Modularity of Donaldson-Thomas invariants on Calabi-Yau threefolds

Boris Pioline

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Main references

- S. Alexandrov, S. Feyzbakhsh, A. Klemm, BP, T. Schimannek, *Quantum geometry, stability and modularity*, Comm. Num. Theo. Phys (2024), arXiv:2301.08066
- C. Doran, BP, T. Schimannek, *Enumerative Geometry and Modularity in Two-moduli K3-fibered Calabi-Yau threefolds*, arXiv:2407.nnnn
- See also Sergey Alexandrov's talk on Thursday morning.

Introduction

- A driving force in high energy theory has been the quest for a microscopic explanation of the Bekenstein-Hawking entropy of black holes.

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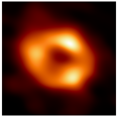

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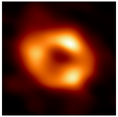

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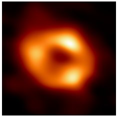

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- Besides confirming the consistency of string theory as a theory of quantum gravity, this has opened up many fruitful connections with mathematics.

BPS indices and Donaldson-Thomas invariants

- In the context of type IIA strings compactified on a Calabi-Yau three-fold X, BPS states are described mathematically by stable objects in the derived category of coherent sheaves $\mathcal{C} = D^b\mathrm{Coh}X$. The Chern character $\gamma = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2, \mathrm{ch}_3)$ is identified as the electromagnetic charge, or D6-D4-D2-D0-brane charge.

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- The problem becomes a question in Donaldson-Thomas theory: for fixed $\gamma \in K(X)$, compute the generalized DT invariant $\Omega_z(\gamma)$ counting (semi)stable objects of class γ for a Bridgeland stability condition $z \in \mathrm{Stab}\,\mathcal{C}$, and determine its growth as $|\gamma| \to \infty$.

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- Physical arguments predict that suitable generating series of rank 0 DT invariants (counting D4-D2-D0 bound states) should have specific (mock) modular properties. This gives very good control on their asymptotic growth, and allows to test whether it agrees with the BH prediction $\Omega_z(\gamma) \simeq e^{S_{BH}(\gamma)}$.

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  2. II. vertical D4-D2-D0 invariants in two-parameter K3-fibered models

     Doran BP Schimannek [arXiv:2407.nnnnn]

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Gromov-Witten invariants

- Let X be a smooth, projective CY threefold. The Gromov-Witten invariants $n_\beta^{(g)}$ count genus g curves Σ with $[\Sigma] = \beta \in H_2^{\rm eff}(X, \mathbb{Z})$. They depend only on the symplectic structure (or Kähler moduli) of X and take rational values.

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- Physically, they determine certain protected couplingsof the form $F_g(t) R^2 W^{2g-2}$ in the low energy effective action, which depend only on the complexified Kähler moduli t and receive worldsheet instanton corrections: $F_g(t) = \sum_\beta n_\beta^{(g)} e^{2\pi i t\cdot\beta}$

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- The first two F_0 and F_1 can be computed using mirror symmetry. Holomorphic anomaly equations along with boundary conditions near the discriminant locus and MUM points allow to determine $F_{g\geq 2}$ up to a certain genus $g_{\rm int}$ ($= 53$ for the quintic threefold X_5).

Bershadsky Cecotti Ooguri Vafa'93; Huang Klemm Quackenbush'06

Gopakumar-Vafa invariants

- While GW invariants take rational values, the Gopakumar-Vafa invariants $GV_\beta^{(g)}$ defined by the 'multicover' formula

$$\sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t) = \sum_{g=0}^{\infty} \sum_{k=1}^{\infty} \sum_{\beta} \frac{GV_\beta^{(g)}}{k} \left(2 \sin \frac{k\lambda}{2}\right)^{2g-2} e^{2\pi \mathrm{i} k t \cdot \beta}$$

take integer values. For $g = 0$, $n_\beta^{(0)} = \sum_{k|\beta} \frac{1}{k^3} GV_{\beta/k}^{(0)}$. Moreover, $GV_\beta^{(g)}$ vanishes for large enough $g \geq g_{\max}(\beta)$ *[Ionel Parker'13]*

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- Physically, $GV_\beta^{(0)}$ counts BPS bound states of D2-branes with charge β, and arbitrary number of D0-branes, while $GV_\beta^{(g\geq 1)}$ keep track of their angular momentum (more on this below).
- The formula above arises from a one-loop Schwinger-type computation of the effective action in a constant graviphoton background $W \propto \lambda$ [Gopakumar Vafa'98]

GV invariants and 5D black holes

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- Keeping track of $m = j_L^z$ only, the number of states is

$$\Omega_{5D}(\beta, m) = \sum_{g=0}^{g_{\max}(\beta)} \binom{2g+2}{g+1+m} GV_\beta^{(g)}$$

Amazingly, it appears that $\Omega(\beta, m) \sim e^{2\pi\sqrt{\beta^3 - m^2}}$ for large β keeping m^2/β^3 fixed, in agreement with the Bekenstein-Hawking entropy of 5D black holes ! *[Klemm Marino Tavanfar'07]*.

GV invariants and D6-brane bound states

- Instead of considering $M/X \times S^1 \times \mathbb{R}^4$, one may take $M/X \times TN \times \mathbb{R}$, where TN is a unit charge Taub-NUT space. This descends to a D6-brane on $X \times \mathbb{R}^{3,1}$.

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- D6-D2-D0 bound states of charge $(1, 0, \beta, n)$ are described mathematically by stable pairs $E : \mathcal{O}_X \xrightarrow{s} F$ where F is a pure 1-dimensional sheaf with $\mathrm{ch}_1 F = \beta$ and $\chi(F) = n$ and s has zero-dimensional kernel *[Pandharipande Thomas'07]*. The PT invariant $PT(\beta, n)$ is defined as the (weighted) Euler characteristic of the corresponding moduli space.

