Indefinite theta series, generalized error functions and D-instantons

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Introduction I

- Theta series for Euclidean lattices are an important source of holomorphic modular forms: for $Q(x) = B(x,x)$ a positive definite quadratic form on $\mathbb{R}^n$,

$$\Theta_Q(\tau, v) = \sum_{k\in\mathbb{Z}^n} q^{\frac{1}{2}Q(k)} e^{2\pi i B(v,k)} \,, \quad q = e^{2\pi i \tau}$$

  is a holomorphic Jacobi form of weight $n/2$ under a suitable congruence subgroup of $SL(2,\mathbb{Z})$.

- Let $Q(x)$ a signature $(r, n-r)$ quadratic form and $\mathcal{C}$ an open cone in $\mathbb{R}^{n-r,r}$ such that $x \in \mathcal{C} \Rightarrow Q(x) > 0$, then

$$\Theta_{Q,\mathcal{C}}(\tau, v) = \sum_{k\in\mathbb{Z}^n\cap\mathcal{C}} q^{\frac{1}{2}Q(k)} e^{2\pi i B(v,k)}$$

  defines a holomorphic q-series, but is it modular ?

Introduction II

- Such indefinite theta series occur in many contexts:
  - Partition functions of coset models / branching functions for affine Lie algebras: *Kac Peterson 1984*
  - Characters of superconformal field theories / super-Lie algebras: *Eguchi Taormina 1988, Kac Wakimoto 2000, Semikhatov Taormina Tipunin 2003*
  - Donadson and Vafa-Witten invariants of 4-manifolds: *(Zagier 1991), Vafa Witten 1994, Goettsche 1996, Goettsche-Zagier 1998, Manschot 2010*
  - Quantum invariants of knows and 3-manifolds: *Lawrence Zagier 1999, Hikami 2007, Hikami Lovejoy 2014*
  - Combinatorics of partitions: *Bringmann Ono 2005*
  - Mirror symmetry for elliptic curves and Abelian varieties: *Polischchuk 1998*
  - Gromov-Witten invariants of Landau-Ginzburg orbifolds: *Lau Zhou 2014, Bringmann Rolen Zwegers 2015*

Introduction III

- Examples of modular theta series of signature $(1,1)$ were studied by Kronecker (around 1890) and Hecke (1925).
- Non-modular examples were studied by Appell (1886) and Lerch (1892): the Appell-Lerch sum can be written as a signature $(1,1)$ theta series, e.g. for $|q| < |y| < 1$, $y = e^{2\pi i v}$,

$$\theta_1(\tau, v)\, \mu(\tau, v) \equiv \sum_{n\in\mathbb{Z}} (-1)^n \frac{q^{\frac{1}{2}n(n+1)} y^{n+\frac{1}{2}}}{1 - yq^n}$$
$$= \Big(\sum_{n,m\geq 0} - \sum_{n<0,m\leq 0} \Big) (-1)^n\, q^{\frac{1}{2}n(n+1)+mn} y^{n+m+\frac{1}{2}}$$

This Appell-Lerch sum and generalizations thereof underlies many of the examples mentioned earlier.

Introduction IV

- In his ground-breaking PhD thesis (2002), Zwegers showed how to correct $\mu(\tau, v)$ into a non-holomorphic, real-analytic, Jacobi form of weight 1. Schematically, one replaces

$$\tfrac{1}{2}\,[\mathrm{sign}(m) + \mathrm{sign}(n)] \mapsto \frac{1}{2}\left[E_1(m\sqrt{2\tau_2}) + \mathrm{sign}(n)\right]$$

  in the summand, where $E_1(u) \equiv \mathrm{Erf}(u\sqrt{\pi})$ is the error function.
- The difference is a theta series with an insertion of $M_1(m\sqrt{2\tau_2})$, where M_1 is the complementary error function,

$$M_1(u) = E_1(u) - \mathrm{sign}(u) = -\mathrm{sign}(u)\mathrm{Erfc}(|u|\sqrt{\pi})\ ,$$

  which can be written as an Eichler integral of an ordinary unary theta series. From this the modular behavior of $\mu(\tau, v)$ follows.

