The global scattering diagram for local $\mathbb{P}^2$

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Workshop on BPS states, mirror symmetry, and exact WKB II
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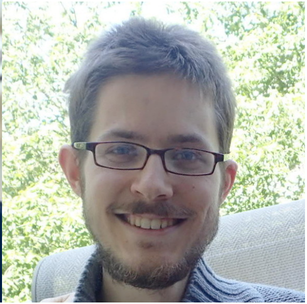
Work in progress with Pierrick Bousseau, Pierre Descombes and Bruno Le Floch

My amazing co-authors





Introduction

- $X = K_{\mathbb{P}^2}$ is one of the simplest example of CY 3-folds
- BPS states in type IIA/X are described by objects in $\mathcal{C} = D^b \,\mathrm{Coh}(X)$
- Stable objects are counted by the Donaldson-Thomas invariants $\Omega_\sigma(\gamma)$ with $\gamma \in K(X)$ and $\sigma \in \mathrm{Stab}\,\mathcal{C}$
- Physicists mostly care about the slice of Π stability conditions $\Pi \subset \mathrm{Stab}\,\mathcal{C}$, isomorphic to (universal cover of) Kähler moduli space $\widetilde{\mathcal{M}}_K(X)$, defined by VHS on the mirror Y.
- In general, $\dim_{\mathbb{C}} \Pi = b_2(X)$ is less than $\dim_{\mathbb{C}}[\mathrm{Stab}\,\mathcal{C}/GL(2,\mathbb{R})^+] = b_{\mathrm{even}}(X) - 2$, but the two agree for $X = K_S$ for any complex surface S.

Scattering diagrams

- The scattering diagram $\mathcal{D}_\psi \subset \mathrm{Stab}\,\mathcal{C}$ is (roughly) the union over $\gamma \in K(X)$ of active rays

$$\mathcal{R}_\psi(\gamma) = \{\arg Z_\sigma(\gamma) = \psi + \tfrac{\pi}{2}\,, \Omega_\sigma(\gamma) \neq 0\}$$

  equipped with some element $\mathcal{U}_\sigma(\gamma)$ in some pro-unipotent group keeping track of $\Omega_\sigma(\gamma)$.
- The consistency of $\mathcal{D}_\psi$ allows to compute all $\Omega_\sigma(\gamma)$ from initial rays.
- Scattering diagrams appear to be the correct mathematical framework for attacking the Split Attractor Flow Tree Conjecture *[Denef'00]*, at least for local CY3 such that Z_σ is holomorphic on Π

Attractor Flows

- Attractor flow of charge γ along Π: $\frac{\mathrm{d}z^a}{\mathrm{d}\mu} = -g^{a\bar{b}}\partial_{\bar{b}}|Z(\gamma)|^2$
- $|Z(\gamma)|^2$ decreases along the flow, until it reaches a local minimum at $z = z_*(\gamma)$, or hits the boundary of Π. The attractor index is $\Omega_*(\gamma) := \Omega_{z_*(\gamma)}(\gamma)$ (There could be different basins of attraction)
- A split attractor flow tree is a rooted binary tree T, decorated with charges γ_e along edges, embedded in Π along the flow lines of $|Z(\gamma_e)|$ along each edge, satisfying at each vertex
  1. charge conservation: $\gamma_{p(v)} = \gamma_{L(v)} + \gamma_{R(v)}$
  2. phase alignment: $\mathrm{Im}[Z_v(\gamma_{L(v)})\overline{Z_v(\gamma_{R(v)})}] = 0, \mathrm{Re}[Z_v(\gamma_{L(v)})\overline{Z_v(\gamma_{R(v)})}] > 0$
  3. stability: $\langle \gamma_{L(v)}, \gamma_{R(v)} \rangle \, \mathrm{Im}[Z_{p(v)}(\gamma_{L(v)})\overline{Z_{p(v)}(\gamma_{R(v)})}] > 0$

Split Attractor Flow Tree Conjecture

- Let $\mathcal{T}(\{\gamma_i\}, z)$ the set of trees rooted at z, with leaves of charge γ_i, and let $\bar{\Omega}(\gamma) := \sum_{k|\gamma} \frac{1}{k^2}\Omega(\gamma/k)$ be the rational DT invariants.
- The Split Attractor Flow Tree Conjecture roughly says

$$\Omega_z(\gamma) = \sum_{\gamma=\sum_i \gamma_i} \frac{1}{|\mathrm{Aut}(\{\gamma_i\})|} \left(\sum_{T\in\mathcal{T}(\{\gamma_i\},z)} \prod_{v\in V_T} \langle \gamma_{L(v)}, \gamma_{R(v)} \rangle \right) \prod_i \bar{\Omega}_\star(\gamma_i)$$

Attractor flow trees and scattering diagrams

- On a local CY, holomorphy of Z implies that $\arg Z(\gamma)$ is conserved along the flow. Hence flow lines lie along rays $\mathcal{R}_\psi(\gamma)$ with $\arg Z(\gamma) = \psi + \frac{\pi}{2}$
- Vertices lie at the intersection of $\mathcal{R}_\psi(\gamma_{L(v)})$ and $\mathcal{R}_\psi(\gamma_{R(v)})$
- Holomorphy of Z also implies that there are no local minima of $|Z(\gamma)|^2$, except on boundary or at points where $Z(\gamma) = 0$
- When $\dim_{\mathbb{C}} \Pi = 1$, flow trees essentially coincide with scattering sequences of initial rays in $\mathcal{D}_\psi \cap \Pi$!

Towards the scattering diagram for $K_{\mathbb{P}^2}$

- Our aim is to construct the scattering diagram $\mathcal{D}_\psi^\Pi$ for $\mathcal{C} = D^b \operatorname{Coh} K_{\mathbb{P}^2}$, and use it to demonstrate the Split Attractor Flow Tree Conjecture in that simple case.
- We build on *[Bridgeland'16]* on scattering diagrams for quivers with potential, and *[Bousseau'19]* for the scattering diagram for coherent sheaves on $\mathbb{P}^2$.
- This construction provides an algorithm to compute BPS indices for local $\mathbb{P}^2$ at any point in $\operatorname{Stab} \mathcal{C}$, and new insights on the microscopic structure of BPS states (BPS dendroscopy)
- Hopefully similar ideas can be used for other local CY3, and perhaps compact CY3.