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- Since TN is locally flat, one expects the same low energy effective action as in flat space. This suggests a relation of the form

$$\sum_{\beta,n} PT(\beta, n) e^{2\pi \mathrm{i} t\cdot\beta} q^n \simeq \exp\left(\sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t)\right)$$

Dijkgraaf Vafa Verlinde'06

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- More precisely, PT invariants are related to GV invariants by *[Maulik Nekrasov Okounkov Pandharipande'06]*

$$\sum_{\beta,n} PT(\beta,n)\, e^{2\pi \mathrm{i} t\cdot\beta} q^n = \prod_{\beta,g,\ell} \left(1-(-q)^{g-\ell-1} e^{2\pi \mathrm{i} t\cdot\beta}\right)^{(-1)^{g+\ell}\binom{2g-2}{\ell} GV_\beta^{(g)}}$$

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- For n close to the Castelnuovo bound, one has $PT(\beta,n)=\sum_{g=1}^{g_{\max}(\beta)}\binom{2g-2}{g-1-n} GV_\beta^{(g)} + \mathcal{O}(GV^2)$, similar to (but distinct from) $\Omega_{5D}(\beta,m)=\sum_{g=0}^{g_{\max}(\beta)}\binom{2g+2}{g+1+m} GV_\beta^{(g)}$.

Generalized Donaldson-Thomas invariants

- More generally, D6-D4-D2-D0 bound states are described by stable objects in the bounded derived category of coherent sheaves $D^b\mathrm{Coh}(X)$ *[Kontsevich'95, Douglas'01]*. Objects are bounded complexes $E = (\dots \to \mathcal{E}_{-1} \to \mathcal{E}_0 \to \mathcal{E}_1 \to \dots)$ carrying charge $\gamma(E) = \sum_K (-1)^k \,\mathrm{ch}\, \mathcal{E}_k$.

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- The space of stability conditions $\mathrm{Stab}\, \mathcal{C}$ is a complex manifold of dimension $\dim K_{\mathrm{num}}(X) = 2b_2(X) + 2$, *unless it is empty*.
- For X a projective CY3, stability conditions are only known to exist for the quintic threefold X_5 and a couple of other examples *[Li'18, Koseki'20, Liu'21]*

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- $\bar{\Omega}_\sigma(\gamma)$ is roughly the weighted Euler number of the moduli space of semi-stable objects $M_\sigma(\gamma)$, where semi-stability means that $\arg Z(E') \leq \arg Z(E)$ for any subobject $E' \subset E$.

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- For $\gamma = (0, 0, \beta, n)$ and $\gamma = (1, 0, \beta, n)$, $\Omega_\sigma(\gamma)$ coincides with $GV_\beta^{(0)}$ and $PT(\beta, n)$ *or* $DT(\beta, n)$ at large volume, respectively.

D4-D2-D0 indices as rank 0 DT invariants

- The main interest in this talk will be on rank 0 DT invariants $\Omega(0, p, \beta, n)$ counting D4-D2-D0 brane bound states supported on a divisor $\mathcal{D}$ with class $[\mathcal{D}] = p \in H_4(X, \mathbb{Z})$.

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- Viewing IIA=M/S^1, they arise from M5-branes wrapped on $\mathcal{D} \times S^1$. In the limit where S^1 is much larger than X, they are described by a two-dimensional superconformal field theory with $(0, 4)$ SUSY.

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- D4-D2-D0 indices (in suitable chamber) occur as Fourier coefficients in the elliptic genus:

$$\mathrm{Tr}(-1)^{2J_3} q^{L_0 - \frac{c_L}{24}} \bar{q}^{\bar{L}_0 - \frac{c_R}{24}} e^{2\pi \mathrm{i} q_a z^a} = \sum_{\mu \in \Lambda/\Lambda^*} h_{p,\mu}(\tau) \Theta_\mu(\tau, \bar{\tau}, z)$$

$$h_{p^a,\mu_a}(\tau) := \sum_n \Omega(0, p^a, \mu_a, n)\, q^{n + \frac{1}{2}\mu_a \kappa^{ab} \mu_b - \frac{1}{2} p^a \mu_a - \frac{\chi(\mathcal{D})}{24}}$$

and $\Lambda = H_4(X, \mathbb{Z})$ equipped with the quadratic form $\kappa_{abc} p^c$.

Modularity of rank 0 DT invariants

- When $\mathcal{D}$ is very ample, there are no walls extending to large volume, so the choice of chamber is moot. The central charges are given by *[Maldacena Strominger Witten'97]*

$$\begin{cases} c_L = & p^3 + c_2(TX) \cdot p = \chi(\mathcal{D}) \,, \\ c_R = & p^3 + \frac{1}{2} c_2(TX) \cdot p = 6\chi(\mathcal{O}_{\mathcal{D}}) \end{cases}$$

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  Cardy's formula predicts a growth $\Omega(0,p,\beta,n\to\infty)\sim e^{2\pi\sqrt{p^3\,n}}$ in perfect agreement with Bekenstein-Hawking formula

- The generating series $h_{p^a,\mu_a}(\tau)$ should be a vector-valued, weakly holomorphic modular form of weight $w=-\frac12 b_2(X)-1$ in the Weil representation of the lattice Λ. It is then completely determined by its polar coefficients, with $n+\frac12\mu_a\kappa^{ab}\mu_b-\frac12 p^a\mu_a<\frac{\chi(\mathcal{D})}{24}$.

Mock modularity of rank 0 DT invariants

- When $\mathcal{D}$ is reducible, the generating series $h_{p^a,\mu_a}(\tau)$ of DT invariants $\Omega_*(0,p,\beta,n)$ in a suitable chamber is expected to be a vector-valued mock modular form of higher depth (see S. Alexandrov's talk and *[Alexandrov BP Manschot'16-20]*)

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- When X is K3-fibered, modularity is known to hold for vertical D4-brane charge, using the relation to Noether-Lefschetz invariants (more on this in part II). In that case, no modular anomaly due to $\kappa_{ab}p^b = 0$. *[Bouchard Creutzig Diaconescu Doran Quigley Sheshmani'16]*

I. Testing modularity for one-parameter models

- Our first aim is to test this prediction for CY threefolds with Picard rank 1, by computing the first few coefficients in the q-expansion and determine the putative vector-valued modular form.

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- This was first attempted by *[Gaiotto Strominger Yin '06-07]* for the quintic threefold X_5 and a few other hypergeometric models. They were able to guess the first few terms for unit D4-brane charge, and find a unique modular completion.
- We shall compute many terms rigorously, using recent results by *[Soheyla Fezbakhsh and Richard Thomas'20-22]* relating rank r DT invariants (including $r = 0$, counting D4-D2-D0 bound states) to rank 1 DT invariants, hence to GV invariants.