Introduction V

- Recall that given a modular form $F(w)$ of weight h, the Eichler integral

$$G(\bar{\tau}) = \int_{\tau}^{\mathrm{i}\infty} \frac{F(w)\,\mathrm{d}w}{(w-\bar{\tau})^{2-h}}$$

  transforms as (the complex conjugate of) a modular form of weight $2-h$, up to an inhomogeneous term proportional to a period integral of F,

$$G\left(\frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right) = (c\bar{\tau}+d)^{2-h}\left[G(\bar{\tau}) - \int_{-d/c}^{\mathrm{i}\infty} \frac{F(w)\,\mathrm{d}w}{(w-\bar{\tau})^{2-h}}\right]$$

- $G(\bar{\tau})$ is a mock modular form of weight $2-h$ and shadow $F(\tau)$.

Introduction VI

- More generally, Zwegers showed that the modular completion of an indefinite theta series of signature $(n-1,1)$ of the form

$$\Theta_{C,C'}(\tau,v)=\frac12\sum_{k\in\mathbb{Z}^n+b}\left[\mathrm{sign}B(k,C)-\mathrm{sign}B(k,C')\right]q^{\frac12 Q(k)}e^{-2\pi\mathrm{i}B(c,k-\frac12 b)}$$

  where C, C' are a pair of vectors with $Q(C), Q(C'), B(C,C')<0$ and $v=b\tau-c$, is obtained by replacing

$$\mathrm{sign}B(k,C)\mapsto E_1\left(B(k,C)\sqrt{\frac{2\tau_2}{-Q(C)}}\right)$$

- Our goal will be to generalize Zwegers' construction to arbitrary signature $(n-r,r)$. *For consistency with our paper, I will henceforth flip the sign of Q: $(n-r,r)\to(r,n-r)$.*

Vignéras' Theorem (1977) I

- Let $\Lambda \subset \mathbb{R}^n$ be an n-dimensional lattice with a signature $(r, n-r)$ quadratic form $Q(x) = B(x,x)$, such that $Q(k) \in 2\mathbb{Z}$ for $k \in \Lambda$. For any $\mu \in \Lambda^*/\Lambda$, $\lambda \in \mathbb{Z}$, $\Phi : \mathbb{R}^n \to \mathbb{C}$ such that $\Phi(x) e^{\frac{\pi}{2} Q(x)} \in L_1(\mathbb{R}^n)$,

$$\vartheta_{\boldsymbol{\mu}}[\Phi, \lambda](\tau, b, c) = \tau_2^{-\lambda/2} \sum_{k \in \Lambda + \boldsymbol{\mu} + b} \Phi(\sqrt{2\tau_2} k)\, q^{-\frac{1}{2} Q(k)}\, e^{2\pi \mathrm{i} B(c, k - \frac{1}{2} b)}$$

satisfies the quasi-periodicity conditions

$$\begin{aligned} \vartheta_{\boldsymbol{\mu}}[\Phi, \lambda]\,(\tau, b, c) =& e^{\mathrm{i}\pi B(c,k)}\, \vartheta_{\boldsymbol{\mu}}[\Phi, \lambda]\,(\tau, b+k, c) \\ =& e^{-\mathrm{i}\pi B(b,k)}\, \vartheta_{\boldsymbol{\mu}}[\Phi, \lambda]\,(\tau, b, c+k) \\ =& e^{\mathrm{i}\pi Q(\mu)}\, \vartheta_{\boldsymbol{\mu}}[\Phi, \lambda]\,(\tau+1, b, c+b) \end{aligned}$$

Vignéras' Theorem (1977) II

- If in addition Φ satisfies

$$\left[B^{-1}(\partial_x,\partial_x)+2\pi x\partial_x\right]\Phi(x)=2\pi\lambda\,\Phi(x),$$

$$\begin{cases} R(x)\Phi(x)e^{\frac{\pi}{2}Q(x)}\in L^2(\mathbb{R}^n)\cap L^1(\mathbb{R}^n) \\ R(\partial)\Phi(x)e^{\frac{\pi}{2}Q(x)}\in L^2(\mathbb{R}^n)\cap L^1(\mathbb{R}^n) \end{cases}$$

for any quadratic polynomial R, then $\vartheta_{\boldsymbol{\mu}}[\Phi,\lambda](\tau,b,c)$ transforms as a vector-valued Jacobi form of weight $(\lambda+\frac{n}{2},0)$. Namely,

