A string theorist viewpoint on the genus-two Kawazumi-Zhang invariant

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MCM2016, OIST, Okinawa

based on 1405.6226 with E. D'Hoker, M. Green and R. Russo;

1502.03377; 1504.04182; 1510.02409 with R. Russo

Introduction

- The KZ invariant $\varphi(\Sigma)$, introduced around 2008 by N. Kawazumi and S-W. Zhang, is a canonical invariant of a compact Riemann surface Σ. When Σ is hyperelliptic, it is related to the Faltings invariant $\delta(\Sigma)$ and to the discriminant $\Delta(\Sigma)$.
- In 2013, D'Hoker and Green noticed that the KZ invariant $\varphi(\Sigma)$ of genus-two curves arises in the integrand of the scattering amplitude of 4 gravitons at two-loop in type II string theories at NNLO in the low-energy expansion.
- Understanding the constraints from U-duality on the low-energy effective action of type II strings compactified on a torus T^d has lead us to uncover unexpected properties of the KZ invariant of genus-two curves.

Outline

1. The genus-two KZ invariant from two-loop string amplitudes

2. The KZ invariant as a Theta lift

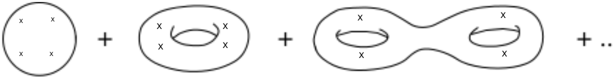
3. Some applications

Outline

Four-graviton scattering in string theory I

- Scattering amplitudes in string theory have a topological expansion

$$\mathcal{A}(\{p_a\}) = \sum_{h=0}^{\infty} g_s^{2h-2} \mathcal{A}_h + \mathcal{O}(e^{-1/g_s}) \,, \quad \mathcal{A}_h = \int_{\mathfrak{M}_{h,n}} \omega_{h,n}$$



where $\omega_{h,n}$ is a top form on the moduli space of Riemann surfaces $\mathcal{M}_{h,n}$ (more precisely, super-Riemann surfaces, but this will not matter for us). Non-perturbative $\mathcal{O}(e^{-1/g_s})$ corrections are known to occur in general but hard to compute.

- I will focus on four-graviton scattering in type II strings on $\mathbb{R}^{1,D-1} \times T^d$, where we have good control both on perturbative and non-perturbative effects. Needless to say, $D + d = 10$.

Four-graviton scattering in type II strings, tree-level

- At tree-level ($h=0$), in terms of the Lorentz invariants $s=-p_1\cdot p_2$, $t=-p_1 p_3$, $u=-p_1 p_4$ (note: $s+t+u=0$),

$$\mathcal{A}^{(0)} \propto \frac{\Gamma(1-s)\,\Gamma(1-t)\,\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)}$$
$$= \frac{3}{stu} + 2\zeta(3) + \zeta(5)(s^2+t^2+u^2) + \frac{2}{3}[\zeta(3)]^2(s^3+t^3+u^3) + \dots$$

Green Schwarz 1981, Gross and Witten 1986, ...

- We denote the coefficients in this Taylor expansion by

$$f^{(0)}_{\mathcal{R}^4} = 2\zeta(3)\,, \quad f^{(0)}_{D^4\mathcal{R}^4} = \zeta(5) \quad f^{(0)}_{D^6\mathcal{R}^4} = \frac{2}{3}[\zeta(3)]^2$$

- These correspond to higher-derivative corrections to the Einstein-Hilbert action, describing gravitational interactions,

$$S = \int \mathrm{d}^D x\,\sqrt{-g}[\mathcal{R} + f_{\mathcal{R}^4}\mathcal{R}^4 + f_{D^4\mathcal{R}^4} D^4\mathcal{R}^4 + f_{D^6\mathcal{R}^4} D^6\mathcal{R}^4 + \dots]$$

