BPS amplitudes, theta lifts and the Kawazumi-Zhang invariant

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Introduction I

- String amplitudes at h-loop typically involve an integral $\int_{\mathcal{M}_h} F \, \mathrm{d}\mu_h$ over the moduli space of compact Riemann surfaces of genus h. For $1 \leq h \leq 3$, this is the same as a fundamental domain $\mathcal{F}_h$ in the Siegel upper half plane of degree h.
- In the case of toroidal vacua, the integrand decomposes as $F = \Phi \times \Gamma_{d+k,d,h}$, where $\Gamma_{d+k,d,h}$ is the Siegel-Narain theta series. The modular integral produces an automorphic form on the orthogonal Grassmannian $G_{d+k,d} = \frac{O(d+k,d)}{O(d+k)\times O(d)}$, which parametrizes the metric and B-field on the torus T^d.
- For $h = 1$, $d = 2$ and F weak holomorphic, this is the regularized theta lift considered by Borcherds and Harvey-Moore (1995). String theory offers many more examples of theta lifts, both at genus one, $d > 2$ and at genus $h = 2, 3$.

Introduction II

- In this talk, I will focus on $\mathcal{R}^4$, $D^4\mathcal{R}^4$ and $D^6\mathcal{R}^4$ couplings in the low energy effective action of type II strings compactified on T^d, since these terms are strongly constrained by supersymmetry and invariance under U-duality group $E_{d+1}(\mathbb{Z})$.
- In particular, D'Hoker and Green (2013) have shown that the two-loop $D^6\mathcal{R}^4$ coupling is proportional to the theta lift of the Kawazumi-Zhang invariant φ, an invariant of compact Riemann surfaces closely related to Faltings invariant.
- By studying various physical constraints on the $D^6\mathcal{R}^4$ coupling, we shall discover that φ is itself a Borcherds' lift of a simple weak Jacobi form, giving unlimited access to this previously elusive invariant !

Four-graviton scattering in type II strings, tree-level

- The study of the four-graviton scattering amplitude in type II string theories has a long history. At tree-level, with $s = -\alpha' p_1 \cdot p_2/2$, $t = -\alpha' p_1 p_3/2$, $u = - - \alpha' p_1 p_4/2$ (hence $s + t + u = 0$)

$$\mathcal{A}^{(0)} \propto \frac{\Gamma(1-s)\,\Gamma(1-t)\,\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)}$$
$$= \frac{3}{stu} + 2\zeta(3) + \zeta(5)(s^2 + t^2 + u^2) + \frac{2}{3}[\zeta(3)]^2(s^3 + t^3 + u^3) + \dots$$

Green Schwarz 1981, Gross and Witten 1986, ...

- These terms generate higher-derivative corrections of the form

$$\int \mathrm{d}^D x \sqrt{-g}\, e^{-2\phi} \left[2\zeta(3)\,\mathcal{R}^4 + \zeta(5)\, D^4 \mathcal{R}^4 + \frac{2}{3}[\zeta(3)]^2 D^6 \mathcal{R}^4 + \dots \right]$$

to the low energy effective action.

- Each of these couplings receives quantum corrections. Denote the h-loop contribution by $f^{(h)}_{\mathcal{R}^4}$, so that $f_{\mathcal{R}^4} \propto \sum_{h\geq 0} f^{(h)}_{\mathcal{R}^4} e^{(2h-2)\phi} + \mathrm{n.p.}$

One-loop correction to four-graviton scattering

- At one-loop, a simple computation gives

$$
\begin{aligned}
f^{(1)}_{\mathcal{R}^4} &= \pi\, \mathrm{R.N.} \int_{\mathcal{F}_1} \mathrm{d}\mu_1\, \Gamma_{d,d,1}(G,B;\tau) \\
f^{(1)}_{D^4\mathcal{R}^4} &= 2\pi\, \mathrm{R.N.} \int_{\mathcal{F}_1} \mathrm{d}\mu_1\, \Gamma_{d,d,1}(G,B;\tau)\, \mathcal{E}^\star_1(2;\tau) \\
f^{(1)}_{D^6\mathcal{R}^4} &= \tfrac{\pi}{3}\, \mathrm{R.N.} \int_{\mathcal{F}_1} \mathrm{d}\mu_1\, \Gamma_{d,d,1}(G,B;\tau)\, \left(5\, \mathcal{E}^\star_1(3;\tau) + \zeta(3)\right)
\end{aligned}
$$

