correspond to regions where Ω_2 becomes large in a diagonal block of size $1 \leq h_2 \equiv h - h_1 \leq h$: in this region,

$$\mathcal{F}_h \to \mathcal{F}_{h_1} \times \frac{\mathcal{P}_{h_2}}{GL(h_2, \mathbb{Z})} \times \tilde{T}^{2h_1 h_2}/\mathbb{Z}_2 \times T^{h_2(h_2+1)/2}$$

The integral over $\mathcal{P}_{h_2}/GL(h_2, \mathbb{Z})$ is potentially divergent, corresponding to an infrared subdivergence at h_2-loop. Eg. for $h = 2$:



Rankin-Selberg method at higher genus V

- The renormalized integral $\mathrm{R.N.} \int_{\mathcal{F}_h} \mathcal{E}_h(s)\, F$ can be defined by imposing an infrared cut-off $\max(\Omega_{\alpha\beta}) < \Lambda$, subtracting Λ-dependent subdivergences, and taking the limit $\Lambda \to \infty$:

$$\mathrm{R.N.} \int_{\mathcal{F}_h} \mathcal{E}_h(s)\, F = \lim_{\Lambda\to\infty} \left[\int_{\mathcal{F}_h^\Lambda} \mathcal{E}_h(s)\, F - \sum_{1\le h_2 \le h} a_{h_2} \Lambda^{\alpha_{h_2}} \right]$$

- Similarly, the renormalized Rankin-Selberg transform $\mathcal{R}^\star_h(F;s)$ is defined by subtracting the non-decaying part of $F_0(\Omega_2)$:

$$\mathcal{R}^\star_h(F;s) = \mathcal{N}_h(s) \int_{GL(h,\mathbb{Z})\backslash\mathcal{P}_h} \mathrm{d}\Omega_2\, |\Omega_2|^{s-h-1}\, [F_0(\Omega_2) - \varphi(\Omega_2)]$$

- Under suitable assumptions, using differential operators one can show $\mathrm{R.N.} \int_{\mathcal{F}_h} \mathcal{E}_h(s)\, F = \mathcal{R}^\star_h(F;s)$.

Florakis BP 2016

Rankin-Selberg method at higher genus VI

- For $F = \Gamma_{d,d,h}$, the RS transform keeps only momenta of maximal rank,

$$\mathcal{R}_h(\Gamma_{d,d,h};s) = \int_{GL(h,\mathbb{Z})\backslash\mathcal{P}_h} \frac{\mathrm{d}\Omega_2}{|\Omega_2|^{h+1-s-\frac{d}{2}}} \sum_{\mathrm{BPS}} e^{-\pi \mathrm{Tr}(\mathcal{M}^2\Omega_2)}$$

$$= \Gamma_h(s - \tfrac{h+1-d}{2}) \sum_{\mathrm{BPS}} \left[\det \mathcal{M}^2\right]^{\frac{h+1-d}{2}-s}$$

where

$$\sum_{\mathrm{BPS}} = \sum_{p\in\Lambda_{d,d}^{\otimes h},\mathrm{Rk}p=h}, \qquad \Gamma_h(s) = \pi^{\frac{1}{4}h(h-1)} \prod_{k=0}^{h-1} \Gamma(s - \tfrac{k}{2})$$

Rankin-Selberg method at higher genus VII

- For $d > h$, this is recognized as the Langlands-Eisenstein series of $SO(d,d,\mathbb{Z})$ with infinitesimal character $\rho - 2(s - \frac{h+1-d}{2})\lambda_h$, associated to $\Lambda^h V$ where V is the defining representation,

$$\mathcal{R}_h(\Gamma_{d,d}; s) = \mathcal{E}^{\star, SO(d,d)}_{\Lambda^h V}(s - \tfrac{h+1-d}{2}) \qquad (h > d)$$

- The pole structure and functional equation predicted from the RS method reproduces the known analytic structure of the Langlands-Eisenstein series
- The modular integral of $\Gamma_{d,d,h}$ is proportional to the residue of $\mathcal{R}_h(\Gamma_{d,d,h}; s)$ at $s = \frac{h+1}{2}$, up to a computable correction δ.
- For $d < h$, $\mathcal{R}_h(\Gamma_{d,d}; s) = 0$ and the integral of $\Gamma_{d,d,h}$ entirely comes from the subtraction δ.

Rankin-Selberg method at higher genus VIII

- For $d = 1$, any h,

$$\mathcal{A}_h = \mathcal{V}_h(R^h + R^{-h}) \,, \quad \mathcal{V}_h = \int_{\mathcal{F}_h} \mathrm{d}\mu_h = 2 \prod_{j=1}^{h} \zeta^\star(2j)$$

- For $h = d = 2$, either by computing the BPS sum, or by unfolding the Siegel-Narain theta series, one finds

$$\mathcal{R}_2^\star(\Gamma_{2,2}, s) = 2\zeta^\star(2s)\zeta^\star(2s-1)\zeta^\star(2s-2) \times [\mathcal{E}_1^\star(T; 2s-1) + \mathcal{E}_1^\star(U; 2s-1)]$$

hence

$$\mathcal{A}_2 = 2\zeta^\star(2)\,[\mathcal{E}_1^\star(T; 2) + \mathcal{E}_1^\star(U; 2)]$$

proving the conjecture by Obers and BP (1999).

Conclusion - Outlook

- Modular integrals can be efficiently computed using Rankin-Selberg type methods. The result is expressed as a field theory amplitude with BPS states running in the loop.
- The RSZ method also works at genus 2 and 3, for integrands whose only singularities correspond to boundaries of the Siegel modular domain. It would be useful to extend it to integrands with singularities on separating degeneration locus.
- Our results confirm predictions from S-duality, which requires that certain loop integrals are expected in terms of Langlands-Eisenstein series. It also opens up the way to construct new types of automorphic forms...
- Non-BPS amplitudes are challenging ! So are amplitudes with $h \geq 4$!