Four-graviton scattering in type II strings, one-loop I

- At one-loop, the 4-graviton amplitude takes the form

$$\mathcal{A}^{(1)} \propto \int_{\mathcal{F}_1} \mathrm{d}\mu_1 \int_{\Sigma^4} \prod_{a=1\dots4} \frac{\mathrm{d}z_a \mathrm{d}\bar{z}_a}{\tau_2}\, e^{-\sum_{a<b} p_a \cdot p_b\, G_{ab}}\, \Gamma_{d,d,1}(G,B;\tau)$$

where $G_{ab} = G(z_a, z_b)$ is the scalar Green function. Expanding in powers of s, t, u gives, as functions of the torus moduli (G, B),

$$\begin{aligned} f^{(1)}_{\mathcal{R}^4} =& \pi\, \mathrm{R.N.} \int_{\mathcal{F}_1} \mathrm{d}\mu_1\, \Gamma_{d,d,1}(G,B;\tau) \\ f^{(1)}_{D^4\mathcal{R}^4} =& 2\pi\, \mathrm{R.N.} \int_{\mathcal{F}_1} \mathrm{d}\mu_1\, \Gamma_{d,d,1}(G,B;\tau)\, \mathcal{E}^\star_1(2;\tau) \\ f^{(1)}_{D^6\mathcal{R}^4} =& \frac{\pi}{3}\, \mathrm{R.N.} \int_{\mathcal{F}_1} \mathrm{d}\mu_1\, \Gamma_{d,d,1}(G,B;\tau)\, \left(5\, \mathcal{E}^\star_1(3;\tau) + \zeta(3)\right) \end{aligned}$$

Green Vanhove 1999; Green Russo Vanhove 2008

Notations:

1. $\mathcal{F}_h$ is a fundamental domain for the action of $Sp(2h,\mathbb{Z})$ on the Siegel upper-half plane $\mathcal{H}_h$ of degree h; $\mathrm{d}\mu_h = \prod_{I<J} \frac{\mathrm{d}\Omega_{IJ}\mathrm{d}\bar{\Omega}_{IJ}}{|\Omega_2|^{h+1}}$
2. $\Gamma_{d,d,h}$ is the genus-h Siegel-Narain theta series, invariant under $Sp(2h,\mathbb{Z})\times O(d,d,\mathbb{Z})$,

$$\Gamma_{d,d,h}(G,B,\Omega\equiv\Omega_1+\mathrm{i}\Omega_2) = |\Omega_2|^{d/2} \sum_{(m_i^I,n^{i,I})\in\mathbb{Z}^{2hd}} e^{-\pi\mathcal{L}^{IJ}\Omega_{2,IJ}+2\pi\mathrm{i}m_i^I n^{i,J}\Omega_{1,IJ}}$$

$$\mathcal{L}^{IJ} = (m_i^I + B_{ij}n^{j,I})G^{ik}(m_k^J+B_{kl}n^{l,J}) + n^{i,I}G_{ij}n^{j,J} \ .$$

   where $G_{ij}=G_{ji}>0$ and $B_{ij}=-B_{ji}$. Physically, G_{ij} and B_{ij} denote the flat metric and Kalb-Ramond two-form on T^d. Mathematically (G_{ij},B_{ij}) parametrize the Grassmannian $O(d,d)/O(d)\times O(d)$.
3. $\mathcal{E}_h^\star(s;\Omega)$ is the non-holomorphic Eisenstein series for $Sp(2h,\mathbb{Z})$;
4. R.N. a suitable renormalization prescription

Four-graviton scattering in type II strings, two-loop I

- At two loops, the 4-graviton amplitude takes the form

$$\mathcal{A}^{(2)} \sim \int_{\mathcal{M}_2} \mathrm{d}\mu_2 \int_{\Sigma^4} |\mathcal{Y}_S|^2 \, e^{-\sum_{a<b} p_a \cdot p_b \, G_{ab}} \, \Gamma_{d,d,2}(G,B;\Omega)$$

where $\mathcal{Y}_S$ is a (1,0)-form in each z_a, linear in s, t, u.

- Taylor expanding in s, t, u gives

$$\begin{aligned} f^{(2)}_{\mathcal{R}^4} =& 0, \\ f^{(2)}_{D^4\mathcal{R}^4} =& \frac{\pi}{2} \, \mathrm{R.N.} \int_{\mathcal{F}_2} \mathrm{d}\mu_2 \, \Gamma_{d,d,2}(G,B;\Omega) \, \mathcal{B}^{(2)}_{D^4\mathcal{R}^4}(\Omega) \\ f^{(2)}_{D^6\mathcal{R}^4} =& \pi \, \mathrm{R.N.} \int_{\mathcal{F}_2} \mathrm{d}\mu_2 \, \Gamma_{d,d,2}(G,B;\Omega) \, \mathcal{B}^{(2)}_{D^6\mathcal{R}^4}(\Omega) \end{aligned}$$