Green Vanhove 1999; Green Russo Vanhove 2008

1. $\mathcal{F}_h$ is a fundamental domain for the action of $Sp(2h,\mathbb{Z})$ on the Siegel upper-half plane of degree h;
2. $\Gamma_{d,d,h}(G,B;\tau)$ is the genus-h Narain lattice partition function, a non-holomorphic Theta series parametrized by the constant metric $G_{ij} = G_{ji} > 0$ and Kalb-Ramond field $B_{ij} = -B_{ji}$ on the torus T^d;
3. $\mathcal{E}^\star_h(s;\tau)$ is the non-holomorphic Eisenstein series for $Sp(2h,\mathbb{Z})$;
4. $\mathrm{R.N.}$ a suitable renormalization prescription - see next

About UV and IR divergences I

- Loop amplitudes in string theory are automatically free of UV divergences. They are also free of IR divergences when $D > 4$ i.e. $d < 6$.
- Near $(s, t, u) \to 0$, the amplitude is non-analytic, and dominated by massless supergravity modes. Decompose

$$\mathcal{A}^{(h)}(s,t,u) = \mathcal{A}^{(h)}_{SUGRA}(s,t,u,\Lambda) + \mathcal{A}^{(h)}_{an}(s,t,u,\Lambda)$$

where the first term is the SUGRA contribution (plus string theory counterterms at lower genus), cut-off at Λ, and $\mathcal{A}^{(h)}_{an}(s,t,u,\Lambda)$ is the remainder. The running scale Λ serves as a UV cut-off for SUGRA modes and IR Wilsonian cut-off for string modes.

About UV and IR divergences II

- The local couplings $f^{(h)}_{D^6\mathcal{R}^4}$ are obtained by Taylor expanding $\mathcal{A}^{(h)}_{an}$ in (s,t,u), subtracting powerlike terms in Λ, and sending $\Lambda \to \infty$.
- For example, at one-loop,

$$\mathrm{R.N.} \int_{\mathcal{F}_1} \mathrm{d}\mu_1\, \Gamma_{d,d,1} = \lim_{\Lambda\to\infty} \left[\int_{\mathcal{F}_1^\Lambda} \mathrm{d}\mu_1\, \Gamma_{d,d,1} - 2\frac{\Lambda^{\frac{d}{2}-1}}{\frac{d}{2}-1} \right]$$

where $\mathcal{F}_1^\Lambda$ is the usual fundamental domain, cut-off at $\mathrm{Im}\tau < \Lambda$.
- These modular integrals can be computed using the Rankin–Selberg-Zagier method, and expressed as Langlands-Eisenstein series for $SO(d,d,\mathbb{Z})$.

Dixon Kaplunovsky Louis 1991, Angelantonj Florakis BP 2011

Two-loop correction to four-graviton scattering I

- At two loops, a much harder computation shows

$$
\begin{aligned}
f^{(2)}_{\mathcal{R}^4} &= 0 \\
f^{(2)}_{D^4\mathcal{R}^4} &= \tfrac{\pi}{2}\,\mathrm{R.N.}\int_{\mathcal{F}_2} \mathrm{d}\mu_2\, \Gamma_{d,d,2}(G,B;\tau) \\
f^{(2)}_{D^6\mathcal{R}^4} &= \pi\,\mathrm{R.N.}\int_{\mathcal{F}_2} \mathrm{d}\mu_2\, \Gamma_{d,d,2}(G,B;\tau)\,\varphi(\tau)
\end{aligned}
$$

  where $\varphi(\tau)$ is the Kawazumi-Zhang invariant !