Outline

Outline

Stability conditions on local $\mathbb{P}^2$

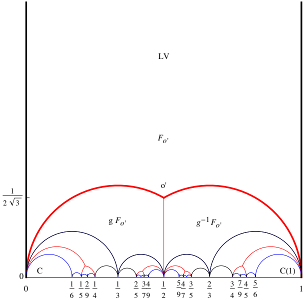
- The category $\mathcal{C} = D^b \,\mathrm{Coh}\, K_{\mathbb{P}^2}$ is graded by $\Gamma = \mathbb{Z}^3$

$$E \mapsto \gamma(E) = [r, d, \chi) \,, \quad \chi = r + \tfrac{3}{2} d + \mathrm{ch}_2$$

- A stability condition σ is a pair $(Z, \mathcal{A})$ such that
  1. $Z : \Gamma \to \mathbb{C}$ linear map
  2. $\mathcal{A}$ heart of t structure
  3. $\forall 0 \neq E \in \mathcal{A}, \mathrm{Im} Z(E) > 0$ or $(\mathrm{Im} Z(E) = 0 \text{ and } \mathrm{Re}(Z) < 0)$
  4. Harder-Narasimhan + support properties
- $\mathrm{Stab}\, \mathcal{C}/\mathbb{C}^\times$ is a complex manifold of dim 2, parametrized locally by (T, T_D) such that $Z(\gamma) = -rT + dT_D - \mathrm{ch}_2$
- Since $\mathrm{Im} Z(\gamma) = r \mathrm{Im}\, T(\mu - s)$ with $\mu = \frac{d}{r}$ and $s = \frac{\mathrm{Im}\, T_D}{\mathrm{Im}\, T}$, we can take $\mathcal{A} = \{E \to F\}$ with $\mu(E) \leq s$ and $\mu(F) > s$. *[Bayer Macri'11]*

Kähler moduli space

- The Kähler moduli space of $K_{\mathbb{P}^2}$ is the modular curve $\mathbb{H}/\Gamma_1(3)$ parametrizing elliptic curves with level structure.



- It admits two cusps at the large volume and conifold points, and one orbifold point $\tau_o = -\frac{1}{2} + \frac{\mathrm{i}}{2\sqrt{3}}$ of order 3.

Π stability slice inside Stab($\mathcal{C}$)

- The universal cover of Π is embedded in Stab($\mathcal{C}$)$/\mathbb{C}^\times$ via

$$Z_\tau(\gamma) = -rT_D(\tau) + dT(\tau) - \mathrm{ch}_2$$

$$\begin{pmatrix} T \\ T_D \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/3 \end{pmatrix} + \int_{\tau_o}^{\tau} \begin{pmatrix} 1 \\ u \end{pmatrix} C(u)\,\mathrm{d}u$$

$$C(\tau) = \frac{\eta(\tau)^9}{\eta(3\tau)^3} = 1 - 9 \sum_{n=0}^{\infty} \frac{(\frac{n}{-3})q^n}{1-q^n}$$

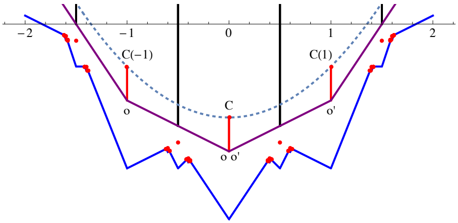
- The group $\Gamma_1(3)$ acts by auto-equivalences of Stab($\mathcal{C}$), generated by $T : E \mapsto E(1) = E \otimes \mathcal{O}_X$and $V : E : ST_{\mathcal{O}}(E)$ – the Seidel-Thomas twist with respect to the spherical object $\mathcal{O} = \mathcal{O}_{\mathbb{P}^2}$

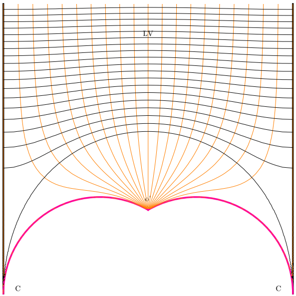
$$\begin{pmatrix} 1 \\ T \\ T_D \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ m & d & c \\ m_D & b & a \end{pmatrix} \begin{pmatrix} 1 \\ T \\ T_D \end{pmatrix}, \quad \begin{pmatrix} r \\ d \\ \mathrm{ch}_2 \end{pmatrix} \mapsto \begin{pmatrix} d & c & 0 \\ b & a & 0 \\ m' & m'_D & 1 \end{pmatrix} \begin{pmatrix} r \\ d \\ \mathrm{ch}_2 \end{pmatrix}$$

Π stability slice inside Stab($\mathcal{C}$)/$GL(2,\mathbb{R})^+$

$$s := \frac{\mathrm{Im}\, T_D}{\mathrm{Im}\, T}\ , \quad q := -\frac{\mathrm{Im}(T\bar{T}_D)}{\mathrm{Im}\, T} := \frac{1}{2}(s^2+t^2)$$

$$Z(\gamma) \approx (rq - \mathrm{ch}_2) + \mathrm{i}(d - sr) \approx -\frac{r}{2}(s+\mathrm{i}t)^2 + d(s+\mathrm{i}t) - \mathrm{ch}_2$$





Outline

Quivers with potential

- Let (Q, W) be a quiver with potential, $J(Q, W)$ the Jacobian path algebra $\mathbb{C}Q/\partial_W$, $\mathcal{A}$ the Abelian category of representations of $J(Q, W)$, graded by the dimension vector $\gamma \in \Gamma = \mathbb{N}^{Q_0}$.
- Stability conditions are parametrized by $Z(\gamma_i) \in \mathbb{H}$ for simple representations attached to the node i.
- Let $\mathcal{M}_Z(\gamma)$ be the moduli space of semi-stable representations with dimension vector γ (i.e. $\arg Z(\gamma') \leq \arg(Z_\gamma)$ for any $E' \subset E$), and $\Omega_Z(\gamma)$ its motivic DT invariant. Informally, $\Omega_Z(\gamma, y) = (-y)^{\dim \mathcal{M}_Z(\gamma)} \sum_n b_n(\mathcal{M}_Z(\gamma))(-y)^n$.
- Let $\bar\Omega_Z(\gamma, y) = \sum_{k|\gamma} \frac{y-1/y}{k(y^k-1/y^k}\Omega_Z(\gamma/k, y^k)$ be the rational DT invariant.