D'Hoker Phong 2001-05; D'Hoker Gutperle Phong 2005

Four-graviton scattering in type II strings, two-loop II

- Here, the integrands $\mathcal{B}^{(2)}_{D^4\mathcal{R}^4}$, $\mathcal{B}^{(2)}_{D^6\mathcal{R}^4}$ are given by

$$\mathcal{B}^{(2)}_{D^4\mathcal{R}^4}(\Omega) = \frac{1}{64}\int_{\Sigma^4} \frac{|\Delta_{12}\Delta_{34}|^2}{|\Omega_2|^2}$$

$$\mathcal{B}^{(2)}_{D^6\mathcal{R}^4}(\Omega) = -\frac{1}{192}\int_{\Sigma^4} \frac{|\Delta_{12}\Delta_{34}-\Delta_{14}\Delta_{23}|^2}{|\Omega_2|^2}\,(G_{12}+G_{34}-G_{13}-G_{24})$$

  where $\Delta_{ab} = \omega_1(z_a)\omega_2(z_b) - \omega_2(z_a)\omega_1(z_b)$

- After integrating over z_a, one finds a remarkably simple result,

$$\mathcal{B}^{(2)}_{D^4\mathcal{R}^4} = 1\ , \qquad \mathcal{B}^{(2)}_{D^6\mathcal{R}^4} = \varphi_2(\Omega)\ ,$$

D'Hoker Green 2013

Four-graviton scattering in type II strings, two-loop III

- ...where $\varphi_2(\Omega)$ is the genus-two Kawazumi-Zhang invariant, defined for arbitrary genus by

$$\varphi_h(\Omega) = -\frac{1}{4h} \int_{\Sigma\times\Sigma} P(z_1, z_2)\, G(z_1, z_2)$$

Kawazumi 2008, S-W Zhang 2010

- Here, $P(z_1, z_2)$ is a $(1,1)$-form in each variable, symmetric under exchange of z_1, z_2, whose integral wrt. z_1 or z_2 vanishes,

$$P(z_1, z_2) = \left[-\Omega_2^{IJ} \Omega_2^{KL} + h\, \Omega_2^{IL} \Omega_2^{JK} \right] \omega_I(z_1) \wedge \overline{\omega_J(z_1)} \wedge \omega_K(z_2) \wedge \overline{\omega_L(z_2)}$$

Other representations of the KZ invariant I

- Spectral formula:

$$\varphi(\Sigma) = \sum_{\lambda>0} \frac{2}{\lambda} \sum_{m,n=1}^{h} \left| \int_\Sigma \phi_\lambda \, \omega_m \, \bar{\omega}_n \right|^2$$

  where $(\omega_1, \dots, \omega_h)$ is an orthonormal basis of holomorphic differentials on Σ, and λ runs over positive eigenvalues of the Arakelov Laplacian, $(\Delta_\Sigma - \lambda)\phi_\lambda = 0$.

  Zhang 2010

- For hyperelliptic curves, the KZ invariant φ, Faltings invariant δ and modular discriminant Δ are related by

$$\varphi(\Sigma) = -\frac{2h+1}{2h-2}\,\delta(\Sigma) - \frac{3h(h+1)!(h-1)!}{(2h-2)(2h)!}\log||\Delta(\Sigma)|| - \frac{8h(2h+1)}{2h-2}\log 2\pi$$

  de Jong, 2013

- Rk: all genus two curves are hyperelliptic.

Other representations of the KZ invariant II

- The Faltings invariant is the regularized Laplacian

$$\delta(\Sigma) = -6 \log \frac{\det' \Delta_\Sigma}{\mathrm{Area}(\Sigma)} + \mathrm{cte}$$

Alvarez-Gaumé, Bost, Moore, Nelson, Vafa, 1987

- In genus two, δ can be expressed as an integral over the Jacobian,

$$\delta(\Sigma) = -\log ||\Delta|| - \int_{J(\Sigma)} \mu \wedge \mu \log ||\theta||^2$$

Bost, 1987

- Later in this talk, we shall give completely different formulae for $\varphi(\Sigma)$ and $\Delta(\Sigma)$...

