BPS Dendroscopy on Local Calabi-Yau Threefolds

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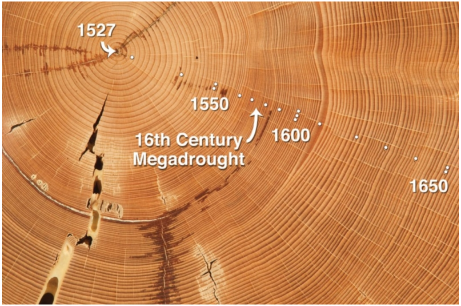
Work in progress with Pierrick Bousseau, Pierre Descombes and Bruno Le Floch

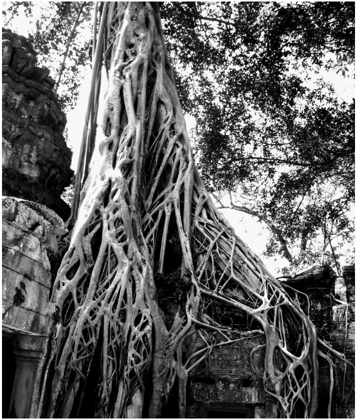
δενδρον= tree

Dentrology

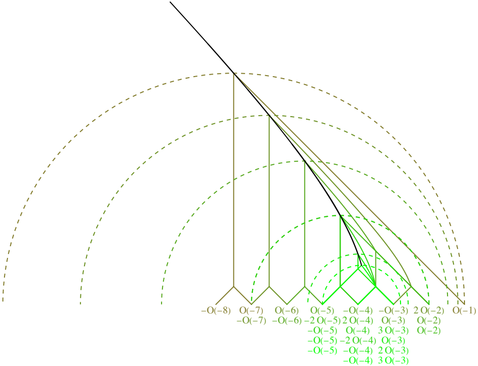


Dendrochronology

δενδροσκοπια= analyzing the BPS spectrum in terms of attractor flow trees



Ta Prohm, Angkor



BPS bound states on $K_{\mathbb{P}^2}$

Introduction

- In type IIA string theory compactified on a Calabi-Yau threefold X, the BPS spectrum consists of bound states of D6-D4-D2-D0 branes with electromagnetic charge $\gamma \in \Gamma \subset H_{\rm even}(X, \mathbb{Q})$.
- The BPS index $\Omega_z(\gamma)$ counts BPS states with charge γ and complexified Kähler moduli $z \in \mathcal{M}$ at spatial infinity.
- $\Omega_z(\gamma)$ is locally constant on $\mathcal{M}$, but can jump across real codimension one walls of marginal stability $\mathcal{W}(\gamma_L, \gamma_R) \subset \mathcal{M}$, where the central phases $Z(\gamma_L)$ and $Z(\gamma_R)$ with $\gamma = \gamma_L + \gamma_R$ become aligned. The jump is governed by a universal wall-crossing formula *[Kontsevich Soibelman '08, Joyce Song'08]*
- Physically, multi-centered black hole solutions (dis)appear across the wall *[Denef Moore '07, Manschot BP Sen '11]*. In contrast, single-centered black holes do not decay.

DT invariants and Bridgeland stability conditions

- Mathematically, $\Omega_z(\gamma)$ are generalized Donadson-Thomas invariants for the derived category of coherent sheaves $\mathcal{C} = D^b\mathrm{Coh}(X)$. *[Douglas 2010]*
- The Kähler moduli z determine a point $(Z, \mathcal{A})$ in the space Stab($\mathcal{C}$) of Bridgeland's stability conditions, where Z is the central charge function and $\mathcal{A}$ a suitable Abelian subcategory of $\mathcal{C}$ such that $\mathrm{Im}Z(\gamma) \geq 0$ for all objects in $\mathcal{A}$ (and $\mathrm{Re}Z(\gamma) < 0$ if $\mathrm{Im}Z(\gamma) = 0$).
- The image of the embedding $\mathcal{M} \hookrightarrow \mathrm{Stab}(\mathcal{C})$ defines the physical slice of Π-stability conditions, of complex codimension $b_{even}(X) - b_2(X) = 1 + b_4(X) + b_6(X)$.
- There is an action of $\widetilde{GL^+}(2,\mathbb{R})$ on Stab($\mathcal{C}$) by rescaling/rotating/stretching Z. This allows to extend Π-stability conditions to a slice of complex codimension $b_4(X) + b_6(X) - 1$.