$$\vartheta_{\boldsymbol{\mu}}[\Phi,\lambda]\left(-\tfrac{1}{\tau},c,-b\right)=\frac{(-\mathrm{i}\tau)^{\lambda+\frac{n}{2}}}{\sqrt{|\Lambda^*/\Lambda|}}\sum_{\boldsymbol{\nu}\in\Lambda^*/\Lambda}e^{2\pi\mathrm{i}B(\boldsymbol{\mu},\boldsymbol{\nu})}\vartheta_{\boldsymbol{\nu}}[\Phi,\lambda]\left(\tau,b,c\right)$$

Vignéras' Theorem (1977) III

- Remark 1: The transformations are those of a Jacobi theta series with $v = 0$ and characteristics $(b, c) \in \Lambda \otimes \mathbb{R}$. To obtain the usual Jacobi form, set $\vartheta_{\boldsymbol{\mu}}(\tau, b, c) = e^{i\pi B(b, b\tau - c)} \, \tilde{\vartheta}_\mu(\tau, v = b\tau - c)$.
- Remark 2: Under the Maass raising and lowering operators,

$$\tau_2^2 \partial_{\bar\tau} \, \vartheta_{\boldsymbol{\mu}}[\Phi, \lambda] = \vartheta_{\boldsymbol{\mu}} \left[\tfrac{i}{4} \left(x \partial_x \Phi - \lambda \Phi \right), \lambda - 2 \right],$$

$$\left(\partial_\tau - \tfrac{i(\lambda + \frac{n}{2})}{2\tau_2} \right) \vartheta_{\boldsymbol{\mu}}[\Phi, \lambda] = \vartheta_{\boldsymbol{\mu}} \left[-\tfrac{i}{4} \left(x \partial_x \Phi + [\lambda + n + 2\pi Q(x)] \Phi \right), \lambda + 2 \right].$$

We call $\tau_2^2 \partial_{\bar\tau}$ and $x \partial_x - \lambda$ the 'shadow' operators.

Vignéras' Theorem (1977) IV

- Remark 3: For any r-dimensional positive plane $\mathcal{P} \subset \mathbb{R}^n$, let $x = x_+ + x_-$ with $x_+ \in \mathcal{P}, x_- \in \mathcal{P}^\perp$. The function $\Phi(x) = e^{-\pi Q(x_+)}$ satisfies the assumptions of the theorem with $\lambda = -r$, and leads to the usual Siegel theta series, also known as Narain lattice partition function $\Gamma_{r,n-r}$.
- Remark 4: In order to get a holomorphic q-series, one needs $x\partial_x \Phi = \lambda \Phi$, but a homogeneous polynomial of degree λ will not satisfy the assumptions of the theorem. Thus, there is tension between holomorphy and modularity.
- To achieve mock modularity, we shall take Φ to be a homogeneous local polynomial of degree λ. For simplicity, take a locally constant function ($\lambda = 0$).

Indefinite theta series of Lorentzian signature I

Let $Q(x)$ a signature $(1, n-1)$ quadratic form, and C, C' linearly independent vectors with $Q(C) = Q(C') = 1$, $B(C, C') > 0$,

$$\begin{aligned}
\Phi_1(x) =& \tfrac{1}{2}(\mathrm{sign}[B(C,x)] - \mathrm{sign}[B(C',x)])\ , \\
\widehat{\Phi}_1(x) =& \tfrac{1}{2}(E_1[B(C,x)] - E_1[B(C',x)]) \\
\Psi_1(x) =& \tfrac{\mathrm{i}}{4}(B(C,x)e^{-\pi[B(C,x)]^2} - B(C',x)e^{-\pi[B(C',x)]^2})
\end{aligned}$$