Stability scattering diagram

- Let $\mathcal{G}$ be the associative graded algebra spanned by $\{x_\gamma, \gamma \in \Gamma\}$ with $x_\gamma x_{\gamma'} = (-y)^{\langle \gamma,\gamma' \rangle} x_{\gamma+\gamma'}$, where $\langle \gamma, \gamma' \rangle = \sum_{(i\to j)\in Q_1} (n'_i n_j - n_i n'_j)$.
- Let $G = \lim_{k\to\infty} \exp(\mathcal{G}_k)$ where $\mathcal{G}_k = \mathcal{G}/\{x_\gamma, \sum_i n_i \leq k\}$.
- For γ primitive and $\psi \in \mathbb{R}$, define the active ray as the locus in $\mathcal{S}_\psi = \{Z : \Gamma \to \mathbb{H}^n, \mathrm{Im}(e^{-\mathrm{i}\psi} Z(\gamma_i)) > 0\}$

$$\mathcal{R}_\psi(\gamma) = \{Z : \mathrm{Re}(e^{-\mathrm{i}\psi} Z(\gamma)) = 0, \exists k \geq 1 \Omega_Z(k\gamma) \neq 0\}$$

endowed at any point $Z \in \mathcal{R}_\psi(\gamma)$ with the automorphism

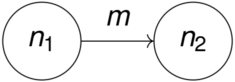
$$\mathcal{U}_Z(\gamma) = \exp\left(\sum_{k=1}^{\infty} \frac{\bar{\Omega}(k\gamma, y)}{y^{-1} - y} x_{k\gamma}\right) = \mathrm{Exp}\left(\sum_{k=1}^{\infty} \frac{\Omega(k\gamma, y)}{y^{-1} - y} x_{k\gamma}\right)$$

- Elementary ray: $\overline{\mathcal{R}}_\psi(\gamma) = \{\mathrm{Re}(e^{-\mathrm{i}\psi} Z(\gamma)) = 0, \bar{\Omega}_Z(\gamma) \neq 0\}$ equipped with $\overline{\mathcal{U}}_Z(\gamma) = \exp\left(\frac{\bar{\Omega}(\gamma,y)}{y^{-1}-y} x_\gamma\right)$

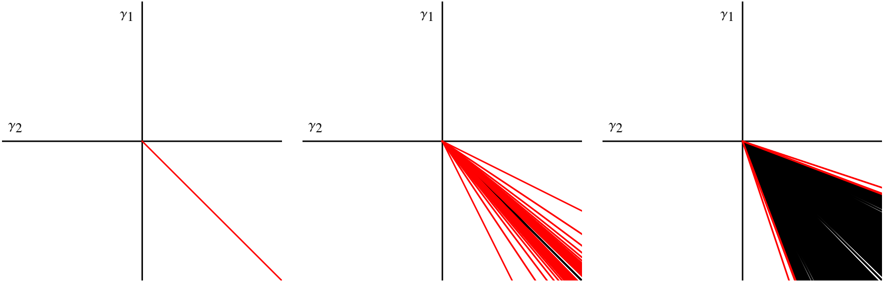
Stability scattering diagram

- The scattering diagram $\mathcal{D}_\psi = \{\mathcal{R}_\psi(\gamma), \gamma \in \Gamma_{\rm prim}\}$ is consistent: for any path $[0,1] \to \mathbb{H}^n$ crossing $\mathcal{R}_\psi(\gamma_a)$ at $t = t_a$, $\prod_i \mathcal{U}_{\sigma(t_a)}(\gamma_a)^{\epsilon_a} = 1$
- Let $\theta_i = \mathrm{Re}(e^{-\mathrm{i}\psi} Z(\gamma_i))$. For $\psi = \arg Z(\gamma) - \frac{\pi}{2}$, the semi-stability condition reduces to King's stability condition, $(\theta, \gamma') \leq 0$ for any $R' \subset R$. Let $\pi : \mathbb{H}^n \to \mathbb{R}^n$ be the projection $Z \mapsto \theta$.
- The scattering diagram $\pi(\mathcal{D}_\psi) = \cup_\gamma \{\theta : (\theta, \gamma) = 0, \Omega_\theta(\gamma) \neq 0\}$ is a complex of convex rational polyhedral cones. *[Bridgeland'16]*
- It is uniquely determined by the initial rays, i.e. those which contain the self-stability condition $\theta = \langle -, \gamma \rangle$. *[Kontsevich Soibelman]*

Scattering diagram for Kronecker quiver



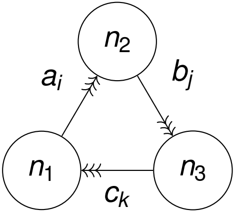
$$\theta_1 > 0, \theta_2 < 0 : \quad \dim \mathcal{M} = m n_1 n_2 - n_1^2 - n_2^2 + 1$$



m=1 m=2 m=3

Quiver for $K_{\mathbb{P}^2}$

- Near the orbifold point $\tau_o = -\frac{1}{2} + \frac{\mathrm{i}}{2\sqrt{3}}$, the BPS spectrum is governed by a quiver with potential:



$$W = \sum \epsilon_{ijk} X_i Y_j Z_k$$

$$\begin{array}{lll} E_1 = \mathcal{O} & & \gamma_1 = [1,0,0] \\ E_2 = \Omega(1)[1], & & \gamma_2 = [-2,1,\tfrac{1}{2}] \\ E_3 = \mathcal{O}(-1)[2] & & \gamma_3 = [1,-1,\tfrac{1}{2}] \end{array}$$

$$\begin{array}{ll} r = & 2n_2 - n_1 - n_3 \\ d = & n_3 - n_2 \\ \mathrm{ch}_2 = & -\tfrac{1}{2}(n_2 + n_3) \end{array}$$