Three-loop correction to four-graviton scattering

- At three-loop, using Berkovits' pure spinor formulation,

$$f^{(3)}_{\mathcal{R}^4} = f^{(3)}_{D^4\mathcal{R}^4} = 0\ , \qquad f^{(3)}_{D^6\mathcal{R}^4} = \frac{5}{16} \int_{\mathcal{F}_3} \mathrm{d}\mu_3\, \Gamma_{d,d,3}$$

Gomez Mafra 2014

- In addition, these couplings may receive non-perturbative corrections, of order $\mathcal{O}(e^{-1/g_s})$ and (for $d \geq 6$) $\mathcal{O}(e^{-1/g_s^2})$.
- The full non-perturbative couplings must be invariant under the U-duality group $E_{d+1}(\mathbb{Z})$, which extends the T-duality group $O(d,d,\mathbb{Z})$, and satisfy certain differential equations required by supersymmetry.
- This predicts in particular that $f_{D^{r\leq 6}\mathcal{R}^4}$ do not get any further perturbative contribution, $f^{(h>1)}_{\mathcal{R}^4} = f^{(h>2)}_{D^4\mathcal{R}^4} = f^{(h>3)}_{D^6\mathcal{R}^4} = 0$!

Supersymmetry constraints I

- Supersymmetry requires that $f_{\mathcal{R}^4}, f_{D^4\mathcal{R}^4}, f_{D^6\mathcal{R}^4}$ satisfy

$$\left(\Delta_{E_{d+1}} - \frac{3(d+1)(2-d)}{8-d}\right) f_{\mathcal{R}^4} = 0$$

$$\left(\Delta_{E_{d+1}} - \frac{5(d+2)(3-d)}{8-d}\right) f_{D^4\mathcal{R}^4} = 0$$

$$\left(\Delta_{E_{d+1}} - \frac{6(4-d)(d+4)}{8-d}\right) f_{D^6\mathcal{R}^4} = -\left(f_{\mathcal{R}^4}\right)^2$$

where $\Delta_{E_{d+1}}$ is the Laplace-Beltrami operator on the moduli space E_{d+1}/K_{d+1}.

BP 1998; Green Sethi 1998; Green Vanhove J. Russo 2010;
Bossard Verschinin 2014; Wang Yin 2015

Supersymmetry constraints II

- In truth, due to infrared divergences, the equations are a little more complicated:

$$\left(\Delta_{E_{d+1}} - \frac{3(d+1)(2-d)}{8-d}\right) f_{\mathcal{R}^4} = 6\pi\,\delta_{d,2}\ ,$$

$$\left(\Delta_{E_{d+1}} - \frac{5(d+2)(3-d)}{8-d}\right) f_{D^4\mathcal{R}^4} = 40\,\zeta(2)\,\delta_{d,3} + 7\,f_{\mathcal{R}^4}\,\delta_{d,4}$$

$$\left(\Delta_{E_{d+1}} - \frac{6(4-d)(d+4)}{8-d}\right) f_{D^6\mathcal{R}^4} = -\left(f_{\mathcal{R}^4}\right)^2 - \beta_6\,\delta_{d,4} - \beta_5\,f_{\mathcal{R}^4}\,\delta_{d,5} - \beta_4\,f_{D^4\mathcal{R}^4}\,\delta_{d,6}$$

with known real coefficients $\beta_4, \beta_5, \beta_6$. This will matter later.

BP 2015; Bossard Kleinschmidt 2015

Supersymmetry constraints III

- Inserting the genus expansion, one gets T-duality invariant differential constraints on $f^{(h)}_{D^r\mathcal{R}^4}(G,B)$, e.g.

$$
\begin{aligned}
\left[\Delta_{SO(d,d)} + d(d-2)/2\right] f^{(1)}_{\mathcal{R}^4} &= 4\pi\,\delta_{d,2} \\
\left[\Delta_{SO(d,d)} + d(d-3)\right] f^{(2)}_{D^4\mathcal{R}^4} &= 24\zeta(2)\,\delta_{d,3} + 4\mathcal{E}^{(d,1)}_{(0,0)}\delta_{d,4} \\
\left[\Delta_{SO(d,d)} - (d+2)(5-d)\right] f^{(2)}_{D^6\mathcal{R}^4} &= -\left(f^{(1)}_{\mathcal{R}^4}\right)^2 - \tfrac{\pi}{3} f^{(1)}_{\mathcal{R}^4}\,\delta_{d,2} \\
&\quad + \tfrac{70}{3}\zeta(3)\delta_{d,5} + \tfrac{20}{\pi} f^{(1)}_{D^4\mathcal{R}^4}\delta_{d,6}
\end{aligned}
$$

- Using $\left[\Delta_{SO(d,d)} - 2\Delta_\Omega + \frac{1}{2}dh(d-h-1)\right]\,\Gamma_{d,d,h} = 0$ and integrating by parts, these constraints are all seen to be satisfied, save for the last one above.