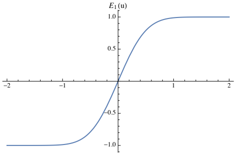
1. $\Theta_\mu[\Phi_1, 0]$, $\Theta_\mu[\widehat{\Phi}_1, 0]$, $\Theta_\mu[\widehat{\Phi}_1, 0]$ are all convergent;
2. $\Theta_\mu[\widehat{\Phi}_1, 0]$ is real-analytic vector-valued Jacobi form of weight $n/2$;
3. $\Theta_\mu[\Phi_1, 0]$ is a holomorphic in τ and in z, but not modular;
4. Their difference is proportional to the Eichler integral of Ψ_1,

$$\Theta_\mu[\widehat{\Phi}_1 - \Phi_1, 0]\,(\tau, b, c) = -4\int_{-\mathrm{i}\infty}^{\bar\tau} \frac{\mathrm{d}\bar w}{(\tau-\bar w)^2}\,\Theta_\mu[\Psi_1, -2]\,(\tau, \bar w, b, c)\,.$$

Sketch of proof:

1. $\mathrm{sign}[B(C,x)] = \mathrm{sign}[B[(C',x)]$ unless $Q(x) < 0$;
2. $\Phi_1(x)$ is locally constant so $\Theta_\mu[\Phi_1, 0]$ is holomorphic;
3. $E_1(u)$ satisfies the 1D Vignéras equation,

$$(\partial_u^2 + 2\pi u \partial_u) E_1(u) = 0$$



so $\widehat{\Phi}_1$ satisfies the assumptions of Vigneras theorem;

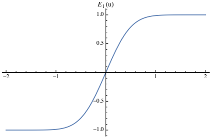
4. Ψ_1 is proportional to the shadow of $\widehat{\Phi}_1$.

Remark: In the limit where C becomes null, then $E_1[B(C,x)] \to \mathrm{sign}[B(C,x)]$. If both C and C' are null and $B(C,C') > 0$, then $\Theta_\mu[\Phi_1]$ is a holomorphic vector-valued Jacobi form of weight $n/2$.

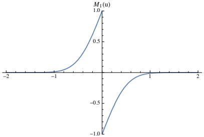
Integral representations for error functions

To prepare the ground for higher signature case, note the following:

$$E_1(u) = \int_{\mathbb{R}} \mathrm{d}u' \, \mathrm{sign}(u') \, e^{-\pi(u-u')^2}$$



$$M_1(u) = \frac{\mathrm{i}}{\pi} \int_{\mathbb{R}-\mathrm{i}u} \frac{\mathrm{d}z}{z} \, e^{-\pi z^2 - 2\pi \mathrm{i} z u}$$



which make it clear that M_1, E_1 are solutions to 1D Vignéras equation,

$$[\partial_u^2 + 2\pi u \partial_u] M_1 = \frac{\mathrm{i}}{\pi} \int_{\mathbb{R}-\mathrm{i}u} \frac{\mathrm{d}z}{z} \, 2\pi z \partial_z [e^{-\pi z^2 - 2\pi \mathrm{i} z u}] = 0$$

$$[\partial_u^2 + 2\pi u \partial_u] E_1 = \int_{\mathbb{R}} \mathrm{d}u' \, \mathrm{sign}(u') \, [\partial_u \partial_{u'} + 2\pi u \partial'_u] \, e^{-\pi(u-u')^2} = 0$$

Indefinite theta series of signature $(r, n-r)$ I

- In signature $(r, n-r)$, consider the locally constant function

$$\Phi_r(x) = \frac{1}{2^r} \prod_{i=1}^{r} \left(\mathrm{sign}[B(C_i, x)] - \mathrm{sign}[B(C'_i, x)] \right)$$

  where C_i, C'_i are chosen such that $Q(x) < 0$ whenever $\Phi_r(x) \neq 0$. To find its modular completion, we need a C^∞ solution $\widehat{\Phi}_r$ of Vignéras equation which asymptotes to Φ_r as $|x| \to \infty$.
- For $r = 2$, Alexandrov Banerjee Manschot BP (2016) found sufficient conditions for the convergence of $\vartheta[\Phi_r, 0]$. Kudla (2016) gave weaker conditions which work for arbitrary r.

Indefinite theta series of signature $(r, n-r)$ II

- To state Kudla's conditions, note that the space $\mathcal{D}$ of positive oriented r planes in $\mathbb{R}^{r,n-r}$ has two disconnected components $\mathcal{D}^{\pm}$. For any $I \subset \{1, \dots r\}$, let $C_I = (C''_1, \dots C''_r)$ where $C''_i = C_i$ if $i \in I$, and $C''_i = C'_i$ if $i \notin I$.