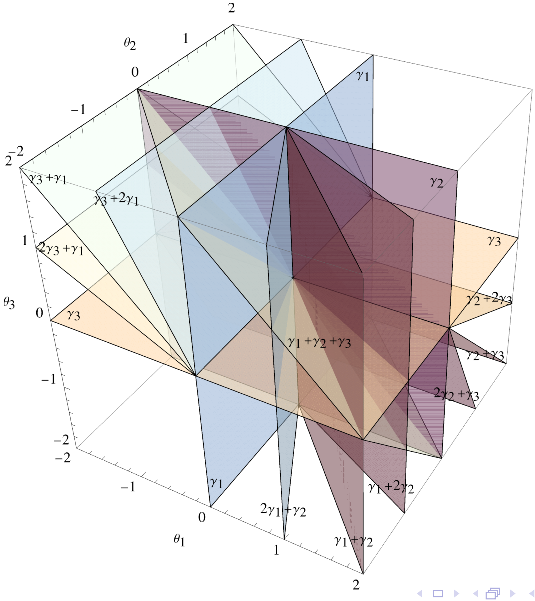
- More precisely, $Z_\tau(\gamma_i) = \frac{1}{3} + \mathcal{O}(\tau - \tau_o)$, so the heart $\mathcal{A}_\tau$ is related to the category of quiver representations by tilting $Z \to \mathrm{i}Z$. To ensure $\mathrm{Im}[e^{-\mathrm{i}\psi} Z(\gamma_i)] > 0$, take $-\frac{\pi}{2} < \psi < 0$.

Initial data for the orbifold quiver

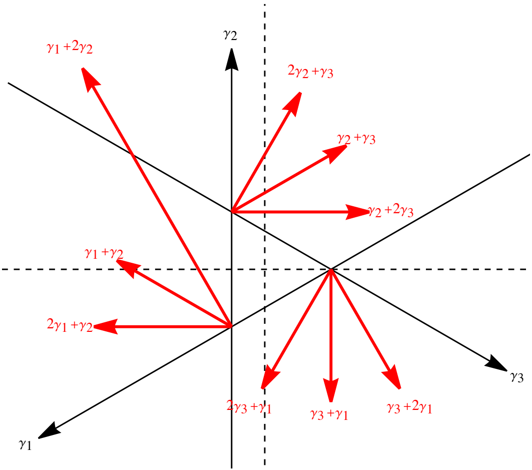
Thm (Beaujard Manschot BP'20; P.Descombes'22): The initial rays are $\mathcal{R}(\gamma)$ with $\gamma \in \{(1,0,0),(0,1,0),(0,0,1),(n,n,n)\}$

- For R a stable representation with $\gamma \neq (n,n,n)$, all cycles vanish.
- In chamber $\theta_1 > 0, \theta_3 < 0$, all arrows $c_k = 0$
- The moduli space $\mathcal{M}$ of representations of Beilinson quiver (a_i, b_j) with relations $\partial_{c_k} W = 0$ is cut out by $3n_1 n_3$ relations in a smooth space of dimension $3n_1 n_2 + 3n_2 n_3 - n_1^2 - n_2^2 - n_3^2 + 1$
- These relations intersect transversally, otherwise there would exist non-zero $\tilde{c}_k : V_3 \to V_1$ such that $(a_i, b_j, \tilde{c}_k)$ is a stable representation of (Q, W).
- Hence $\dim \mathcal{M} = 3n_1 n_2 + 3n_2 n_3 - 3n_1 n_3 - n_1^2 - n_2^2 - n_3^2 + 1 = 1 - \frac{1}{2}[(n_1-n_2)^2 + (n_2-n_3)^2 + (n_3-n_1)^2] - 2n_1(n_3-n_2) - 2n_3(n_1-n_2)$
- For self-stability condition $\theta_1 > 0, \theta_3 < 0$ implies $n_1, n_3 > n_2$ so $\dim \mathcal{M} < 0$ except for $(1,0,0),(0,1,0),(0,0,1)$.

Orbifold scattering diagram

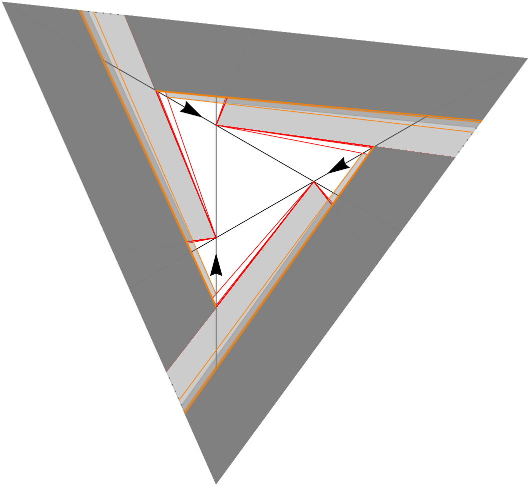


A 2D slice of the orbifold scattering diagram

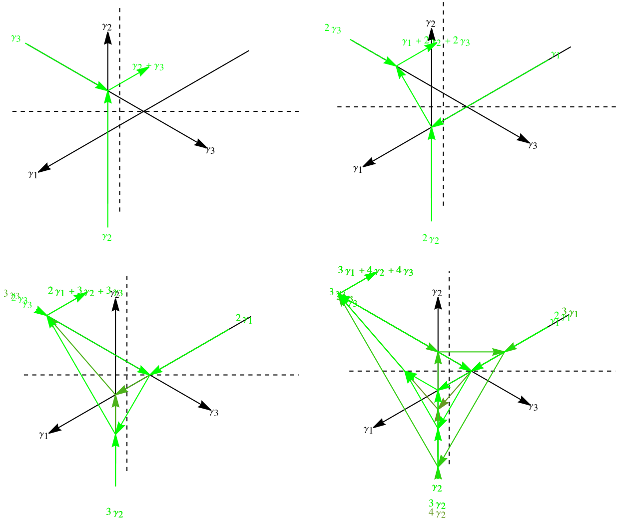


$$\theta_1 = u - v\sqrt{3} - \frac{1}{3}, \theta_2 = -2u - \frac{1}{3}, \theta_3 = u + v\sqrt{3} - \frac{1}{3}, \quad \theta_1 + \theta_2 + \theta_3 = -1$$