  D'Hoker Phong 2001-05; D'Hoker Gutperle Phong 2005; D'Hoker Green 2013

- The integrand is obtained by expanding $|\mathcal{Y}_S|^2\, e^{-\frac{\alpha'}{2}\sum_{i<j} p_i\cdot p_j\, G(z_i,z_j)}$ in powers of α', and integrating over the location of the four vertex operators z_i on the genus-two curve.

Two-loop correction to four-graviton scattering II

- At $\mathcal{O}(\alpha'^0)$, corresponding to $D^4\mathcal{R}^4$, the integral over z_i gives a constant. At $\mathcal{O}(\alpha')$, two of the integrations can be done easily, leaving an integral of the form

$$\varphi(\tau) = \int_{\Sigma\times\Sigma} P(z_1,z_2)\, G(z_1,z_2)$$

  where $G(z_1,z_2)$ is the scalar Green function and $P(z_1,z_2)$ is a canonical form of degree $(1,1)$ in z_1 and in z_2. This is recognized as one of the defining formulae for the KZ invariant !

D'Hoker Green 2013

- Using a higher-loop version of the RSZ method, $f^{(2)}_{D^4\mathcal{R}^4}$ can be expressed as Langlands-Eisenstein series of $SO(d,d)$. But $f^{(2)}_{D^6\mathcal{R}^4}$ is a very different type of automorphic form !

Florakis BP 2016

Other definitions of the KZ invariant I

- Spectral formula:

$$\varphi(\Sigma) = \sum_{\ell>0} \frac{2}{\lambda_\ell} \sum_{m,n=1}^{h} \left| \int_\Sigma \phi_\ell\, \omega_m\, \bar\omega_n \right|^2$$

  where $(\omega_1, \dots, \omega_h)$ is an orthonormal basis of holomorphic differentials on Σ, $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \dots$ are the eigenvalues of the Arakelov Laplacian, $(\Delta_\Sigma - \lambda_\ell)\phi_\ell = 0$.

Kawazumi, Zhang 2010

- For hyperelliptic curves, φ, δ and Δ are related by

$$\varphi(\Sigma) = -\tfrac{2h+1}{2h-2}\,\delta(\Sigma) - \tfrac{3h(h+1)!(h-1)!}{(2h-2)(2h)!} \log||\Delta(\Sigma)|| - \tfrac{8h(2h+1)}{2h-2} \log 2\pi$$

de Jong, 2013

- Rk: all genus two curves are hyperelliptic.

Other definitions of the KZ invariant II

- The Faltings invariant is

$$\delta(\Sigma) = -6 \log \frac{\det' \Delta_\Sigma}{\mathrm{Area}(\Sigma)} + \mathrm{cte}$$

Alvarez-Gaumé, Bost, Moore, Nelson, Vafa, 1987

- In genus two,

$$\delta(\Sigma) = -\log ||\Psi_{10}|| - \int_{J(\Sigma)} \mu \wedge \mu \log ||\theta||^2$$

Bost, 1987

- Later in this talk, we shall prove (BP, 2015) [Ω: period matrix of Σ]

$$\boxed{\varphi(\Omega) = -\frac{1}{2} \int_{\mathcal{F}_1} \mathrm{d}\mu_1(\tau) \left[\Gamma^{\mathrm{even}}_{3,2}(\Omega;\tau)\, D_\tau \tilde{h}_0(\tau) + \Gamma^{\mathrm{odd}}_{3,2}(\Omega;\tau)\, D_\tau \tilde{h}_1(\tau) \right]}$$

where $\tilde{h}_0(\tau) = \frac{\theta_2(2\tau)}{\eta^6}$, $\tilde{h}_1(\tau) = -\frac{\theta_3(2\tau)}{\eta^6}$, $D_\tau = \frac{\mathrm{i}}{\pi}(\partial_\tau + \frac{5\mathrm{i}}{4\tau_2})$.