- Theorem (Kudla 2016): Assume that all such collections C_I span distinct positive oriented r planes in the same component, say $\mathcal{D}^{\pm}$. Then $\vartheta[\Phi_r, 0]$ is absolutely convergent.
- For $r = 2$, this reduces to

$$Q(C_1), Q(C'_1), Q(C_2), Q(C'_2) > 0, \quad \Delta_{12}, \Delta_{12'}, \Delta_{1'2}, \Delta_{1'2'} > 0$$
$$B(C_{1\perp 2}, C_{1'\perp 2}), B(C_{1\perp 2'}, C_{1'\perp 2'}) > 0$$
$$B(C_{2\perp 1}, C_{2'\perp 1}), B(C_{2\perp 1'}, C_{2'\perp 1'}) > 0$$

where $\Delta_{12} = Q(C_1)Q(C_2) - B(C_1, C_2)^2$, $C_{1\perp 2} = C_1 - \frac{B(C_1,C_2)}{Q(C_2)} C_2$, etc

Indefinite theta series of signature $(r, n-r)$ III

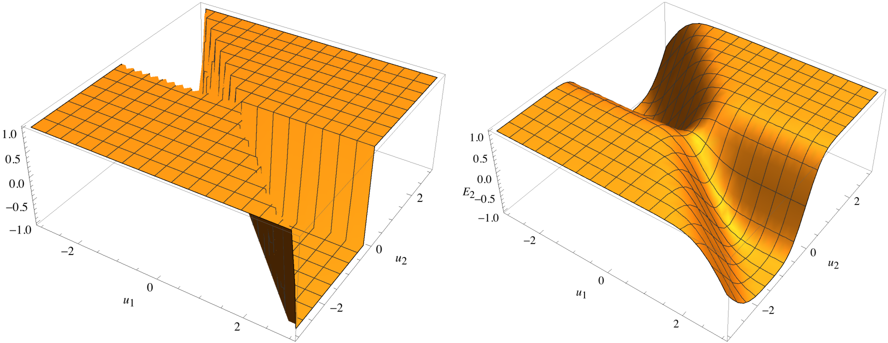
- To construct $\widehat{\Phi}_r$, consider the natural generalizations of the error functions $M_1(u)$ and $E_1(u)$,

$$E_r(\{C_i\};x) = \int_{\langle C_1,\dots,C_r\rangle} \mathrm{d}^r y \prod_{i=1}^{r} \mathrm{sign} B(C_i,y)\, e^{-\pi Q(y-x_+)},$$

$$M_r(\{C_i\};x) = \left(\tfrac{\mathrm{i}}{\pi}\right)^r \int_{\langle C_1,\dots,C_r\rangle - \mathrm{i}x} \mathrm{d}^r z\, \frac{\sqrt{\Delta(\{C_i\})}\, e^{-\pi Q(z)-2\pi \mathrm{i} B(z,x)}}{\prod_{i=1}^{r} B(C_i,z)}$$

where $\mathrm{d}^r y$ is the uniform measure on the plane $\langle C_1,\dots,C_r\rangle$, normalized such that $\int_{\langle C_1,\dots,C_r\rangle} \mathrm{d}^r y\, e^{-\pi Q(y)} = 1$, and x_+ is the orthogonal projection of x on the same plane.

Indefinite theta series of signature $(r, n-r)$ IV



Indefinite theta series of signature $(r, n-r)$ V

Proposition: (ABMP 2016; Nazaroglu 2016)

- $E_r(\{C_i\}; x)$ is a C^∞ solution of Vignéras' equation with $\lambda = 0$, which asymptotes to $\prod_{i=1}^r \mathrm{sign} B(C_i, x)$ as $|x| \to \infty$.
- $M_r(\{C_i\}; x)$ is a C^∞ solution of Vignéras' equation with $\lambda = 0$, away from the walls $B(C_i, x) = 0$, exponentially suppressed in all directions.
- The difference $E_r(\{C_i\}) - \prod_{i=1}^r \mathrm{sign} B(C_i, x)$ is a linear combination of $M_{r'}$ functions with $1 \le r' \le r$, with locally constant coefficients.
- The shadow of $E_r(\{C_i\})$ is a linear combination of $E_{r'}$ functions with $0 \le r' < r$, with Gaussian coefficients.