Supersymmetry constraints IV

- Since $(d+2)(5-d)+d(d-3)=10$, the constraint

$$\left[\Delta_{SO(d,d)} - (d+2)(5-d)\right] f^{(2)}_{D^6\mathcal{R}^4} = -\left(f^{(1)}_{\mathcal{R}^4}\right)^2 + \dots$$

would be satisfied if $\varphi(\Omega)$ was an eigenmode of Δ_Ω, up to a delta function source on the separating degeneration locus $S : \Omega_{12} \to 0$:

$$\boxed{[\Delta_\Omega - 5]\,\varphi \overset{?}{=} -2\pi\,\delta_S} \qquad [*]$$

D'Hoker Green BP R. Russo 2014

- The delta function source agrees from known behavior in the separating degeneration limit $v \equiv \Omega_{12} \to 0$,

$$\varphi(\Omega) = -\log\left|2\pi v\,\eta^2(\rho)\eta^2(\sigma)\right| + \mathcal{O}(|v|^2 \log|v|)\,.$$

Wentworth 1991

Closing in on the KZ invariant I

- Further support for comes by studying the tropical limit (i.e. maximal non-separating degeneration): parametrizing $\Omega_2 = \begin{pmatrix} L_1 + L_3 & L_3 \\ L_3 & L_2 + L_3 \end{pmatrix}, 0 < L_3 < L_1 < L_2$, two-loop SUGRA gives

$$\varphi(\Omega) \overset{L_i \to \infty}{\to} \varphi_t(L_i) = \frac{\pi}{6}\left[L_1 + L_2 + L_3 - \frac{5\, L_1 L_2 L_3}{L_1 L_2 + L_2 L_3 + L_3 L_1} \right]$$

  which is indeed annihilated by $\Delta_\Omega - 5$!

- [$*$] can in fact be established using standard deformation theory of complex structures on a Riemann surface. Genus 2 is crucial !
- The two-loop $D^6\mathcal{R}^4$ coupling in $D = 10$ is now easily computed:

$$f^{(2),d=0}_{D^6\mathcal{R}^4} \propto \int_{\mathcal{F}_2} \mathrm{d}\mu_2\, \varphi = \frac{1}{5} \lim_{\epsilon \to 0} \int_{\mathcal{F}_2^\epsilon} \mathrm{d}\mu_2\, \Delta_\Omega \varphi = \frac{2\pi^3}{45}$$

  D'Hoker Green BP Russo 2014

  in agreement with S-duality predictions !

Closing in on the KZ invariant II

- Additional source terms in the differential equation for $f^{(2)}_{D^6\mathcal{R}^4}$ in $d=4,5,6$ can be seen to arise with the right coefficient, provided φ behaves in the maximal non-separating degeneration as,

$$\varphi(\Omega)=\varphi_t(L_i)+\frac{5\zeta(3)}{4\pi^2(L_1L_2+L_2L_3+L_3L_1)}+\mathcal{O}(e^{-L_i})$$

and in the minimal non-separating degeneration as

$$\varphi(\Omega)=\frac{\pi}{6}t-\log\left[e^{-\pi v_2^2/\rho_2}\left|\frac{\theta_1(\rho,v)}{\eta(\rho)}\right|\right]+\frac{\varphi_1}{t}+\mathcal{O}(e^{-t})$$

where $\Omega=\begin{pmatrix}\rho & v\\ v & \sigma_1+\mathrm{i}(t+v_2^2/\rho_2)\end{pmatrix}$ and $\varphi_1(\rho,v)$ is a specific real-analytic Jacobi form of index 0 and weight 0.

BP and R. Russo, 2015

- With some hindsight, this is enough to demystify φ !