Three-loop correction to four-graviton scattering

- At three-loop, using Berkovits' pure spinor formulation,

$$f^{(3)}_{\mathcal{R}^4} = f^{(3)}_{D^4\mathcal{R}^4} = 0 \,,$$
$$f^{(3)}_{D^6\mathcal{R}^4} = \frac{5}{16} \int_{\mathcal{F}_3} \mathrm{d}\mu_3 \, \Gamma_{d,d,3}(G,B;\tau)$$

Gomez Mafra 2014

- In addition, these couplings may receive non-perturbative corrections, of order $\mathcal{O}(e^{-1/g_s})$ and (for $d \geq 6$) $\mathcal{O}(e^{-1/g_s^2})$.
- These are not computable from first principle yet, however they are fixed by requiring supersymmetry and invariance under the U-duality group $E_{d+1}(\mathbb{Z})$.
- This predicts that $f_{D^{r\leq 6}\mathcal{R}^4}$ do not get any further perturbative contribution, $f^{(h>1)}_{\mathcal{R}^4} = f^{(h>2)}_{D^4\mathcal{R}^4} = f^{(h>3)}_{D^6\mathcal{R}^4} = 0$!

Supersymmetry constraints I

- Supersymmetry requires that $f_{\mathcal{R}^4}, f_{D^4\mathcal{R}^4}, f_{D^6\mathcal{R}^4}$ satisfy the Laplace-type equations

$$\left(\Delta_{E_{d+1}} - \frac{3(d+1)(2-d)}{(8-d)}\right) f_{\mathcal{R}^4} = 6\pi\, \delta_{d,2} \,,$$

$$\left(\Delta_{E_{d+1}} - \frac{5(d+2)(3-d)}{(8-d)}\right) f_{D^4\mathcal{R}^4} = 40\,\zeta(2)\,\delta_{d,3} + 7\, f_{\mathcal{R}^4}\,\delta_{d,4}$$

$$\left(\Delta_{E_{d+1}} - \frac{6(4-d)(d+4)}{8-d}\right) f_{D^6\mathcal{R}^4} = -\,(f_{\mathcal{R}^4})^2 - \beta_6\,\delta_{d,4} - \beta_5\, f_{\mathcal{R}^4}\,\delta_{d,5} - \beta_4\, f_{D^4\mathcal{R}^4}\,\delta_{d,6}$$

where $\Delta_{E_{d+1}}$ is the Laplace-Beltrami operator on the moduli space E_{d+1}/K_{d+1}.

BP 1998; Green Sethi 1998; Green Vanhove J. Russo 2010;

Bossard Verschinin 2014; Wang Yin 2015; BP 2015; Bossard Kleinschmidt 2015

Supersymmetry constraints II

- Inserting the genus expansion, one gets T-duality invariant differential constraints on $f^{(h)}_{D^r\mathcal{R}^4}(G,B)$, e.g.

$$\left[\Delta_{SO(d,d)} + d(d-2)/2\right] f^{(1)}_{\mathcal{R}^4} = 4\pi\,\delta_{d,2}$$

$$\dots$$

$$\left[\Delta_{SO(d,d)} + d(d-3)\right] f^{(2)}_{D^4\mathcal{R}^4} = 24\zeta(2)\,\delta_{d,3} + 4\mathcal{E}^{(d,1)}_{(0,0)}\delta_{d,4}$$

$$\left[\Delta_{SO(d,d)} - (d+2)(5-d)\right] f^{(2)}_{D^6\mathcal{R}^4} = -\left(f^{(1)}_{\mathcal{R}^4}\right)^2 - \tfrac{\pi}{3} f^{(1)}_{\mathcal{R}^4}\,\delta_{d,2} + \tfrac{70}{3}\zeta(3)\delta_{d,5} + \tfrac{20}{\pi} f^{(1)}_{D^4\mathcal{R}^4}\delta_{d,6}$$

$$\dots$$

where $\Delta_{SO(d,d)}$ is the Laplace-Beltrami operator on $SO(d,d)/SO(d)\times SO(d)$