Theorem:

- The modular completion of $\vartheta[\Phi_r, 0]$ is the non-holomorphic theta series $\vartheta[\widehat{\Phi}_r, 0]$ with kernel

$$\widehat{\Phi}_r(x) = \frac{1}{2^r} \sum_{I \subset \{1,\dots r\}} (-1)^{r-|I|} E_r(C_I; x)$$

- Its shadow $\vartheta[\widehat{\Psi}_r, -2]$ is a linear combination of indefinite theta series of signature $(r-1, n-r+1)$.
- The difference $\vartheta[\widehat{\Phi}_r - \Phi_r, 0]$ is an Eichler integral of $\vartheta[\widehat{\Psi}_r, -2]$, giving access to the modular properties of the holomorphic theta series $\vartheta[\Phi_r, 0]$.

Other approaches to indefinite theta series I

- An alternative approach to indefinite theta series was developped in the 80s by Kudla and Millson, who constructed a closed r-form $\theta_r^{KM}(z,\tau)$ on $\mathcal{D}\times\mathcal{H}$, which is invariant under a (finite index subgroup) of $\mathrm{Aut}(\Lambda)$ and has modular weight $n/2$ in τ.
- Kudla (2016) proposes (and shows for $r=1$ and $r=2$) that

$$\vartheta[\widehat{\Phi}_r, 0] \propto \int_S \theta_r^{KM}(z,\tau)$$

where S is a geodesic hypercube in $\mathcal{D}^+$, parametrized by

$$\phi : [0,1]^r \to S \ , \ [s_1, ..., s_r] \mapsto \langle B_1(s_1), \dots B_r(s_r)\rangle$$

where $B_i(s_i) = (1-s_i)C_i + s_i C_i'$.

Other approaches to indefinite theta series II

- The shadow is proportional to the integral of $\theta^{KM}_{r-1}(z,\tau)$ on the faces of the hypercube, hence a linear combination of indefinite theta series of signature $(r-1, n-r+1)$.
- Any compact geodesic polyhedron S in $\mathcal{D}^+$ similarly leads to a non-holomorphic theta series, which is the modular completion of a holomorphic indefinite theta series.
- Any such S can be decomposed as a sum of geodesic simplexes. Each term corresponds to a theta series $\Theta_{Q,\mathcal{C}}$ where $\mathcal{C}$ is a tetrahedral cone.

Westerholt-Raum

Physics application: S-duality in HM moduli space I

In type IIB string theory compactified on a CY threefold X, the moduli space decomposes into two factors:

$$\mathcal{M} = \mathcal{M}_V(X) \times \mathcal{M}_H(X)$$

- The vector-multiplet moduli space $\mathcal{M}_V(X)$ describes the complex structure moduli of X. It carries a Kähler metric derived from the Kähler potential $\mathcal{K}_V = -\log \int_X \Omega \wedge \bar{\Omega}$.
- The hypermultiplet moduli space $\mathcal{M}_H(X)$ describes the Kähler moduli of X, along with the string coupling $g = e^{\phi}$, the RR axions $C \in H_{\rm even}(X,\mathbb{R})/H_{\rm even}(X,\mathbb{Z})$ and the NS axion $\sigma \in \mathbb{R}/\mathbb{Z}$. It carries a quaternion-Kähler metric (which is NOT a Kähler, not even complex !). In the weak coupling limit $g \to 0$,

$$\mathcal{M}_H(X) \sim_{g\to 0} \text{c-map}(\mathcal{M}_V(\hat{X}))$$

Physics application: S-duality in HM moduli space II

- For g finite, one expects $\mathcal{O}(e^{-1/g})$ corrections from Euclidean D-branes, i.e. stable objects in the derived category of stable sheaves on X, weighted by the corresponding generalized Donaldson-Thomas invariants $\Omega(\gamma)$; as well as $\mathcal{O}(e^{-1/g^2})$ corrections from NS five-branes, whose mathematical description is unknown.
- The same generalized DT invariants count D6-D4-D2-D0 black holes in type IIA string theory compactified on X.

Physics application: S-duality in HM moduli space III

- The effect of Euclidean D-branes on the QK metric on $\mathcal{M}_H(X)$ is well understood in terms of the twistor space $\mathcal{Z}$, a $\mathbb{P}^1$ bundle over $\mathcal{M}_H(X)$ equipped with a canonical complex contact structure.