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Automorphic forms from theta lifts I

- In a separate project, Angelantonj Florakis and I (2011-...) studied heterotic one-loop modular integrals of the form

$$\text{R.N.} \int_{\mathcal{F}_1} \mathrm{d}\mu_1 \, \Gamma_{d+k,d}(G,B,Y)\, D^n \Phi(\tau)$$

where $\Phi(\tau)$ is a weakly holomorphic modular form of weight $-2n - \frac{k}{2}$ and $D_w = \partial_\tau - \frac{\mathrm{i}w}{2\tau_2}$. This provides automorphic forms on the Grassmannian $O(d+k,d)/[O(d+k) \times O(d)]$, which are eigenmodes of $\Delta_{O(d+k,d)}$, and have logarithmic singularities in real codimension d.

Harvey Moore 1995, Borcherds 1997, Kiritsis Obers 1997

- For $(d+k,d) = (3,2)$, noting that $O(3,2) = Sp(4)$, one obtains a large supply of real-analytic Siegel modular forms of degree 2 !

Automorphic forms from theta lifts II

- For example, the Igusa cusp-form Ψ_{10} is obtained from

$$\log ||\Psi_{10}||(\Omega) = -\frac{1}{4}\int_{\mathcal{F}_1} \mathrm{d}\mu_1 \left[\Gamma^0_{3,2}(\Omega;\tau)\, h_0 + \Gamma^1_{3,2}(\Omega;\tau)\, h_1 - 20\,\tau_2\right] + \mathrm{cte}$$

where $\chi_{K3}(\tau,z) = h_0(\tau)\,\theta_3(2\tau,2z) + h_1(\tau)\,\theta_2(2\tau,2z)$ and $\Gamma^{0|1}_{3,2}$ is the (genus-one, vector-valued) Siegel-Narain theta series for an even lattice of signature (3,2).

Kawai 1996

- The singularity at $\Omega_{12} = 0$ reflects the appearance of new 'massless states': $\log ||\Psi_{10}|| \overset{v\to 0}{\to} \log |\rho_2^5 \sigma_2^5 v^2 \eta^{24}(\rho)\eta^{24}(\sigma)|$.

Automorphic forms from theta lifts III

- Evaluating the integral using the unfolding method leads to the product formula (Gritsenko Nikulin 1997)

$$\Psi_{10}(\Omega) = e^{2\pi i(\rho+\sigma-v)} \prod_{(k,\ell,b)>0} \left(1 - e^{2\pi i(k\sigma+\ell\rho+bv)}\right)^{c(4k\ell-b^2)}$$

where $c(m)$ are the Fourier coefficients of $h(\tau) = h_0(4\tau) + h_1(4\tau) = 2q^{-1} + 20 - 128q^3 + \dots$.

The genus-two KZ invariant as a theta lift I

- Choosing $\frac{\theta_1^2(\tau,z)}{\eta^6} = \tilde{h}_0(\tau)\,\theta_3(2\tau,2z) + \tilde{h}_1(\tau)\,\theta_2(2\tau,2z)$, the theta lift

$$\tilde{\varphi}(\Omega) = -\frac{1}{2}\int_{\mathcal{F}_1} \mathrm{d}\mu_1 \left[\Gamma^0_{3,2}(\Omega;\tau)\,D_\tau \tilde{h}_0(\tau) + \Gamma^1_{3,2}(\Omega;\tau)\,D_\tau \tilde{h}_1(\tau)\right] ,$$

  can be shown to satisfy the same Laplace equation and degeneration limits as $\varphi(\Omega)$.
- The difference $\varphi(\Omega) - \tilde{\varphi}(\Omega)$ is square-integrable, and eigenmode of Δ_Ω with strictly positive eigenvalue (5). Thus $\varphi(\Omega) = \tilde{\varphi}(\Omega)$!

BP 2015

The genus-two KZ invariant as a theta lift II

- Using the relation between the KZ invariant, Faltings invariant δ and discriminant $\Delta = \Psi_{10}$,

$$\varphi(\Omega) = -3 \log ||\Psi_{10}||(\Omega) - \tfrac{5}{2}\delta(\Omega) - 40 \log 2\pi$$

a theta lift representation for the Faltings invariant $\delta(\Omega)$ follows:

$$\delta(\Omega) = \int_{\mathcal{F}_1} \mathrm{d}\mu_1 \left[\Gamma^0_{3,2}(\Omega;\tau)\, \tfrac{2\hat{E}_2 \tilde{h}_0 + 7h_0}{24} + \Gamma^1_{3,2}(\Omega;\tau)\, \tfrac{2\hat{E}_2 \tilde{h}_1 + 7h_1}{24} - 6\,\tau_2 \right] + 6 \log\left(\tfrac{4}{3\sqrt{3}} e^{1-\gamma_E} \right) - 10 \log 2\pi \; .$$