Supersymmetry constraints III

- The strategy to show that these equations hold is to act with $\Delta_{SO(d,d)}$ on the regularized integral, use

$$\left[\Delta_{SO(d,d)} - 2\Delta_\tau + \frac{dh(d-h-1)}{2}\right] \Gamma_{d,d,h} = 0$$

and integrate by parts:

$$\left[\Delta_{SO(d,d)} + \frac{dh(d-h-1)}{2}\right] \int_{\mathcal{F}_h^\Lambda} \mathrm{d}\mu_h \, \Phi \, \Gamma_{d,d,h} = 2 \int_{\mathcal{F}_h^\Lambda} \mathrm{d}\mu_h \, \Phi \, \Delta_\tau \Gamma_{d,d,h}$$
$$= 2 \int_{\mathcal{F}_h^\Lambda} \mathrm{d}\mu_h \, \Gamma_{d,d,h} \, \Delta_\tau \Phi + 2 \int_{\partial\mathcal{F}_h^\Lambda} \left[\Phi \star \mathrm{d}\Gamma_{d,d,h} - \Gamma_{d,d,h} \star \mathrm{d}\Phi\right]$$

- If Φ is an eigenmode of Δ_τ, then $\mathrm{R.N.} \int_{\mathcal{F}_h} \mathrm{d}\mu_h \, \Phi \, \Gamma_{d,d,h}$ is an eigenmode of $\Delta_{SO(d,d)}$, up to a source term coming from $\partial\mathcal{F}_h$.

Supersymmetry constraints IV

- Since $(d+2)(5-d)+d(d-3)=10$, the constraint

$$\left[\Delta_{SO(d,d)}-(d+2)(5-d)\right] f^{(2)}_{D^6\mathcal{R}^4} = -\left(f^{(1)}_{\mathcal{R}^4}\right)^2 + \dots$$

will be satisfied if $\varphi(\tau)$ is an eigenmode of Δ_τ, up to a delta function source on the separating degeneration locus,

$$\boxed{[\Delta_\tau - 5]\,\varphi \stackrel{?}{=} -2\pi\,\delta_S} \qquad [*]$$

D'Hoker Green BP R. Russo 2014

- The delta function source agrees with the known behavior of φ in the separating degeneration limit $\tau_{12}\to 0$,

$$\varphi(\tau) = -\log\left|2\pi\tau_{12}\,\eta^2(\tau_{11})\eta^2(\tau_{22})\right| + \mathcal{O}(|\tau_{12}|^2\log|\tau_{12}|)\ .$$

Wentworth 1991

Closing in on the KZ invariant I

- Further support for [*] comes by studying the SUGRA limit: parametrizing $\mathrm{Im}\tau = \begin{pmatrix} L_1 + L_3 & L_3 \\ L_3 & L_2 + L_3 \end{pmatrix}$, $0 < L_3 < L_1 < L_2$,

$$\varphi(\tau) \overset{L_i\to\infty}{\to} \varphi_t(L_i) = \frac{\pi}{6}\left[L_1 + L_2 + L_3 - \frac{5\,L_1 L_2 L_3}{L_1 L_2 + L_2 L_3 + L_3 L_1}\right]$$

  which is indeed annihilated by $\Delta_\tau - 5$!
- $[*]$ can in fact be established using standard deformation theory of complex structures on a Riemann surface.
- The modular integral of φ over $\mathcal{F}_2$ is now easily computed:

$$\int_{\mathcal{F}_2} \mathrm{d}\mu_2\, \varphi = \frac{1}{5} \lim_{\epsilon\to 0} \int_{\mathcal{F}_2^\epsilon} \mathrm{d}\mu_2\, \Delta_\tau \varphi = \frac{2\pi^3}{45}$$

  in agreement with S-duality predictions for $f^{(2)}_{D^6\mathcal{R}^4}$ in $D = 10$!