Alexandrov BP Saueressig Vandoren 2008; Gaiotto Moore Neitzke 2008

- A canonical system of Darboux coordinates Ξ on $\mathcal{Z}$ is obtained by solving the integral equations

$$\Xi = \Xi_{\rm sf} + \frac{1}{8\pi^2} \sum_{\gamma} \gamma\, \Omega(\gamma, z^a) \int_{\ell_\gamma} \frac{\mathrm{d}t'}{t'} \frac{t+t'}{t-t'} \log\left[1 - e^{-2\pi\mathrm{i}\langle \Xi(t'), \gamma\rangle}\right],$$

where $\Xi_{\rm sf}$ are the semi-flat coordinates and $\ell(\gamma) = \{t \,:\, Z(\gamma, z^a)/t \in \mathrm{i}\mathbb{R}^+\}$ are the BPS rays.

- The QK metric is smooth across walls of marginal stability.

Physics application: S-duality in HM moduli space IV

- An key constraint is that $\mathcal{M}_H(X)$ should admit a smooth QK manifold with an isometric action of $SL(2,\mathbb{Z})$, originating from S-duality in type IIB string theory. Equivalently, the contact structure on the twistor space must be invariant under $SL(2,\mathbb{Z})$.

Robles-Llana, Rocek, Saueressig, Theis, Vandoren 2006

- S-duality relates D1-instantons to F1-instantons (hence DT invariants to GW invariants); D5-instanton to NS5-instantons; but maps D3-instantons to themselves. Thus it gives an important constraint on DT-invariants for pure dimension 2 sheaves supported on a divisor $\mathcal{P} \subset X$, with $p = [\mathcal{P}] \in H_4(\mathcal{X},\mathbb{Z}) \equiv \Lambda$.

Physics application: S-duality in HM moduli space V

- Indeed, these same invariants count D4-D2-D0 black hole microstates in type IIA/X, and are described by given by the (modified) elliptic genus of a suitable $N = (4,0)$ SCFT:

$$Z_p(\tau, c) = \sum_{q_a \in \Lambda^*, q_0 \in \mathbb{Z}} \Omega^{\mathrm{MSW}}(p^a, q_a, q_0)\, e^{-2\pi\tau_2 |Z(\gamma)| - 2\pi\mathrm{i}(\tau_1 q_0 + c^a q_a)}$$

  is a Jacobi form of weight $(-3/2, 1/2)$.

Maldacena Strominger Witten 1997

- Here, the MSW invariants are defined as the generalized DT invariants in the 'large volume attractor chamber':

$$\Omega^{\mathrm{MSW}}(p^a, q_a, q_0) = \lim_{\lambda \to +\infty} \Omega\left(0, p^a, q_a, q_0; -\kappa^{ab} q_b + \mathrm{i}\lambda\, p^a\right)$$

de Boer Denef El Showk Messamah van den Bleeken 2008

Physics application: S-duality in HM moduli space VI

- Invariance under spectral flow $(q_a, q_0) \mapsto (q_a - \kappa_{abc} p^b \epsilon^c, q_0 - \epsilon^a q_a + \frac{1}{2}\kappa_{abc} p^a \epsilon^b \epsilon^c)$ implies the theta series decomposition

$$Z_p(\tau, c) = \sum_{\mu \in \Lambda^*/\Lambda} h_{p,\mu}(\tau)\, \theta_{p,\mu}(\tau, c)$$

where $h_{p,\mu}(\tau)$ is a vector-valued holomorphic modular form of weight $-\frac{b_2}{2} - 1$, and $\theta_{p,\mu}(\tau, c)$ a Siegel theta series of signature $(1, b_2 - 1)$ and weight $(\frac{b_2 - 1}{2}, \frac{1}{2})$.

Denef Moore; Gaiotto Strominger Yin; de Boer Cheng Dijkgraaf Manschot Verlinde

Physics application: S-duality in HM moduli space VII

- Explicitly,

$$\theta_{p,\mu}(\tau, c) = \sum_{k\in\Lambda+\mu} e^{-\mathrm{i}\pi\tau Q(k_-)-\mathrm{i}\pi\bar\tau Q(k_+)-2\pi\mathrm{i}c^a k_a}$$

where $Q(k) = \frac{1}{2}\kappa_{abc}p^a k^b k^c$ is a quadratic form of signature $(1, b_2 - 1)$, k_+ is the projection of k along the timelike vector $t \in \mathbb{R}^{1,b_2-1}$ parametrizing the Kähler cone, and $k_- = k - k_+$.