Bridgeland stability conditions on local surfaces

- For a non-compact CY3 of the form $X = K_S$ where S is a complex surface, the derived category $D^b_c(X)$ of compactly supported coherent sheaves coincides with $D^b\,\mathrm{Coh}(S)$.
- An object in $D^b\,\mathrm{Coh}(S)$ with Chern vector $\gamma = [r, c_1, \mathrm{ch}_2]$ lifts to a bound state of $Q_4 = r$ D4-branes wrapped on S, $Q_2 \sim c_1$ D2-branes and $Q_0 \sim \mathrm{ch}_2$ D0-branes.
- The central charge for Π-stability is determined by local mirror symmetry. At large volume $z \to \mathrm{i}\infty$,

$$Z(\gamma) \sim -\int_S e^{-z^a H_a}\,\mathrm{ch}\,E = -r\, z^a Q_{ab} z^b + z^a\,\mathrm{ch}_{1,a} - \mathrm{ch}_2$$

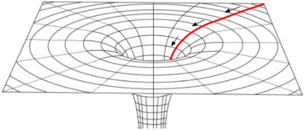
  and $\Omega_z(\gamma) \to \Omega_\infty(\gamma)$ counting Gieseker semi-stable sheaves.
- Since $b_6(X) = 0$ and $b_4(X) = 1$, any subleading corrections to Z_γ can be absorbed by $\widetilde{GL^+}(2,\mathbb{R})$ in a region around large volume.

Dendroscopy on local CY3 manifolds

- Physically, $\Omega_z(\gamma)$ counts BPS states in the 5D-dimensional superconformal field theory engineered by M-theory on X, further reduced along S^1. Macroscopically, they correspond to multi-centered dyon solutions of 4D $\mathcal{N} = 2$ effective field theory.
- Our goal will be to analyze the BPS spectrum in the simplest case $X = K_{\mathbb{P}^2} = \mathbb{C}^3/\mathbb{Z}_3$, corresponding to a non-Lagrangian SCFT in 5D, and categorize it into various types of multi-centered bound states.
- It will emerge that attractor flow trees for non-compact CY3 are closely connected to scattering diagrams in the space of Bridgeland stability conditions.

Attractor flow and attractor indices

- The black hole / dyon picture suggests to reconstruct the full BPS spectrum from the attractor indices $\Omega_\star(\gamma) := \Omega_{z_\star(\gamma)}(\gamma)$, where $z_\star(\gamma)$ is the endpoint of the attractor flow equations on $\mathcal{M}$,



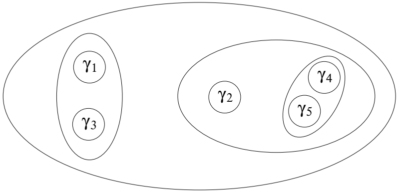
$$\frac{\mathrm{d}z^i}{\mathrm{d}\mu} = -g^{i\bar{j}}\partial_{\bar{j}}|Z(\gamma)|^2$$

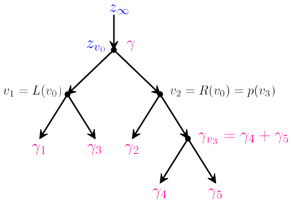
Ferrara Kallosh Strominger'95

- For spherically symmetric solutions, the moduli $z^i(\mu)$ reaches an attractor point $z_\star(\gamma)$ as $\mu \to \infty$ ($r \to 0$), which is independent of the initial values of the moduli, at least within a given basin of attraction.

Split attractor flows

- For a large class of multi-centered solutions with hierarchical structure, the flow splits on walls of marginal stability, leading to split attractor flow trees,





Denef '00; Denef Greene Raugas '01; Denef Moore'07; Manschot '10

Such solutions cease to exist when z_∞ crosses $\mathcal{W}(\gamma_{L(v_0)}, \gamma_{R(v_0)})$.

- A notable exception are scaling solutions, which do not exhibit any hierarchical structure nor walls of marginal stability. These solutions contribute to the attractor index $\Omega_\star(\gamma) = \Omega_S + \mathrm{scaling}$.

The split attractor flow conjecture

- The Attractor Flow Tree Formula postulates that the BPS index $\Omega_z(\gamma)$ can be computed by summing over all possible flow trees: schematically,

$$\Omega_z(\gamma) = \sum_{\gamma=\gamma_1+\dots+\gamma_n} \left(\sum_{T\in\mathcal{T}_z(\{\gamma_i\})} \prod_{v\in V_T} \langle \gamma_{L(v)}, \gamma_{R(v)} \rangle \right) \prod_{i=1}^{n} \Omega_\star(\gamma_i)$$

- Here, a flow tree T is a binary rooted tree, with edges decorated with charges γ_e, embedded in $\mathcal{M}$ along the gradient flow lines of $|Z(\gamma_e)|$. The root vertex maps to (γ, z), the leaves to $(\gamma_i, z_\star(\gamma_i))$ and at each vertex, $\gamma_v = \gamma_{L(v)} + \gamma_{R(v)}, z_v \in \mathcal{W}(\gamma_{L(v)}, \gamma_{R(v)})$.