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Fourier expansion I

- Evaluating the integral using the unfolding trick, one finds

$$\varphi(\Omega) = \frac{\pi}{6}(\rho_2 + \sigma_2 - |v_2|) - \frac{5\pi}{6}\frac{|v_2|(\rho_2 - |v_2|)(\sigma_2 - |v_2|)}{|\Omega_2|} + \frac{5\zeta(3)}{4\pi^2|\Omega_2|}$$
$$- \frac{5}{16\pi^2|\Omega_2|}\sum_{(k,\ell,b)>0} \tilde{c}(4k\ell - b^2)\, D_2\left(e^{2\pi\mathrm{i}(k\sigma+\ell\rho+bv)}\right)$$
$$+ \frac{1}{2}\sum_{(k,\ell,b)>0} (4k\ell - b^2)\,\tilde{c}(4k\ell - b^2)\, D_1\left(e^{2\pi\mathrm{i}(k\sigma+\ell\rho+bv)}\right),$$

where $(k,\ell,b) > 0$ means $(k > 0, \ell \geq 0)$ or $(k = 0, \ell > 0)$ or $(k = \ell = 0, b > 0)$; $D_m(x)$ are single-valued polylogarithms

$$D_1(x) = 2\mathrm{Re}[\mathrm{Li}_1(x)]\,, \quad D_2(x) = -4\mathrm{Re}[\mathrm{Li}_3(x) - \log|x|\ \mathrm{Li}_2(x)]\,.$$
$$\tilde{h}(\tau) = \tilde{h}_0(4\tau) + \tilde{h}_1(4\tau) = \sum_{m\geq -1} \tilde{c}(m) q^m = -\frac{1}{q} + 2 - 8q^3 + \dots$$

Harvey Moore 1995, Borcherds 1995

Numerical investigations I

- These formulae provide an efficient algorithm to evaluate $\varphi(\Sigma)$ to arbitrary accuracy, given the period matrix Ω.
- Numerical searches indicate that the minimal value of φ_2 is attained at the Burnside curve $y^2 = x^5 - x$, with automorphism group $S_4 \times \mathbb{Z}_2$ and period matrix

$$\Omega = \begin{pmatrix} -\frac{1}{2} + \frac{\mathrm{i}}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} + \frac{\mathrm{i}}{\sqrt{2}} \end{pmatrix} , \qquad \begin{aligned} \varphi(\Sigma) &= 0.519860385419959\ldots \\ \delta(\Sigma) &= -16.8264632650009\ldots \end{aligned}$$

Klein Kotokov Korotkin 2005

A Siegel mock modular form underlying φ I

- Using the same idea which allows to write the one-loop correction to heterotic gauge couplings in terms of a prepotential, $\varphi(\Omega)$ can be integrated to a holomorphic function on $\mathcal{H}_2$,

$$\varphi = \mathrm{Re}\left(\square_{-2} F\right)$$

$$F(\Omega) = \sum_{(k,\ell,b)>0} \tilde{c}(4k\ell - b^2)\, \mathrm{Li}_3\left(e^{2\pi\mathrm{i}(k\sigma+\ell\rho+bv)}\right) - \frac{\mathrm{i}\pi^3}{3}\rho\sigma(\rho+\sigma-2v) + \zeta(3)$$

where $\square_w$ is the Maass raising operator, sending M_w to M_{w+2},

$$\square_w = -\frac{1}{\pi^2}\left[\partial_\rho\partial_\sigma - \tfrac{1}{4}\partial_v^2 + \frac{\mathrm{i}(1-2w)}{4(\rho_2\,\sigma_2 - v_2^2)}\left(\tfrac{w}{2\mathrm{i}} + \sigma_2\partial_\sigma + \rho_2\partial_\rho + v_2\partial_v\right)\right] ,$$

A Siegel mock modular form underlying φ II

- F transforms as a Siegel mock modular form of weight -2,

$$F|_{-2}\gamma\,(\Omega) = F(\Omega) + P_\gamma(\Omega)\ ,$$

  where $P_\gamma(\Omega)$ is a polynomial of degree 2 in Ω.