A 2D slice of the orbifold scattering diagram



Attractor Flow Trees for $\gamma = (n-1, n, n) = [1, 0, 1-n]$



$\Omega(n-1, n, n) = 3, 9, 22 = 13 + 9, 51 = 15 + 9 + 27$, respectively

Embeding the quiver scattering diagram inside Π ?

- Recall that the King stability parameters (rescaled such that $\theta_1 + \theta_2 + \theta_3 = -1$) are given by $\theta_i = \frac{\mathrm{Re}(e^{-\mathrm{i}\psi} Z(\gamma_i))}{\cos\psi}$.
- Parametrizing as before

$$\theta_1 = u - v\sqrt{3} - \tfrac{1}{3}, \quad \theta_2 = -2u - \tfrac{1}{3}, \quad \theta_3 = u + v\sqrt{3} - \tfrac{1}{3}$$

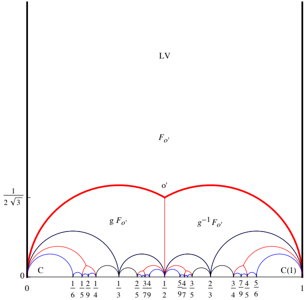
and expanding at first order near τ_o, we can relate

$$\tau \simeq \tau_o - \frac{2\mathrm{i}\sqrt{3}}{C(\tau_o)} e^{\mathrm{i}\psi}(u + \mathrm{i}v) \cos\psi$$

As $\psi \to -\frac{\pi}{2}$, all scatterings take place near τ_o.

- Q: Where do initial rays come from in the full scattering diagram ? Do outgoing rays ever escape to the large volume region ?

Intermission



$$Z_\tau(\gamma) = -rT_D(\tau) + dT(\tau) - \mathrm{ch}_2$$

$$\begin{pmatrix} T \\ T_D \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/3 \end{pmatrix} + \int_{\tau_o}^{\tau} \begin{pmatrix} 1 \\ u \end{pmatrix} C(u)\,\mathrm{d}u$$

Outline

Large volume scattering diagram

- In the large volume region, Z_τ is well approximated by

$$Z_{(s,t)}(\gamma) = -\frac{r}{2}(s+\mathrm{i}t)^2 + d(s+\mathrm{i}t) - \mathrm{ch}_2 \;,$$

  with $\tau = s + \mathrm{i}t$, $\mathcal{A}_s = \{E \to F\}$ with $\mu(E) \leq s, \mu(F) > s$

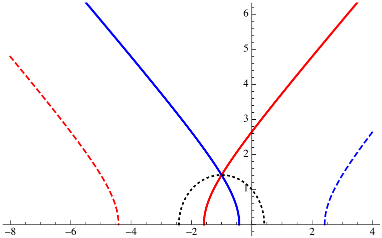
- Geometric rays are easy to describe for $\psi = 0$:

$$\mathrm{Re} Z_{(s,t)}(\gamma) = -\frac{r}{2}(s^2 - t^2) + ds - \mathrm{ch}_2, \quad \mathrm{Im} Z_{(s,t)}(\gamma) = t(d - rs)$$

  hence vertical lines $s = \frac{\mathrm{ch}_2}{d}$ when $r = 0$, or branches of hyperbola asymptoting to $t = \pm(s - \frac{d}{r})$, for $r \neq 0$ (degenerating to straight lines when $\Delta = \frac{1}{2}d^2 - r\,\mathrm{ch}_2 = 0$)

- Walls of marginal stability $\mathrm{Im}[Z(\gamma')\overline{Z(\gamma)}] = 0$ for $\langle \gamma, \gamma' \rangle \neq 0$ are half-circles.

Large volume scattering diagram



- A useful physical analogy: think of $\mathcal{R}(\gamma)$ as the worldline of a fictitious relativistic particle in two-dimensional Minkovski space (s,t), with mass $m^2 = \Delta = \frac{1}{2}d^2 - r\,\mathrm{ch}_2$, electric charge r, immersed in a constant electric field !
- In particular, rays "stay inside the light-cone" and the electric potential $\varphi_s(\gamma) = 2(d - rs)$ increases along each ray.
- For $\psi \neq 0$, just rotate $s \mapsto s - t\tan\psi, t \mapsto t/\cos\psi$.

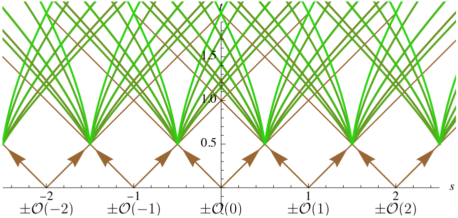
Large volume scattering diagram

Theorem (Arcara, Bertram, Huizenga, Martinez, Wang, Maciocia'13): for fixed charge $\gamma = [r, d, \mathrm{ch}_2]$,

- There is a finite sequence of nested walls, along with a vertical wall at $s = \frac{d}{r}$.
- Outside the innermost wall, $\Omega_{(s,t)}(\gamma)$ agrees with the DT invariant for the moduli space of Gieseker-semi-stable sheaves on $\mathbb{P}^2$
- Across each wall, the moduli space $\mathcal{M}_{(s,t)}(\gamma)$ undergoes birational transformation, until it becomes empty inside the innermost wall
- For $\gamma_m = [1, m, \frac{1}{2}m^2]$, the structure sheaf $\mathcal{O}(m)$ (fluxed D4-brane in physics parlance) is stable whenever $s < m$; its homological shift $\mathcal{O}(m)[1]$ is stable whenever $s > m$. Note that $Z_{(m,0)}(\gamma_m) = 0$.