Improving the split attractor flow conjecture

This formula is oversimplified on many counts:

- Each 2-body interaction contributes $(-1)^{\gamma_{LR}+1}|\gamma_{LR}|$ where $\gamma_{LR} = \langle\gamma_{L)}, \gamma_R\rangle$. More generally, including a fugacity for angular momentum, $(-1)^{\gamma_{LR}+1}\mathrm{sgn}\gamma_{LR}\frac{y^{\gamma_{LR}}-y^{-\gamma_{LR}}}{y-1/y}$
- In order to enforce Bose-Fermi statistics whenever two charges coincide, one should replace $\Omega(\gamma)$ by *[Manschot BP Sen'11]*

$$\bar{\Omega}_z(\gamma, y) = \sum_{d|\gamma} \frac{y - 1/y}{d(y^d - y^{-d})} \Omega_z(\tfrac{\gamma}{d}, y^d) \overset{y \to 1}{\to} \sum_{d|\gamma} \frac{1}{d^2} \Omega_z(\tfrac{\gamma}{d})$$

- When the charges of the constituents are not linearly independent, some splittings can involve higher valency vertices. Those can be resolved into binary trees by perturbing the charges or the moduli.
- One should restrict the possible constituents to those lying in the Abelian subcategory $\mathcal{A} \subset \mathcal{C}$, which changes along the tree.

Flow tree formula for quivers with potential

- A precise version of the Attractor Flow Tree Formula was proposed in the context of quiver quantum mechanics: 0+1 dim SUSY gauge theory with $G = \prod_{i\in Q_0} U(N_i)$, bifundamental matter $\Phi_{i\bar{j}}$ for every arrow $(i \to j) \in Q_1$.
- The dimension vector $\gamma = (N_i)_{i\in Q_0}$ plays the role of the charge vector, with Dirac product $\langle \gamma, \gamma' \rangle = \sum_{i\to j}(N_i N'_j - N'_i N_j)$.
- The index $\Omega_\zeta(\gamma)$ is a locally constant function of the Fayet-Iliopoulos parameters (aka King stability parameters) $\zeta \in \mathbb{R}^{Q_0}/\mathbb{R}^+$ and superpotential $W = \sum_{w\in Q_2} \lambda_w w$, with jumps in real codimension 1 and 2, respectively.

Flow tree formula for quivers with potential

- The continuous attractor flow is replaced by a discrete version

$$(\zeta_v, \cdot) = (\zeta_{p(v)}, \cdot) - \frac{\langle \gamma_v, \cdot \rangle}{\langle \gamma_v, \gamma_{L(v)} \rangle} (\zeta_{p(v)}, \gamma_{L(v)})$$

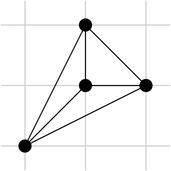
Manschot'10; Alexandrov BP'18

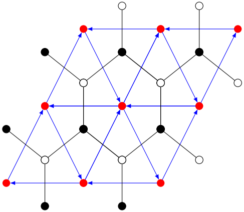
  Since $N_i \geq 0$, only a finite number of trees contribute. When $N_i > 1$, a small perturbation of ζ is necessary so as to restrict to binary trees.
- The attractor point becomes the self-stability condition $(\zeta_\star(\gamma), \cdot) = \langle \gamma, \cdot \rangle$. The attractor indices $\Omega_\star(\gamma_i)$ depend on W.
- The formula is manifestly consistent with wall-crossing. In addition, there are fake walls where the contributing trees jump but $\Omega(\gamma)$ stays constant, thanks to Jacobi-type identity

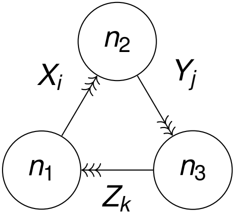
$$\langle \gamma_1, \gamma_2 \rangle \langle \gamma_1 + \gamma_2, \gamma_3 \rangle + \text{cyc.} = 0$$

Quivers from tiltings and tilings

- For non-compact CY3 admitting a tilting sequence $(E_1, \dots, E_K)$, there is an isomorphism $D^b_c \mathrm{Coh}(X) \simeq D^b \mathrm{Rep} J(Q, W)$, where nodes of Q correspond to the objets E_i, and arrows $i \to j$ to $\mathrm{Ext}^1(E_i, E_j)$.
- The abelian category of representations $\mathcal{A} = \mathrm{Rep} J(Q, W)$ agrees with the category of D-branes in the vicinity of 'orbifold points' where $Z(\mathrm{ch}\, E_i)$ are nearly aligned.
- Such tilting sequences exist for crepant resolutions of toric CY3 singularities, where (Q, W) can be read off from the brane tiling.