Alexandrov, Feyzbakhsh, Klemm, BP, Schimannek'23

From rank 1 to rank 0 DT invariants

- The key idea is to use wall-crossing in a family of weak stability conditions (aka tilt-stability) parametrized by $b + \mathrm{i}t \in \mathbb{H}$, with central charge[1]

$$Z_{b,t}(E) = \frac{\mathrm{i}}{6} t^3 \,\mathrm{ch}_0^b(E) - \frac{1}{2} t^2 \,\mathrm{ch}_1^b(E) - \mathrm{i}t \,\mathrm{ch}_2^b(E) + 0 \,\mathrm{ch}_3^b(E)$$

  with $\mathrm{ch}_k^b(E) = \int_X H^{3-k} e^{-bH} \,\mathrm{ch}(E)$. The heart $\mathcal{A}_b$ is generated by length-two complexes $\mathcal{E}_{-1} \to \mathcal{F}_0$ with $(\mathcal{E}, \mathcal{F})$ slope semi-stable sheaves with $\mathrm{ch}_1^b(\mathcal{E}) > 0, \mathrm{ch}_1^b(\mathcal{F}) \leq 0$.

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- The KS/JS wall-crossing formulae hold for this family of weak stability conditions. In fact, tilt-stability provides the first step in constructing genuine stability conditions near the large volume point *[Bayer Macri Toda'11]*

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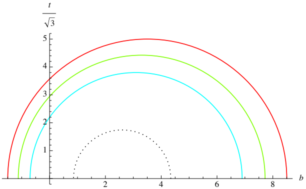
Rank 0 DT invariants from GV invariants

- Tilt stability agrees with Gieseker stability at large volume, but the chamber structure is much simpler: walls are nested half-circles in the Poincaré upper half-plane spanned by $z = b + \mathrm{i}\frac{t}{\sqrt{3}}$.



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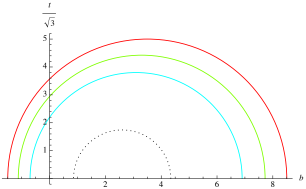

- Importantly, for any $\nu_{b,w}$-semistable object E there is a conjectural inequality on Chern classes $C_i := \int_X \mathrm{ch}_i(E).H^{3-i}$ *[Bayer Macri Toda'11; Bayer Macri Stellari'16]*

$$(C_1^2 - 2C_0C_2)(\tfrac{1}{2}b^2 + \tfrac{1}{6}t^2) + (3C_0C_3 - C_1C_2)b + (2C_2^2 - 3C_1C_3) \geq 0$$

Rank 0 DT invariants from GV invariants

- By studying wall-crossing between the empty chamber provided by BMT bound and large volume, *[Feyzbakhsh Thomas'20-22]* show that D4-D2-D0 indices can be computed from PT invariants, which are in turn related to GV invariants.

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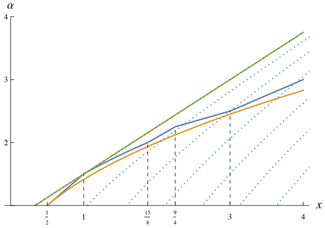
- Let (X, H) be a smooth polarised CY threefold with $\mathrm{Pic}(X) = \mathbb{Z}.H$ satisfying the BMT conjecture. Aim: compute $PT(\beta, m)$ inductively.
- Fix $m \in \mathbb{Z}, \beta \in H_2(X, \mathbb{Z})$ and define $x = \frac{\beta.H}{H^3}, \quad \alpha = -\frac{3m}{2\beta.H}$

$$f(x) := \begin{cases} x + \frac{1}{2} & \text{if } 0 < x < 1 \\ \sqrt{2x + \frac{1}{4}} & \text{if } 1 < x < \frac{15}{8} \\ \frac{2}{3}x + \frac{3}{4} & \text{if } \frac{15}{8} \leq x < \frac{9}{4} \\ \frac{1}{3}x + \frac{3}{2} & \text{if } \frac{9}{4} \leq x < 3 \\ \frac{1}{2}x + 1 & \text{if } 3 \leq x \end{cases}$$



A new explicit formula (S. Feyzbakhsh'23)

Theorem (wall-crossing for $\gamma = (-1, 0, \beta, -m)$):

- If $\alpha > f(x)$ then the stable pair invariant $PT(\beta, m)$ equals

$$\sum_{(m',\beta')} (-1)^{\chi_{m',\beta'}} \chi_{m',\beta'} PT(\beta', m')\, \Omega\left(0, 1, \frac{H^2}{2} - \beta' + \beta\,, \frac{H^3}{6} + m' - m - \beta'.H\right)$$

where $\chi_{m',\beta'} = \beta.H + \beta'.H + m - m' - \frac{H^3}{6} - \frac{1}{12} c_2(X).H$.

- The sum runs over $(\beta', m') \in H_2(X, \mathbb{Z}) \oplus H_0(X, \mathbb{Z})$ such that

$$0 \leq \beta'.H \leq \frac{H^3}{2} + \frac{3mH^3}{2\beta.H} + \beta.H$$

$$-\frac{(\beta'.H)^2}{2H^3} - \frac{\beta'.H}{2} \leq m' \leq \frac{(\beta.H - \beta'.H)^2}{2H^3} + \frac{\beta.H + \beta'.H}{2} + m$$

In particular, $\beta'.H < \beta.H$.

Corollary (Castelnuovo bound): $PT(\beta, m) = 0$ unless $m \geq -\frac{(\beta.H)^2}{2H^3} - \frac{\beta.H}{2}$

Modularity for one-modulus compact CY

- Using the theorem above and known GV invariants, we could compute a large number of coefficients in the generating series of Abelian (=unit D4-brane charge) rank 0 DT invariants in one-parameter hypergeometric threefolds, including the quintic X_5.

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- Using the theorem above and known GV invariants, we could compute a large number of coefficients in the generating series of Abelian (=unit D4-brane charge) rank 0 DT invariants in one-parameter hypergeometric threefolds, including the quintic X_5.
- In all cases (except $X_{3,2,2}, X_{2,2,2,2}$ where current knowledge of GV invariants is insufficient), we could find a linear combination of the following vv modular forms matching all computed coeffs:

$$\frac{E_4^a E_6^b}{\eta^{4\kappa+c_2}} D^\ell(\vartheta_\mu^{(\kappa)}) \quad \text{with} \quad \vartheta_\mu^{(\kappa)} = \sum_{k \in \mathbb{Z}+\frac{\mu}{\kappa}+\frac{1}{2}} q^{\frac{1}{2}\kappa k^2}, \quad \kappa = H^3$$

where $D = q\partial_q - \frac{w}{12}E_2$, and $4a + 6b + 2\ell - 2\kappa - \frac{1}{2}c_2 = -2$.