D'Hoker Green BP Russo 2014

Closing in on the KZ invariant II

- Additional source terms in the differential equation for $f^{(2)}_{D^6\mathcal{R}^4}$ in $d = 4, 5, 6$ arise with the right coefficient, provided φ behaves in the maximal non-separating degeneration $L_i \to \infty$ as,

$$\varphi(\tau) = \varphi_t(L_i) + \frac{5\zeta(3)}{4\pi^2(L_1L_2+L_2L_3+L_3L_1)} + \mathcal{O}(e^{-L_i})$$

and in the minimal non-separating degeneration $t \to \infty$ as

$$\varphi(\tau) = \frac{\pi}{6}t + \varphi_0 + \frac{\varphi_1}{t} + \mathcal{O}(e^{-t})$$

where $\tau = \begin{pmatrix} \rho & u_1 + \rho u_2 \\ u_1 + \rho u_2 & \sigma_1 + \mathrm{i}(t + \rho_2 u_2^2) \end{pmatrix}$, such that

$$\int_{T^2} \mathrm{d}u_1 \mathrm{d}u_2\, \varphi_0 = 0\,, \quad \int_{T^2} \mathrm{d}u_1 \mathrm{d}u_2\, \varphi_1 = \frac{5}{2\pi}\mathcal{E}^\star_1(2;\rho)\,.$$

Closing in on the KZ invariant III

- Indeed, $\varphi_0 = -\log\left[e^{-\pi u_2^2 \rho_2}\left|\frac{\theta_1(\rho, u_1+\rho u_2)}{\eta(\rho)}\right|\right]$ integrates to zero. The differential constraint $(\Delta_\tau - 5)\varphi = 0$ strongly suggests that φ_1 is given by the Kronecker-Eisenstein series

$$\varphi_1 = \frac{5}{2\pi}\mathcal{E}^\star(2;\rho) - \frac{5}{4\pi^3\rho_2^2}\sum_{m_1,m_2\in\mathbb{Z}^2}' \frac{e^{2\pi \mathrm{i}(m_1 u_1 + m_2 u_2)}}{|m_1\rho + m_2|^4}$$

- These asymptotic estimates strongly suggest that the real-analytic Siegel modular form $\varphi(\tau)$ is a theta lift of an almost holomorphic modular form of depth 1...

Automorphic forms from theta lifts I

- In a separate project with Angelantonj and Florakis (2011-16), we studied one-loop modular integrals of the form

$$\text{R.N.} \int_{\mathcal{F}_1} \mathrm{d}\mu_1 \, \Gamma_{d+k,d}(G,B,Y)\, D^n \Phi(\tau)$$

where $\Phi(\tau)$ is a weakly holomorphic modular form of weight $-2n - \frac{k}{2}$ and $D_w = \partial_\tau - \frac{\mathrm{i}w}{2\tau_2}$. This provides automorphic forms on the Grassmannian $SO(d+k,d)/[SO(d+k) \times SO(d)]$, which are eigenmodes of $\Delta_{SO(d+k,d)}$, and have logarithmic singularities in real codimension d.

Harvey Moore 1995, Borcherds 1997, Kiritsis Obers 1997

- For $(d+k,d) = (3,2)$, noting that $SO(3,2) = Sp(4)$, one obtains a large supply of real-analytic Siegel modular forms of degree 2 !

Automorphic forms from theta lifts II

- For example, the Igusa cusp-form Ψ_{10} is obtained from (Kawai, 1996):

$$\log ||\Psi_{10}||(\Omega) = -\frac{1}{4}\int_{\mathcal{F}_1} \frac{\mathrm{d}^2\tau}{\tau_2^2}\left[\Gamma_{3,2}^{\text{even}}(\Omega;\tau)\, h_0 + \Gamma_{3,2}^{\text{odd}}(\Omega;\tau)\, h_1 - 20\,\tau_2\right] + \text{cte}$$

  where $\chi_{K3}(\tau,z) = h_0(\tau)\,\theta_3(2\tau,2z) + h_1(\tau)\,\theta_2(2\tau,2z)$ and $\Gamma_{3,2}^{\text{even}|\text{odd}}$ is the (genus-one, vector-valued) Siegel-Narain theta series for an even lattice of signature (3,2).

- Physically, the singularity at $\Omega_{12}=0$ reflects the 'appearance of new massless states': $\log ||\Psi_{10}|| \overset{v\to 0}{\to} \log |\rho_2^5\sigma_2^5 v^2 \eta^{24}(\rho)\eta^{24}(\sigma)|$.