Large volume scattering diagram

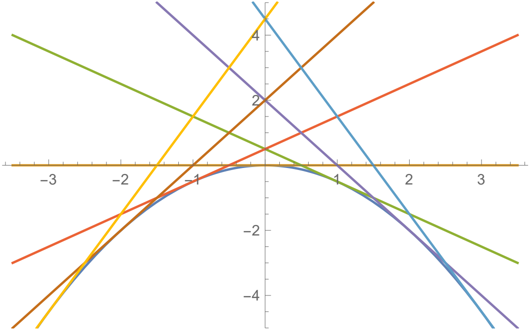
Thm (Bousseau, 2019): the only initial rays are associated to $\mathcal{O}(m)$ and $\mathcal{O}(m)[1]$ emanating from $s = m, t = 0$, with index $\Omega_*(\pm\gamma_m) = 1$



- The absence of incoming rays in intervals $]m, m+1[$ at $t = 0$ follows from quiver description.
- The scattering diagram can be regulated by using monotonicity of $\varphi(\gamma) = 2(d - sr)$ along the rays.

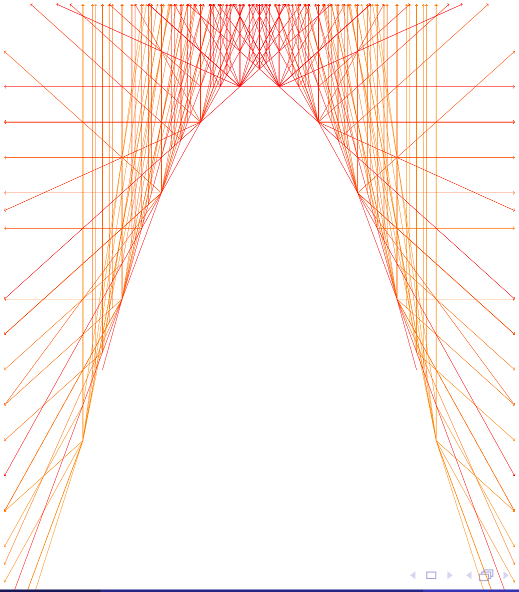
Scattering diagram in x, y plane

- The scattering diagram was originally constructed in coordinates $(x, y) = (s, \frac{1}{2}(t^2 - s^2))$, where rays are straight lines.



- The same scattering diagram arises in the context of Gromov-Witten invariants on $(\mathbb{P}^2, \mathcal{E})$ *[Carl Pumperla Siebert]*

Scattering diagram in x, y plane (T. Graefnitz)



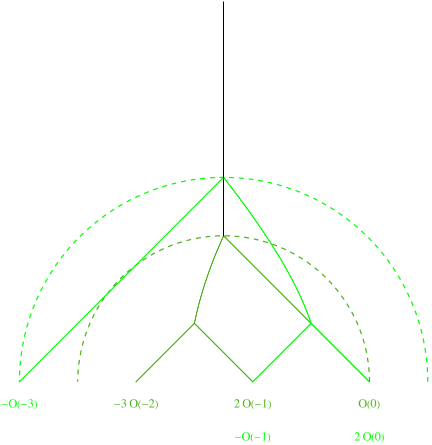
BPS indices from scattering sequences

- To compute $\bar{\Omega}_Z(\gamma)$ at (s,t) such that $\mathrm{Re} Z_{(s,t)}(\gamma)=0$, one must find all sequences of scatterings of initial rays $\{k_i \mathcal{O}(m_i), k'_i \mathcal{O}(m'_i)[1]\}$ which produce an elementary ray of charge γ passing through.
- Unlike for quivers, charge conservation is not sufficient to a finite number of possible splittings $\gamma = \sum_i \gamma_i$:

$$[r,d,c_2] = \sum_i k_i\,[1,m_i,\tfrac12 m_i^2] - \sum_j k'_j\,[1,m'_j,\tfrac12 m'^2_j]$$

- Causality restricts possible slopes $s-t \leq m'_{\rm min} < m_{\rm max} \leq s+t$
- Since $\varphi_s(\gamma_m)\geq 1$ at the first scattering, one also has $\sum_i k_i + \sum_j k'_j + 2(m_{\rm max} - m'_{min}+1) \leq \varphi_s(\gamma)$.
- The contribution of each scattering sequence can be computed using the Attractor Flow Tree formula at each vertex.

Flow trees for $\gamma = [0, 4, 1)$

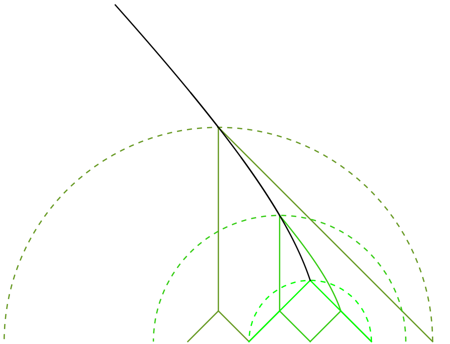


- $\{\{-3\mathcal{O}(-2), 2\mathcal{O}(-1)\}, \mathcal{O}\}$:
  $3\mathcal{O}(-2) \to 2\mathcal{O}(-1) \oplus \mathcal{O} \to E$
  $K_3(2,3)K_{12}(1,1) \to -156$

- $\{-\mathcal{O}(-3), \{-\mathcal{O}(-1), 2\mathcal{O}\}\}$:
  $\mathcal{O}(-3) \oplus \mathcal{O}(-1) \to 2\mathcal{O} \to E$
  $K_3(1,2)K_{12}(1,1) \to -36$