Modularity for one-modulus compact CY

X	χ_X	κ	$c_2(TX)$	$\chi(\mathcal{O}_\mathcal{D})$	n_1	C_1
$X_5(1^5)$	-200	5	50	5	7	0
$X_6(1^4,2)$	-204	3	42	4	4	0
$X_8(1^4,4)$	-296	2	44	4	4	0
$X_{10}(1^3,2,5)$	-288	1	34	3	2	0
$X_{4,3}(1^5,2)$	-156	6	48	5	9	0
$X_{4,4}(1^4,2^2)$	-144	4	40	4	6	1
$X_{6,2}(1^5,3)$	-256	4	52	5	7	0
$X_{6,4}(1^3,2^2,3)$	-156	2	32	3	3	0
$X_{6,6}(1^2,2^2,3^2)$	-120	1	22	2	1	0
$X_{3,3}(1^6)$	-144	9	54	6	14	1
$X_{4,2}(1^6)$	-176	8	56	6	15	1
$X_{3,2,2}(1^7)$	-144	12	60	7	21	1
$X_{2,2,2,2}(1^8)$	-128	16	64	8	33	3

Modular predictions for the quintic threefold

- Using known $GV_\beta^{(g\leq 53)}$ we can compute more than 20 terms:

$$h_0 = q^{-\frac{55}{24}}\Big(\underline{5 - 800q + 58500q^2} + 5817125q^3 + 75474060100q^4 + 28096675153255q^5 + 3756542229485475q^6 + 277591744202815875q^7 + 13610985014709888750q^8 + \ldots\Big),$$

$$h_{\pm 1} = q^{-\frac{55}{24}+\frac{3}{5}}\Big(\underline{0 + 8625q} - 1138500q^2 + 3777474000q^3 + 3102750380125q^4 + 577727215123000q^5 + \ldots\Big)$$

$$h_{\pm 2} = q^{-\frac{55}{24}+\frac{2}{5}}\Big(\underline{0 + 0q} - 1218500q^2 + 441969250q^3 + 953712511250q^4 + 217571250023750q^5 + 22258695264509625q^6 + \ldots\Big)$$

Modular predictions for the quintic threefold

- The space of vv modular forms has dimension 7. Remarkably, all terms above are reproduced by *[Gaiotto Strominger Yin'06]*

$$h_\mu = \frac{1}{\eta^{70}} \Big[-\frac{222887 E_4^8 + 1093010 E_4^5 E_6^2 + 177095 E_4^2 E_6^4}{35831808} + \frac{25\left(458287 E_4^6 E_6 + 967810 E_4^3 E_6^3 + 66895 E_6^5\right)}{53747712} D + \frac{25\left(155587 E_4^7 + 1054810 E_4^4 E_6^2 + 282595 E_4 E_6^4\right)}{8957952} D^2 \Big] \vartheta_\mu^{(5)},$$

Assuming this we can in principle compute all $GV_\beta^{(g\leq 69)}$!

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- For X_{10} , Gaiotto et al predicted

$$h_{1,0} \overset{?}{=} q^{-\frac{35}{24}} \left(\underline{3 - 576q} + 271704q^2 + 206401533q^3 + \cdots \right)$$

  whereas the correct result turns out to be

$$h_{1,0} \overset{!}{=} \frac{203 E_4^4 + 445 E_4 E_6^2}{216\, \eta^{35}} = q^{-\frac{35}{24}} \left(\underline{3 - 575q} + 271955q^2 + \cdots \right)$$

II. Modularity for two-parameter K3-fibered models

- A natural next step is to consider two-parameter CY threefolds. We restrict attention to K3-fibered models X with $h_{1,1}(X) = 2$, whose mirror Y is also K3-fibered. *[Doran BP Schimannek, to appear]*

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- On the A-model side, X is fibered by Picard rank 1 K3-surfaces (Σ_m, L), polarized by a degree $2m$ line bundle L. On the B-model side, Y is fibered by Picard rank 19 K3-surfaces $\hat{\Sigma}_m$, polarized by the lattice $M_m = U \oplus E_8 \oplus E_8 \oplus \langle -2m \rangle$. The fibers $(\Sigma_m, \hat{\Sigma}_m)$ are related by Dolgachev-Nikulin mirror symmetry.

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- The moduli space of M_m-polarized K3 surfaces is the modular curve $X_0(m)^+ = \mathbb{H}/\Gamma_0(m)^+$. The fundamental period of $\hat{\Sigma}_m$, holomorphic at $\lambda = \infty$, is a weight 2 modular form *[Lian Yau'95]*

$$f_m(\lambda) = \sum_{d \geq 0} c_m(d) \lambda^{-d} = E_2^{(m)}(\tau) \, , \quad \lambda = J_m^+(\tau)$$