Automorphic forms from theta lifts III

- Evaluating the integral using the unfolding method leads to the product formula (Gritsenko Nikulin 1997)

$$\Psi_{10}(\Omega) = e^{2\pi i(\rho+\sigma-v)} \prod_{(k,\ell,b)>0} \left(1 - e^{2\pi i(k\sigma+\ell\rho+bv)}\right)^{c(4k\ell-b^2)}$$

  where $c(m)$ are the Fourier coefficients of $h(\tau) = h_0(4\tau) + h_1(4\tau) = 2q^{-1} + 20 - 128q^3 + \dots$.

The genus-two KZ invariant as a theta lift I

- Choosing $\frac{\theta_1^2(\tau,z)}{\eta^6} = \tilde{h}_0(\tau)\,\theta_3(2\tau,2z) + \tilde{h}_1(\tau)\,\theta_2(2\tau,2z)$, the theta lift

$$\tilde{\varphi}(\Omega) = -\frac{1}{2}\int_{\mathcal{F}_1} \frac{\mathrm{d}^2\tau}{\tau_2^2}\,\left[\Gamma_{3,2}^{\rm even}(\Omega;\tau)\,D_\tau \tilde{h}_0(\tau) + \Gamma_{3,2}^{\rm odd}(\Omega;\tau)\,D_\tau \tilde{h}_1(\tau)\right]\,,$$

can be shown to satisfy the same Laplace equation and degeneration limits as $\varphi(\Omega)$.

- The difference $\varphi(\Omega) - \tilde{\varphi}(\Omega)$ is square-integrable, and eigenmode of Δ_Ω with strictly positive eigenvalue (5). Thus $\varphi(\Omega) = \tilde{\varphi}(\Omega)$!

BP, Jour. Num. Theory. 2015

The genus-two KZ invariant as a theta lift II

- Using the unfolding trick following Harvey Moore (1995), one finds the complete Fourier expansion of $\varphi(\Omega)$ near the cusp at infinity,

$$\begin{aligned}\varphi(\Omega) =& \frac{\pi}{6}(\rho_2+\sigma_2-|v_2|) - \tfrac{5\pi}{6}\tfrac{|v_2|(\rho_2-|v_2|)(\sigma_2-|v_2|)}{|\Omega_2|} + \tfrac{5\zeta(3)}{4\pi^2|\Omega_2|} \\ & - \tfrac{5}{16\pi^2|\Omega_2|} \sum_{(k,\ell,b)>0} \tilde{c}(4k\ell-b^2)\, D_2\left(e^{2\pi \mathrm{i}(k\sigma+\ell\rho+bv)}\right) \\ & + \tfrac12 \sum_{(k,\ell,b)>0} (4k\ell-b^2)\,\tilde{c}(4k\ell-b^2)\, D_1\left(e^{2\pi \mathrm{i}(k\sigma+\ell\rho+bv)}\right) \ ,\end{aligned}$$

where $(k,\ell,b)>0$ means $(k>0,\ell\geq 0)$ or $(k=0,\ell>0)$ or $(k=\ell=0,b>0)$;

$$\begin{aligned}D_1(x) =& 2\mathrm{Re}[\mathrm{Li}_1(x)] \ , \quad D_2(x) = -4\mathrm{Re}[\mathrm{Li}_3(x) - \log|x|\,\mathrm{Li}_2(x)] \\ \tilde{h}(\tau) =& \tilde{h}_0(4\tau)+\tilde{h}_1(4\tau) = \sum_{m\geq -1} \tilde{c}(m) q^m = -\frac{1}{q} + 2 - 8q^3 + \dots\end{aligned}$$

The genus-two KZ invariant as a theta lift III

- This provides an efficient algorithm to evaluate $\varphi(\Sigma)$ to arbitrary accuracy, given the period matrix Ω.
- Using the relation between the KZ invariant, Faltings invariant δ and discriminant $\Delta = \Psi_{10}$,

$$\varphi(\Omega) = -3 \log ||\Psi_{10}||(\Omega) - \frac{5}{2}\delta(\Omega) - 40 \log 2\pi$$

a theta lift representation for the Faltings invariant $\delta(\Omega)$ follows.