S-duality in HM moduli space I

- In the large volume limit, at one-instanton order and zooming near $t = 0$, the Darboux coordinates Ξ can be expressed in terms of contour integrals of the form

$$\mathcal{J}_p(t) = \sum_{\substack{q\in\Lambda \\ q_0\in\mathbb{Z}}} \int_{\mathbb{R}} \frac{\mathrm{d}t'}{t'-t}\, \Omega^{\mathrm{MSW}}(\gamma) e^{-2\pi\mathrm{i}\langle \Xi_{\mathrm{sf}}(t'),\gamma\rangle},$$

- Using spectral flow invariance of $\Omega^{\mathrm{MSW}}(\gamma)$, and restricting to $t = 0$ for simplicity, this can be rewritten as

$$\mathcal{J}_p(0) = \sum_{\mu\in\Lambda^*/\Lambda} h_{p,\mu} \left[\sum_{k\in\Lambda+\mu} M_1(k_+\sqrt{\tau_2})\, q^{-\frac{1}{2}Q(k)} \right]$$

S-duality in HM moduli space II

- The series $\sum_k M_1(k_+\sqrt{\tau_2})\, q^{-\frac{1}{2}Q(k)}$ is an Eichler integral of the Gaussian theta series $\sum_k k_+ e^{-\pi\tau_2 k_+^2}\, q^{-\frac{1}{2}Q(k)}$, therefore it transforms non-homogeneously under $SL(2,\mathbb{Z})$.
- However, since it appears in the modular completion of the Zwegers-type indefinite theta series $\sum_k\, [\mathrm{sign}(k_+) - \mathrm{sign}(k'_+)] q^{-\frac{1}{2}Q(k)}$ (where $k'_+ = k_a t'^a$ with $Q(t') = 0$), its modular anomaly is holomorphic, and therefore Ξ transforms by a holomorphic contact transformation !
- This shows that at one-instanton level, D3-instanton corrections are consistent with S-duality, provided the generating function $h_{p,\mu}$ of MSW invariants is a vector-valued modular form of fixed weight and multiplier system, as predicted by MSW.

Alexandrov Manschot BP, 2012

S-duality in HM moduli space III

- At two-instanton level, one has to deal with indefinite theta series of signature $(2, 2b_2 - 2)$, corresponding to sums over fluxes on pairs of D3-branes wrapped on a divisor $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$.
- After a lot of work, one finds that D3-instanton corrections are consistent with S-duality, provided $h_{p,\mu}$ is a vector-valued mock modular form of fixed weight, multiplier system and shadow. More precisely, $\widehat{h}_{p,\mu} \equiv h_{p,\mu} - R_{p,\mu}$ is a vector-valued modular form of weight $-\frac{b_2}{2} - 1$, where $R_{p,\mu}$ is a non-holomorphic function of τ constructed from $h_{p_1,\mu_1}, h_{p_2,\mu_2}$.

Alexandrov Banerjee Manschot BP, 2016

- Such mock modularity is known to appear in the context of rank 2 sheaves on complex surfaces, here we see it arise more generally in the context of DT invariants of pure dimension 2 sheaves on Calabi-Yau threefolds.

Conclusion I

- We have found a very simple prescription for finding the modular completion of a large class of indefinite theta series of arbitrary: replace the locally constant function Φ_r by its image under the heat kernel operator !
- This in principle allows to determine the modular properties of many interesting q-series, e.g. the generating function of Euler numbers of moduli spaces of rank $r+1$ sheaves on rational surfaces...
- This hints a theory of level-r mock modular forms, defined recursively as holomorphic q-series whose shadow is a level $r-1$ mock modular form. Do the connections to harmonic Maass forms and meromorphic Jacobi forms generalize to $r>1$?
- Re: HM multiplet moduli space, a manifestly S-duality invariant. twistorial construction would be very useful, e.g. for incorporating NS5-brane instantons.