Structure of the modular curve $X_0(m)^+$

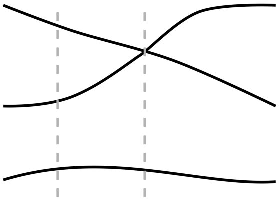
m	Orbifold Type	$\lambda_1, \dots, \lambda_r$	$\tau_1, \dots, \tau_r$
1	$(3; 2; \infty)$	1728	i
2	$(4; 2; \infty)$	256	$\frac{\mathrm{i}}{\sqrt{2}}$
3	$(6; 2; \infty)$	108	$\frac{\mathrm{i}}{\sqrt{3}}$
4	$(\infty; 2; \infty)$	64	$\frac{\mathrm{i}}{2}$
5	$(2; 2, 2; \infty)$	$22 + 10\sqrt{5}, 22 - 10\sqrt{5}$	$\frac{\mathrm{i}}{\sqrt{5}}, \frac{4}{9} + \frac{\mathrm{i}}{9\sqrt{5}}$
6	$(\infty; 2, 2; \infty)$	$17 + 12\sqrt{2}, 17 - 12\sqrt{2}$	$\frac{\mathrm{i}}{\sqrt{6}}, \frac{2}{5} + \frac{\mathrm{i}}{5\sqrt{6}}$
7	$(3; 2, 2; \infty)$	$27, -1$	$\frac{\mathrm{i}}{\sqrt{7}}, \frac{1}{2} + \frac{\mathrm{i}}{2\sqrt{7}}$
8	$(\infty; 2, 2; \infty)$	$12 + 8\sqrt{2}, 12 - 8\sqrt{2}$	$\frac{\mathrm{i}}{\sqrt{8}}, -\frac{2}{11} + \frac{\mathrm{i}}{22\sqrt{2}}$
9	$(\infty; 2, 2; \infty)$	$9 + 6\sqrt{3}, 9 - 6\sqrt{3}$	$\frac{\mathrm{i}}{9}, \frac{1}{2} + \frac{\mathrm{i}}{6}$
10	$(\infty; 2, 2, 2; \infty)$	$9 + 4\sqrt{5}, 1, 9 - 4\sqrt{5}$	$\frac{\mathrm{i}}{\sqrt{10}}, \frac{1}{5}, \frac{4}{7} + \mathrm{i}\frac{\sqrt{10}}{70}$
11	$(2; 2, 2, 2; \infty)$	Roots of $\lambda^3 - 20\lambda^2 + 56\lambda - 44$	$\frac{\mathrm{i}}{\sqrt{11}}, \frac{2}{3} + \mathrm{i}\frac{\sqrt{11}}{33}, \frac{22}{25} + \frac{\mathrm{i}\sqrt{11}}{275}$

$X_0(m)^+$ has a cusp at $\lambda = \infty$, $\mathbb{Z}_2$-orbifold points at $\lambda_1, \dots, \lambda_r$ and a cusp or $\mathbb{Z}_a$-orbifold point at $\lambda = 0$, with $a \in \{2, 3, 4, 6, \infty\}$. J_m^+ maps $\tau_1, \dots \tau_r, \mathrm{i}\infty$ to $\lambda_1, \dots, \lambda_r, \infty$.

Generalized invariant map

- The fibration $\hat{\Sigma} \to Y \to \mathbb{P}^1$ is determined by the generalized invariant map $\Lambda : \mathbb{P}^1 \to X_0(m)^+$, a branched cover over $\mathbb{P}^1$. We assume that the cover is unramified over the $\mathbb{Z}_2$-orbifold points λ_r.

m	$[y_1, y_2]$
1	[1, 1], [1, 2], [2, 2]
2	[1, 1], [1, 2], [1, 4], [2, 2], [2, 4], [4, 4]
3	[1, 1], [1, 2], [1, 3], [2, 2], [2, 3], [3, 3]
4, 5	[1, 1], [1, 2], [2, 2]
6, 8, 9, 11	[1, 1]



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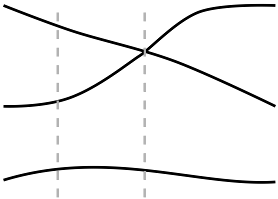
- Possible ramification profiles over $\lambda = 0$ leading to a smooth CY3 were classified by *[Doran Harder Novoseltsev Thompson'17]*.

m	$[y_1, y_2]$
1	$[1,1], [1,2], [2,2]$
2	$[1,1], [1,2], [1,4], [2,2], [2,4], [4,4]$
3	$[1,1], [1,2], [1,3], [2,2], [2,3], [3,3]$
$4, 5$	$[1,1], [1,2], [2,2]$
$6, 8, 9, 11$	$[1,1]$



Fundamental period

- Restricting to CY3 with $h_{1,2}(Y) = 2$, we find that the ramification profile over $\lambda = \infty$ must also be of the form $[i - s, j + s]$ with $0 \leq s \leq j - s$, with two 'excess ramification points' away from $\mathbb{Z}_2$-orbifold points. The covering $\Lambda : \mathbb{P}^1_y \to \mathbb{P}^1_\lambda$ is then given by

$$\lambda(y) = y^{-s}(1 - y)^i (1 - z_2/y)^j / z_1$$

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- The fundamental period of Y follows by integrating the fundamental period of $\hat{\Sigma}_m$ along a contour on $\mathbb{P}^1_y$,

$$\varpi(v, w) = \oint \frac{f_m(\lambda(y))\mathrm{d}y}{y(1-y)(1-v/y)} = \sum_{k,d\geq 0} c_m(d) \frac{(k+id-is)!\,(k+jd)!}{(id)!\,(jd)!k!(k-sd)!} z_1^d z_2^k$$

This allows to extract the Picard-Fuchs ideal, and obtain the basis of periods around the MUM point $z_1, z_2 \to 0$.

Mirror symmetry and Tyurin degenerations

- The mirror $X = X_{m,s}^{[i,j]}$ can be guessed from the explicit form of the period, e.g. for $(m,i,j)=(1,1,1)$

$$c_1(d) = \frac{(6d)!}{(d!)^3(3d)!} \Rightarrow \mathbb{P}\begin{pmatrix} i & j & 1 & 1 & 1 & 3 & -s & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}\begin{bmatrix} 6 & i-s & j \\ 0 & 1 & 1 \end{bmatrix}$$

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- As argued in *[Doran-Harder-Thompson'17]*, the K3-fibration on the B-model side is reflected by the existence of a Tyurin degeneration on the A-model side, where X splits into a union of two Fano threefolds $F_m^{[i]} \cup F_m^{[j]}$, each of Picard rank 1, intersecting over an anticanonical K3 surface Σ_m.

Mirror symmetry and Tyurin degenerations

- The mirror $X = X^{[i,j]}_{m,s}$ can be guessed from the explicit form of the period, e.g. for $(m,i,j)=(1,1,1)$

$$c_1(d) = \frac{(6d)!}{(d!)^3(3d)!} \Rightarrow \mathbb{P}\begin{pmatrix} i & j & 1 & 1 & 1 & 3 & -s & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}\begin{bmatrix} 6 & i-s & j \\ 0 & 1 & 1 \end{bmatrix}$$

- More generally, $X^{[i,j]}_{m,s}$ can be constructed as a complete intersection in a projective bundle over a Fano 4-fold $V^{[i,j]}_m$.
- As argued in [Doran-Harder-Thompson'17], the K3-fibration on the B-model side is reflected by the existence of a Tyurin degeneration on the A-model side, where X splits into a union of two Fano threefolds $F^{[i]}_m \cup F^{[j]}_m$, each of Picard rank 1, intersecting over an anticanonical K3 surface Σ_m.
- The K3-fibration on the A-model side requires the existence of a Tyurin degeneration on the B-model side, which requires $s=0$.