- More generally, the theta lift of a weak Jacobi form $\chi(\tau, z) \in J_{-2n,1}$ produces a real-analytic Siegel modular function φ, which can be integrated to a Siegel mock modular form F of weight $-2n$:

$$\varphi = \mathrm{Re}\left(\square^n_{-2n} F\right)\ , \quad F = F_0 + \sum_{m=1}^{\infty} V_m \cdot \chi(\rho, v)\, e^{2\pi i m\sigma}$$

  where $V_m : J_{w,1} \to J_{w,m}$ is the usual Hecke-like operator. This provides an infinite supply of Siegel mock modular forms...

Poincaré series representation I

- The vector-valued modular form $(\tilde{h}_0, \tilde{h}_1)$ has a Poincaré series representation,

$$\tilde{h}_0 = -\frac{1}{\Gamma(9/2)} \mathcal{F}_0(\tfrac{9}{4}, \tfrac{1}{4}, -\tfrac{5}{2})\,, \quad \tilde{h}_1 = -\frac{1}{\Gamma(9/2)} \mathcal{F}_\infty(\tfrac{9}{4}, \tfrac{1}{4}, -\tfrac{5}{2})$$

where $\mathcal{F}_a(s, \kappa, w; \tau)$ is the Niebur-Poincaré

$$\mathcal{F}_a(s, \kappa, w) = \sum_{\gamma \in \Gamma_a \backslash \Gamma_0(4)} \mathcal{M}_{s,w}(-\kappa \tau_2)\, e^{2\pi i \kappa \tau_1} |_w \gamma$$

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Poincaré series representation II

- Computing the modular integral by unfolding trick, we get a Poincaré series representation for the KZ invariant,

$$\varphi(\Omega) = \sum_{\gamma \in \frac{Sp(4,\mathbb{Z})}{SL(2,\mathbb{Z})\times SL(2,\mathbb{Z})}} f\left(\frac{|v|}{|\Omega_2|^{1/2}}\right)|_\gamma$$

  where

$$f(u) = \frac{\Gamma(5/2)}{4\Gamma(9/2)\,u^5}\,{}_2F_1\left(\frac{5}{2},\frac{5}{2};\frac{9}{2};-1/u^2\right)$$
$$= \frac{2+5u^2}{4}\,\text{arccosh}(u) - \frac{11+15u^2}{12\sqrt{1+u^2}}$$

- Each term in the series is an eigenmode of $\Delta_\Omega - 5$, with a log singularity on the separation locus $v|_\gamma = 0$.

Conclusion - Outlook

- Using insights from string dualities, we discovered completely new formulae for the genus-two Kawazumi-Zhang and Faltings invariant. Can this be pushed to higher genus ?
- Theta lifts of vector-valued modular forms give an infinite supply of mock modular forms on orthogonal Grassmannians $\frac{O(2+k)}{O(2)\times O(k)}$. Can one find their modular completion, etc ?
- String amplitudes at higher order in momentum provide an infinite series of real-analytic functions on $\mathcal{M}_h$. How about $f_{D^8\mathcal{R}^4}$ at two-loop ? three-loop ? Non-perturbatively ?
- Higher loop theta lifts of φ, such as $\int_{\mathcal{F}_2} \mathrm{d}\mu_2 \Gamma_{d,d,2}\varphi$, give rise to new types of automorphic functions. How do they fit in the general framework of automorphic forms ?

Backup: Beyond the KZ invariant

- At next order in the derivative expansion, one finds e.g.

$$f^{(2)}_{D^8\mathcal{R}^4} = \text{R.N.} \int_{\mathcal{F}_2} \mathrm{d}\mu_2 \, \Gamma_{d,d,2}(G,B;\Omega) \, \mathcal{B}^{(2)}_{D^8\mathcal{R}^4}(\Omega)$$

where

$$\mathcal{B}^{(2)}_{D^8\mathcal{R}^4}(\Omega) = \int_{\Sigma^4} \frac{|\Delta_{12}\Delta_{34}|^2}{|\Omega_2|^2} \left(G_{14} + G_{23} - G_{13} - G_{24}\right)^2$$

- More generally, string amplitudes produce infinite families of real-analytic Siegel modular forms, or more generally Teichmuller modular forms. Are some of them relevant for mathematicians ?