Total: $\Omega_\infty(\gamma) = -192 = GV_4^{(0)}$

Flow trees for $\gamma = [1, 0, -3)$



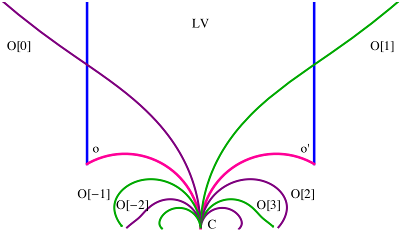
- $\{\{-\mathcal{O}(-5), \mathcal{O}(-4)\}, \mathcal{O}(-1)\}$
  $\mathcal{O}(-5) \to \mathcal{O}(-4) \oplus \mathcal{O}(-1) \to E$
  $K_3(1,1)^2 \to 9$
- $\{\{-\mathcal{O}(-4), \mathcal{O}(-3)\},$
  $\{-\mathcal{O}(-3), 2\mathcal{O}(-2)\}\}$
  $\mathcal{O}(-4) \oplus \mathcal{O}(-3) \to$
  $\mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \to E$
  $K_3(1,1)^2 K_3(1,2) \to 27$
- $\{-\mathcal{O}(-4), 2\mathcal{O}(-2)\}$
  $\mathcal{O}(-4) \to 2\mathcal{O}(-2) \to E$
  $K_6(1,2) \to 15$

Total: $\Omega_\infty(\gamma) = 51 = \chi(\mathrm{Hilb}_4 \mathbb{P}^2)$

Outline

Exact scattering diagram

- The full scattering diagram be invariant under the action of $\Gamma_1(3)$.
- Under $\tau \mapsto \frac{\tau}{3n\tau+1}$ with $n \in \mathbb{Z}$, $\mathcal{O} \mapsto \mathcal{O}[2n]$. Hence we have an doubly infinite family of initial rays associated to $\mathcal{O}(m)[n]$ at $\tau = 0$.



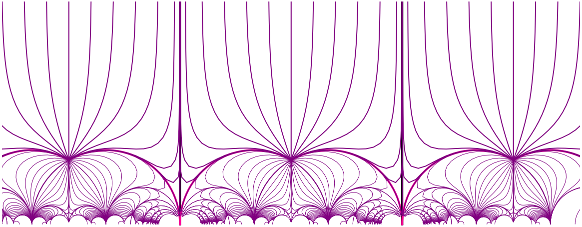
- All $\tau = p/q$ with $q \neq 0 \mod 3$ are related to $\tau = 0$ by $\Gamma_1(3)$. Hence a similar set of ray originates from any such τ.

Massless objects at conifold points

τ	g	γ_C	$\Delta(\gamma_C)$	E
0	1	$[1,0,1)$	0	$\mathcal{O}$
$1/5$	U^2T^{-1}	$-[5,1,6)$	$3/25$	$E \to \Omega(2)[-1] \to \mathcal{O}^{\oplus 3}[2]$
$1/4$	UT	$[4,1,6)$	$-3/32$	$E \to \mathcal{O}(1) \to \mathcal{O}^{\oplus 3}[3]$
$2/5$	UT^{-2}	$-[5,2,6)$	$12/25$	$E \to \mathcal{O}(-2) \to \mathcal{O}^{\oplus 6}$
$3/7$	$UT^{-1}VT$	$[7,3,10)$	$15/49$	$E \to \Omega(0)[1] \to \mathcal{O}^{\oplus 9}[1]$
$1/2$	TVT	$-[2,1,3)$	$3/8$	$\Omega(2)[1]$
$4/7$	$TVTUT^{-1}$	$[7,4,12)$	$15/49$	$\mathcal{O}(1)^{\oplus 9}[-1] \to \Omega(4)[-1] \to E$
$3/5$	TVT^2	$-[5,3,8)$	$12/25$	$\mathcal{O}(1)^{\oplus 6} \to \mathcal{O}(3) \to E$
$3/4$	TVT^{-1}	$[4,3,10)$	$-3/32$	$\mathcal{O}(1)^{\oplus 3}[-3] \to \mathcal{O}(0) \to E$
$4/5$	TV^2T	$-[5,4,12)$	$3/25$	$\mathcal{O}(1)^{\oplus 3}[-2] \to \Omega(2)[1] \to E$
1	T	$[1,1,3)$	0	$\mathcal{O}(1)$

Exact scattering diagram - $\psi = -\frac{\pi}{2}$

- For $\psi = \pm\frac{\pi}{2}$, the diagram $\mathcal{D}^{\Pi}_{\psi}$ simplifies dramatically, since the loci $\mathrm{Im} Z_\tau(\gamma) = 0$ are lines of constant $s := \frac{\mathrm{Im}\, T_D}{\mathrm{Im}\, T} = \frac{d}{r}$, independent of ch_2. They only collide at orbifold points



- Hence, there is no wall-crossing between τ_o and $\tau = \mathrm{i}\infty$ when $-1 \leq \frac{d}{r} \leq 0$, which implies that the Gieseker index $\Omega_\infty(\gamma)$ agrees with the index $\Omega_\theta(\gamma)$ for the anti-self-stability condition $\theta = -\langle -, \gamma \rangle$

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Scattering diagram in affine coordinates

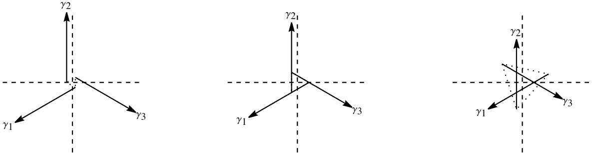
- For $-\frac{\pi}{2} < \psi \le 0$, define affine coordinates

$$x = \frac{\mathrm{Re}\left(e^{-\mathrm{i}\psi} T\right)}{\cos\psi}\,, \quad y = -\frac{\mathrm{Re}\left(e^{-\mathrm{i}\psi} T_D\right)}{\cos\psi}\,, \quad \mathcal{V}_\psi := \mathcal{V}\tan\psi$$

  such that geometric rays are straight lines $ry + dx = \mathrm{ch}_2$. Let $\mathcal{V} = \mathrm{Im}\, T(0) = \frac{27}{4\pi^2}\mathrm{Im}\mathrm{Li}_2(e^{2\pi\mathrm{i}/3}) \simeq .0462758$.
- Initial rays associated to $\pm\mathcal{O}(m)$ are tangent to $y = -\frac{1}{2}x^2$ and emitted at $x_m = m + \mathcal{V}_\psi$
- Initial rays associated to $\pm\Omega(m+1)$ are tangent to $y = -\frac{1}{2}x^2 - \frac{3}{8}$ and emitted at $x_m = n - \frac{1}{2} - 2\mathcal{V}_\psi$
- The orbifold point $\tau_o + n$ is mapped to $x_n = n - \frac{1}{2}$ along the parabola $y = -\frac{1}{2}x^2 - \frac{5}{24}$