A family of Picard rank 2 K3-fibered threefolds $X_m^{[i,j]}$

$$
\begin{aligned}
h_{1,1} &= 2 \\
h_{1,2} &= 22 + m(i^2 + j^2) - 2mij \\
&\quad + h_{1,2}(F_m^{[i]}) + h_{1,2}(F_m^{[j]}) \\
\kappa_{111} &= 2m\left(\frac{1}{i} + \frac{1}{j}\right), \kappa_{112} = 2m, \\
\kappa_{122} &= \kappa_{222} = 0 \\
c_{2,1} &= 2m(i+j) + 24\left(\frac{1}{i} + \frac{1}{j}\right) \\
c_{2,2} &= 24 \\
\mathrm{GV}_{0,1}^{(0)} &= 2mij\,, \quad \mathrm{GV}_{0,k>0}^{(0)} = 0\,.
\end{aligned}
$$

(m,i)	$h_{1,2}(F_m^{[i]})$	Construction of $F_m^{[i]}$
(1,1)	52	$\mathbb{P}_{1,1,1,1,3}[6]$
(1,2)	21	$\mathbb{P}_{1,1,1,2,3}[6]$
(2,1)	30	$\mathbb{P}^4[4]$
(2,2)	10	$\mathbb{P}_{1,1,1,1,2}[4]$
(2,4)	0	$\mathbb{P}^3$
(3,1)	20	$\mathbb{P}^5[2,3]$
(3,2)	5	$\mathbb{P}^4[3]$
(3,3)	0	$\mathbb{P}^4[2]$
(4,1)	14	$\mathbb{P}^6[2,2,2]$
(4,2)	2	$\mathbb{P}^5[2,2]$
(5,1)	10	$X^{2,5}_{\mathcal{O}(1)^{\oplus 2}\oplus\mathcal{O}(2)}$
(5,2)	0	$B_5 = X^{2,5}_{\mathcal{O}(1)^{\oplus 3}}$
(6,1)	7	$X^{2,5}_{\mathcal{S}(1)^\vee\oplus\mathcal{O}(1)}$
(7,1)	5	$X^{2,6}_{\mathcal{O}(1)^{\oplus 5}}$
(8,1)	3	$X^{3,6}_{\bigwedge^2\mathcal{S}^\vee\oplus\mathcal{O}(1)^{\oplus 3}}$
(9,1)	2	$X^{2,7}_{\mathcal{Q}^\vee(1)\oplus\mathcal{O}(1)^{\oplus 2}}$
(11,1)	0	$A_{22} = X^{3,7}_{(\bigwedge^2\mathcal{S}^\vee)^{\oplus 3}}$

Alternative realizations

(m,i,j)	χ_X		CICY	Transition
$(1,1,1)$	-252	$\mathbb{P}^4_{1,1,2,2,6}[12]$		$X_{6,2}$
$(2,1,1)$	-168	$\mathbb{P}^4_{1,1,2,2,2}[8] = \left(\begin{array}{c\|cc} \mathbb{P}^4 & 4 & 1 \\ \mathbb{P}^1 & 0 & 2 \end{array}\right)$	7886, 7888	$X_{4,2}$
$(2,4,1)$	-168	$\left(\begin{array}{c\|cc} \mathbb{P}^4 & 4 & 1 \\ \mathbb{P}^1 & 1 & 1 \end{array}\right)$	7885	X_5
$(2,4,4)$	-168	$\left(\begin{array}{c\|c} \mathbb{P}^3 & 4 \\ \mathbb{P}^1 & 2 \end{array}\right)$	7887	X_8
$(3,1,1)$	-132	$\left(\begin{array}{c\|cccc} \mathbb{P}^6 & 3 & 2 & 1 & 1 \\ \mathbb{P}^1 & 0 & 0 & 1 & 1 \end{array}\right)$	7867, 7869	$X_{3,2,2}$
$(3,2,1)$	-120	$\left(\begin{array}{c\|ccc} \mathbb{P}^5 & 2 & 3 & 1 \\ \mathbb{P}^1 & 1 & 0 & 1 \end{array}\right)$	7840	$X_{3,3}$
$(3,2,2)$	-108	$\left(\begin{array}{c\|cc} \mathbb{P}^4 & 3 & 2 \\ \mathbb{P}^1 & 0 & 2 \end{array}\right)$	7806	$X_{4,3}$
$(3,3,1)$	-140	$\left(\begin{array}{c\|ccc} \mathbb{P}^5 & 2 & 3 & 1 \\ \mathbb{P}^1 & 0 & 1 & 1 \end{array}\right)$	7873	$X_{4,2}$
$(3,3,2)$	-128	$\left(\begin{array}{c\|cc} \mathbb{P}^4 & 3 & 2 \\ \mathbb{P}^1 & 1 & 1 \end{array}\right)$	7858	X_5
$(3,3,3)$	-148	$\left(\begin{array}{c\|cc} \mathbb{P}^4 & 3 & 2 \\ \mathbb{P}^1 & 2 & 0 \end{array}\right)$	7882	$X_{6,2}$
$(4,1,1)$	-112	$\left(\begin{array}{c\|cccc} \mathbb{P}^6 & 2 & 2 & 2 & 1 \\ \mathbb{P}^1 & 0 & 0 & 0 & 2 \end{array}\right)$	7819, 7823	$X_{2,2,2,2}$
$(4,2,1)$	-112	$\left(\begin{array}{c\|cccc} \mathbb{P}^6 & 2 & 2 & 2 & 1 \\ \mathbb{P}^1 & 0 & 0 & 1 & 1 \end{array}\right)$	7817	$X_{3,2,2}$
$(4,2,2)$	-112	$\left(\begin{array}{c\|ccc} \mathbb{P}^5 & 2 & 2 & 2 \\ \mathbb{P}^1 & 0 & 1 & 1 \end{array}\right)$	7816, 7822	$X_{4,2}$

NL invariants for Picard rank one K3 fibrations

- Let $\Sigma \to X \xrightarrow{\pi} B$ be a CY3 fibered by polarized K3 surfaces (Σ, L) of degree $\int_\Sigma L^2 = 2m$. The moduli space of (Σ, L) is

$$\mathcal{M}_m = O(2) \times O(19) \backslash O(2,19) / O(\Gamma_m)$$

where $\Gamma_m = L^\perp = \langle -2m \rangle \oplus H \oplus E_8(-1) \oplus E_8(-1)$.