A Siegel mock modular form underlying φ I

- This modular integral is similar to the one arising when computing the one-loop correction to the holomorphic prepotential in $\mathcal{N}=2$ heterotic string vacua.
- By the same token, $\varphi(\Omega)$ can be integrated to a holomorphic function,

$$\varphi = \mathrm{Re}\left(\Box_{-2} F_1\right)$$

where

$$F_1(\Omega) = \sum_{(k,\ell,b)>0} \tilde{c}(4k\ell - b^2)\,\mathrm{Li}_3\left(e^{2\pi\mathrm{i}(k\sigma+\ell\rho+bv)}\right) - \frac{\mathrm{i}\pi^3}{3}\rho\sigma(\rho+\sigma-2v) + \zeta(3)$$

where $\Box_w$ is the Maass raising operator, sending M_w to M_{w+2}.

A Siegel mock modular form underlying φ II

- F_1 transforms as a Siegel mock modular form of weight -2,

$$F_1|_{-2}\gamma\,(\Omega) = F_1(\Omega) + P_\gamma(\Omega)\ ,$$

  where $P_\gamma(\Omega)$ is a polynomial of degree 2 in Ω.

- More generally, the theta lift of a weak Jacobi form of index 1 and weight $-2n$ produces a real-analytic Siegel modular function φ_n, which can be written as

$$\varphi_n = \mathrm{Re}\,(\Box^n F_n)$$

  where F_n is a Siegel mock modular form of weight $-2n$.

Kiritsis Obers 1997; Lerche Stieberger Warner 1998; Angelantonj Florakis BP, to appear

Exact $D^6\mathcal{R}^4$ coupling I

- The non-perturbative completion of $f_{\mathcal{R}^4}$ and $f_{D^4\mathcal{R}^4}$ couplings is known to be given by Langlands-Eisenstein series $\mathcal{E}^{E_{d+1}(\mathbb{Z})}_{R,s}$ for the duality group. Due to the quadratic source term in the Laplace equation, $f_{D^6\mathcal{R}^4}$ must lie outside this class.
- Using the fact that the U-duality group $SO(5,5)$ in $D=6$ coincides with the T-duality group in $D=5$, a natural candidate for the non-perturbative completion of $f_{D^6\mathcal{R}^4}$ in $D=6$ is (BP, 2015):

$$f_{D^6\mathcal{R}^4} = \pi\,\mathrm{R.N.}\int_{\mathcal{F}_2} \mathrm{d}\mu_2\,\Gamma_{5,5,2}\,\varphi + \frac{8}{189}\hat{\mathcal{E}}^{SO(5,5)}_{[00001],4}$$

This reproduces the correct perturbative terms at weak-coupling. It would be interesting to extract the non-perturbative corrections from 1/8-BPS instantons, and compare with other proposals in the literature.

Green Miller Russo Vanhove; Bossard Kleinschmidt

Conclusion - Outlook

- Using insights from string dualities, we discovered completely new, efficient formulae for the genus-two Kawazumi-Zhang and Faltings invariant, opening the way to numerical experiments. Can this be pushed to higher genus ?
- Theta lifts of vector-valued modular forms give an infinite supply of mock modular forms on orthogonal Grassmannians $\frac{O(2+k)}{O(2)\times O(k)}$. Can one find their modular completion, etc ?
- String amplitudes at higher order in momentum provide an infinite series of real-analytic functions on $\mathcal{M}_h$. How about $f_{D^8\mathcal{R}^4}$ at two-loop ? three-loop ? Non-perturbatively ?
- Higher loop theta lifts of φ, such as $\int_{\mathcal{F}_2} \mathrm{d}\mu_2 \Gamma_{d,d,2}\varphi$, give rise to new types of automorphic forms, beyond Langlands-Eisenstein series. How do they fit in the Langlands program ?