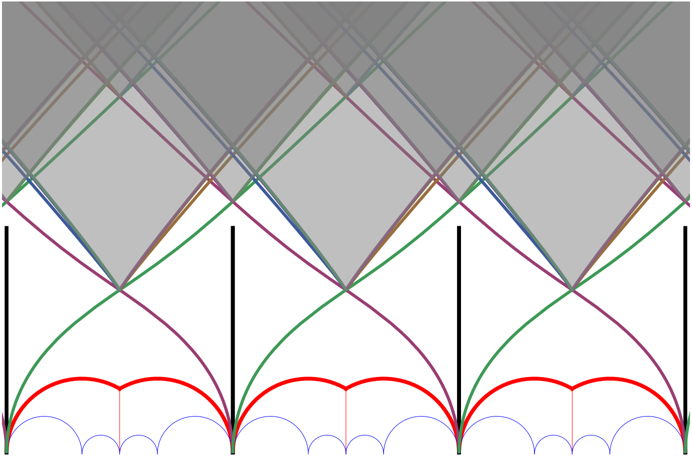
Critical phases

- The coordinates (x, y) are related to (u, v) near τ_o via $u = y - \frac{1}{2}x + \frac{1}{12}, v = -\frac{1}{2\sqrt{3}}(x + \frac{1}{2})$. In particular, initial rays of the orbifold scattering diagram start from finite distance $\simeq \mathcal{V}_\psi$!



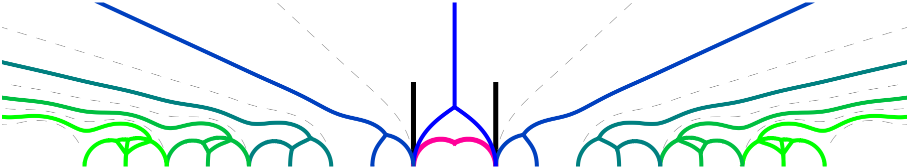
- For $|\mathcal{V}_\psi| < 1/2$, only rays $\mathcal{O}(m)$ and $\mathcal{O}(m)[1]$ escape to $\tau = \mathrm{i}\infty$, as in the large volume scattering diagram.
- For $|\mathcal{V}_\psi| > 1/2$, initial rays from $\Omega(m+1)[1]$ can interact with $\mathcal{O}(m)$ and $\mathcal{O}(m-1)[2]$ near the orbifold point $\tau_o + m$, and produce rays which escape towards the large volume region.
- The topology of the trees jumps for a discrete set of phases $|\mathcal{V}_\psi| = \frac{F_{2k}+F_{2k+2}}{2F_{2k+1}} \in \{\frac{1}{2}, 1, \frac{11}{10}, \frac{29}{26}, \frac{19}{17}, \dots \to \frac{1}{2}\sqrt{5}\}$.

Exact scattering diagram - $\psi = 0$

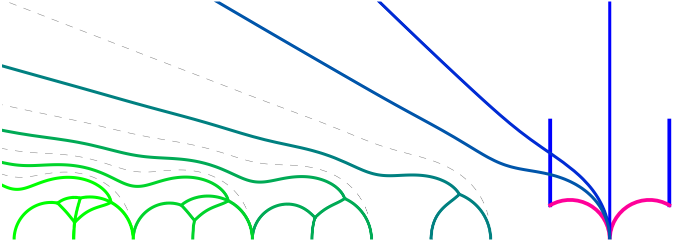


Exact scattering diagram, varying ψ

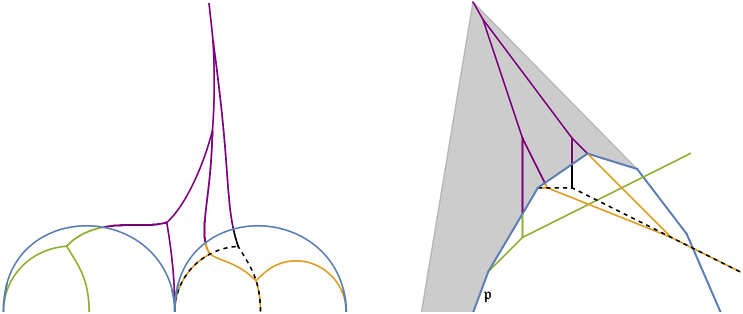
$\gamma = [0, 1, 1) = \mathrm{ch}\, \mathcal{O}_C$:



$\gamma = [1, 0, 1) = \mathrm{ch}\, \mathcal{O}$:



Composite flow trees



$$\{\{\{2\mathcal{O}(-1)[1], \Omega(1)\}, \{4\mathcal{O}[1], \{3\Omega(2), \mathcal{O}(1)[-1]\}\}\}, \{3\mathcal{O}(1), \{\Omega(2), 2\mathcal{O}[1]\}$$

$$\psi = -1.2$$

Conclusion - outlook

- Scattering diagrams appear to be the proper mathematical framework for the attractor flow tree formula in the case of local CY3, due to holomorphy of $Z(\gamma)$.
- They provide an effective way of computing (unframed) BPS invariants in any chamber, and a natural decomposition into elementary constituents. Mathematically, different trees should correspond to different strata in $\mathcal{M}_Z(\gamma)$.
- It would be interesting to extend this description to other toric CY3, such as local del Pezzo surfaces.
- For a compact CY3, $\arg Z(\gamma)$ is no longer constant along the flow and there can be attractor points with $\Omega_\star(\gamma) \neq 0$ at finite distance in Kähler moduli space...

Thanks for your attention !