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- For any $h, \mu \geq 0$, let $D_{h,d} \subset \mathcal{M}_m$ be the divisor supported on the locus where

$$\exists \beta \in \mathrm{Pic}(\Sigma) : \int_\Sigma \beta^2 = 2h - 2, \int_\Sigma \beta \cdot B = \mu.$$

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- The Noether-Lefschetz number $\mathrm{NL}_{h,\mu} := \int_B \iota_\pi [D_{h,\mu}]$ vanishes unless $h \leq \frac{\mu^2}{4m} + 1$, and is invariant under spectral (semi-)flow

$$(h, \mu) \mapsto (h + k\mu + mk^2, \mu + 2km), \; k \geq 0$$

GV/NL relation for K3-fibered CY threefolds

- The vertical GV invariants are related to NL numbers via

$$\mathrm{GV}^{(g)}_{0,d} = \sum_{h\geq g} r_{g,h}\, \mathrm{NL}_{h,d}$$

where $r_{g,h}$ are the reduced GW invariants of K3, given by

$$\sum_{h,g} r_{g,h} \left(2 - y - y^{-1}\right)^g q^h = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{20}(1-yq^n)^2(1-q^n/y)^2} \,.$$

Katz Klemm Vafa'99, Maulik Pandharipande'07

Reduced GW invariants of K3

$g\backslash h$	0	1	2	3	4	5	6	7
0	1	24	324	3200	25650	176256	1073720	5930496
1	0	-2	-54	-800	-8550	-73440	-536860	-3459456
2	0	0	3	88	1401	15960	145214	1118880
3	0	0	0	-4	-126	-2136	-25750	-246720
4	0	0	0	0	5	168	3017	38328
5	0	0	0	0	0	-6	-214	-4056
6	0	0	0	0	0	0	7	264
7	0	0	0	0	0	0	0	-8

Modularity of NL invariants

- The generating series

$$\Phi_\mu(\tau) := \frac{1}{\eta^{24}} \sum_{h \leq \frac{\mu^2}{4m}+1} \mathrm{NL}_{h,\mu}\, q^{\frac{\mu^2}{4m}+1-h} , \quad \mu \in \mathbb{Z}/(2m\mathbb{Z})$$

is known to transform as a vv modular form of weight $-3/2$ under the Weil representation of $\mathbb{Z}[2m]$ *[Kudla Millson'90, Borcherds'99]*.

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- Physically, the GV/NL correspondance follows from Heterotic-type II duality: Φ_d is the new supersymmetric index counting perturbative (Dabholkar-Harvey) BPS states along T^2.

Heterotic-type II duality and Borcherds lift

- At one-loop on heterotic side, these states contribute to the prepotential $F(S,T) = -mST^2 + W(T) + \mathcal{O}(e^{-S})$ via

$$\partial_T^5 W = \int_{\mathcal{F}} \sum_{\mu \in \mathbb{Z}/(2m\mathbb{Z})} \Phi_\mu \, Z_\mu(\tau, T) = \sum_{d \geq 1} d^5 \, GV_{0,d}^{(0)} \, \mathrm{Li}_{-2}\left(e^{2\pi i d T}\right)$$

where $\mathrm{Li}_{-2}(x) = \frac{x(x+1)}{(x-1)^3}$ *[Antoniadis Gava Narain Taylor'95, Marino Moore'98,Enoki Watari'19]*

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- $\partial_T^5 W$ is a meromorphic modular form of weight 6 under $\Gamma_0(m)^+$, with poles at orbifold points in $\mathbb{H}/\Gamma_0(m)^+$. $W(T)$ itself transforms as a meromorphic mock modular form of weight -4.

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- The map $\Phi_\mu(\tau) \to W(T)$ generalizes the standard correspondence between (skew) Jacobi forms of index m, weight w and modular forms of weight $2w-2$ under $\Gamma_0(m)$. *[Shintani, Borcherds,Skoruppa-Zagier]*

GV/NL relation for K3-fibered CY threefolds

- The dimension of the relevant space of vv modular forms is
  [Bruinier'02, Maulik Pandharipande'07]

m	1	2	3	4	5	6	7	8	9	10	11	12
#pol	2	3	4	5	6	7	8	9	10	11	12	13
dim	2	3	4	4	6	7	7	8	9	10	11	12

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- For $m \leq 4$, an overcomplete basis is again given by

$$\frac{E_4^a E_6^b}{\eta^{24}} D^\ell(\vartheta_\mu^{(2m)}) \quad \text{with} \quad \vartheta_\mu^{(2m)} = \sum_{k \in \mathbb{Z} + \frac{\mu}{2m}} q^{mk^2}$$

with $4a + 6b + 2\ell = 10$. For $m \geq 5$, additional generators can be obtained via suitable Hecke-type operators.

Two-parameter K3-fibered models

- In the large base limit $z_2 \to \infty$, the Yukawa couplings (integrated once with respect to T) are given for all models by

$$\partial_T^2 W = -2m\left[\frac{1}{i}+\frac{1}{j}-2r\right]\log J_m^+(T) - 4m\sum_{k=1}^{r}\log(J_m^+(T)-\lambda_k) - 6m\left(\frac{\tilde{f}_{m,i}(J_m^+)+\tilde{f}_{m,j}(J_m^+)}{f_m(J_m^+)}\right)$$

where $\tilde{f}_{m,i}(\lambda)$ is a variant of the Lian-Yau series $\sum_d c_m(d)/\lambda^d$,

$$\tilde{f}_{m,i}(\lambda) := \sum_{d\geq 1} c_m(d) H_{id}\,\lambda^{-d}, \quad H_n := \sum_{k=1}^{n} 1/k$$

Inverting the Shintani lift, one obtains the NL generating series Φ.

Example: $X_1^{[1,1]} = \mathbb{P}_{1,1,2,2,6}[12]$

$$
\begin{aligned}
\chi_X &= -252, \quad \kappa = (4,2,0,0), \quad c_2 = (52,24) \\
J_1^+ &= J = \frac{E_4^3}{\eta^{24}}, \quad c_1(d) = \frac{(6d)!}{(d!)^3(3d)!}, \quad f_1 = \sqrt{E_4} \\
\partial_T^2 W &= -4\log(J-1728) - 6\,\frac{\tilde{f}_1(J)}{\sqrt{E_4}} \\
\partial_T^5 W &= \frac{2E_4^6}{E_6^3} - \frac{23}{9}\frac{E_4^3}{E_6} + \frac{5}{9}E_6 \\
&= 2496q + 7170048q^2 + 9388935936q^3 + \dots \\
\Phi &= \frac{-\frac{5}{3}E_4E_6\vartheta^{(2)} + 8E_4^2 D\vartheta^{(2)}}{\eta^{24}} = -\frac{2}{q} + 252 + 2496q^{1/4} + \dots + \dots
\end{aligned}
$$

Note the rapid growth of Fourier coefficients in $\partial^5 W$, due to pole at $\tau = \mathrm{i}$. *[Kaplunovsky Louis Theisen'95, Antoniadis Gava Narain Taylor'95]*

DT/NL relation on K3-fibered threefolds

- Under a monodromy $T \mapsto -1/(mT)$ in Kähler moduli space, vertical D2-D0 bound states turn into vertical D4-D2-D0 bound states. Under Heterotic/type II duality, these again map to perturbative DH states.

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- For vertical D4-brane charge $p^a = (r, 0)$, the generating series $h_{p,q}(\tau)$ coincides with the NL generating series Φ_d, acted upon by a suitable Hecke operator H_r !

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- In particular, mock modularity does not arise in this case, due to p^a being in the kernel of the quadratic form $\kappa_{ab} = \kappa_{abc} p^c$.
- For non-vertical D4-brane charge $p^a = (r, s)$ with $s > 0$, we expect a vector-valued (mock) modular form of weight -2.

Some open questions

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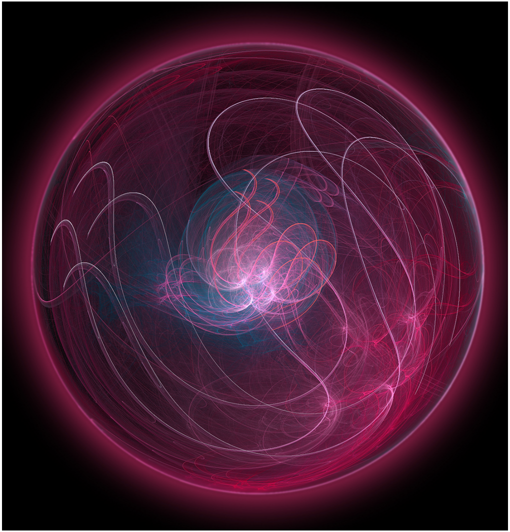
- Higher rank DT invariants can also be computed in terms of GV invariants. Do they define some higher rank version of topological string theory ?

Thanks for your attention !



Back up: a remark on the BMT inequality

$$(C_1^2 - 2C_0C_2)(\tfrac{1}{2}b^2 + \tfrac{1}{6}t^2) + (3C_0C_3 - C_1C_2)b + (2C_2^2 - 3C_1C_3) \geq 0$$

- Requiring the existence of empty chamber, the discriminant at $t = 0$ must be positive:

$$8C_0C_2^3 + 6C_1^3C_3 + 9C_0^2C_3^2 - 3C_1^2C_2^2 - 18C_0C_1C_2C_3 \geq 0$$

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- In terms of the electric and magnetic charges

$$p^0 = C_0/\kappa, \quad p^1 = C_1/\kappa, \quad q_1 = -C_2 - \frac{c_2}{24\kappa}C_0, \quad q_0 = C_3 + \frac{c_2}{24\kappa}C_1$$

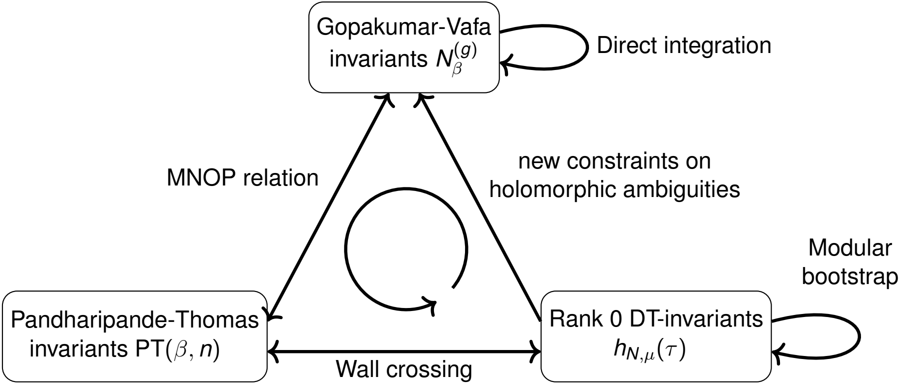
and ignoring the c_2-dependent terms, this becomes

$$\tfrac{8}{9\kappa}p^0q_1^3 - \tfrac{2}{3}\kappa q_0(p^1)^3 - (p^0q_0)^2 + \tfrac{1}{3}(p^1q_1)^2 - 2p^0p^1q_0q_1 \leq 0$$

hence an empty chamber arises when single centered black hole solutions are ruled out !

Quantum geometry from stability and modularity

Conversely, we can use our knowledge of Abelian D4-D2-D0 invariants to compute GV invariants and push the direct integration method to higher genus !



Alexandrov Feyzbakhsh Klemm BP Schimannek'23

Quantum geometry from stability and modularity

X	χ_X	κ	type	$g_{\rm integ}$	$g_{\rm mod}$	$g_{\rm avail}$
$X_5(1^5)$	-200	5	F	53	69	64
$X_6(1^4,2)$	-204	3	F	48	57	48
$X_8(1^4,4)$	-296	2	F	60	80	64
$X_{10}(1^3,2,5)$	-288	1	F	50	70	68
$X_{4,3}(1^5,2)$	-156	6	F	20	24	24
$X_{6,4}(1^3,2^2,3)$	-156	2	F	14	17	17
$X_{6,6}(1^2,2^2,3^2)$	-120	1	K	18	22	22
$X_{4,4}(1^4,2^2)$	-144	4	K	26	34	34
$X_{3,3}(1^6)$	-144	9	K	29	33	33
$X_{4,2}(1^6)$	-176	8	C	50	66	50
$X_{6,2}(1^5,3)$	-256	4	C	63	78	49

http://www.th.physik.uni-bonn.de/Groups/Klemm/data.php