$$W = \sum \epsilon_{ijk} X_i Y_j Z_k$$

$$\begin{aligned} E_1 &= \mathcal{O} & \gamma_1 &= [1,0,0] \\ E_2 &= \Omega(1)[1], & \gamma_2 &= [-2,1,\tfrac{1}{2}] \\ E_3 &= \mathcal{O}(-1)[2] & \gamma_3 &= [1,-1,\tfrac{1}{2}] \end{aligned}$$

$$\begin{aligned} r &= 2n_2 - n_1 - n_3 \\ d &= n_3 - n_2 \\ \mathrm{ch}_2 &= -\tfrac{1}{2}(n_2 + n_3) \end{aligned}$$

Attractor conjecture

- By examining the expected dimension of the moduli space of quiver representations in the attractor chamber *[Beaujard BP Manschot]* conjectured that all attractor invariants $\Omega_\star(\gamma)$ vanish except $\Omega_\star(\gamma_i) = 1$ for $i = 1, 2, 3$ and $\Omega_\star(k(\gamma_1 + \gamma_2 + \gamma_3)) = -3$ for $k \geq 1$.
- Under this assumption, we observed that the index $\Omega_{-\zeta_\star(\gamma)}(\gamma)$ for $\gamma = (n_1, n_2, n_3)$, in the anti-attractor chamber coincides with the index $\Omega_\infty(r, d, \text{ch}_2)$ counting Gieseker semi-stable sheaves provided $r > 0$ and $-r \leq d \leq 0$. But the quiver description is only supposed to be valid near the orbifold point !
- A similar conjecture for $\Omega_\star(\gamma)$ holds for any toric CY3, giving in principle access to DT invariants $\Omega_z(\gamma_i)$ at any point in Kähler moduli space. *[Mozgovoy BP'20; Descombes'21]*

Flow tree formula from scattering diagrams

- The Attractor Flow Tree formula was established rigorously using the framework of scattering diagrams *[Argüz Bousseau '20]*.
- For any quiver with potential (Q, W), the scattering diagram $\mathcal{D}$ is the set of real codimension-one rays $\{\mathcal{R}^+(\gamma), \gamma \in \mathbb{Z}^{Q_0}\} \subset \mathbb{R}^{Q_0}$ defined by $\mathcal{R}^+(\gamma) = \{\zeta : (\zeta, \gamma) = 0, \Omega_\zeta(k\gamma) \neq 0 \text{ for some } k \geq 1\}$ *[Bridgeland'16]*.
- More generally, for any $\psi \in \mathbb{R}/2\pi\mathbb{Z}$ define

$$\mathcal{R}^+_\psi(\gamma) = \{Z : \mathrm{Re}(e^{-\mathrm{i}\psi} Z(\gamma)) = 0, \mathrm{Im}(e^{-\mathrm{i}\psi} Z(\gamma)) > 0, \Omega_\zeta(k\gamma) \neq 0\}$$

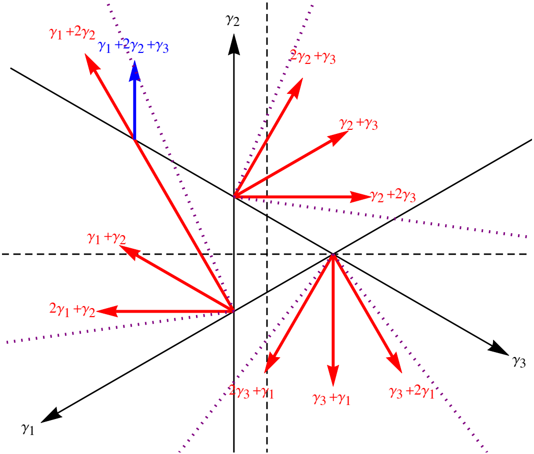
- Each point along $\mathcal{R}^+_\psi(\gamma)$ is endowed with an automorphism of the quantum torus algebra, (assume γ primitive)

$$\mathcal{U}(\gamma) = \exp\Big(\sum_{m=1}^{\infty} \frac{\bar\Omega_\zeta(k\gamma, y)}{y^{-1} - y} \mathcal{X}_{k\gamma}\Big) , \quad \mathcal{X}_\gamma \mathcal{X}_{\gamma'} = (-y)^{\langle \gamma, \gamma' \rangle} \mathcal{X}_{\gamma+\gamma'}$$

Flow tree formula from scattering diagrams

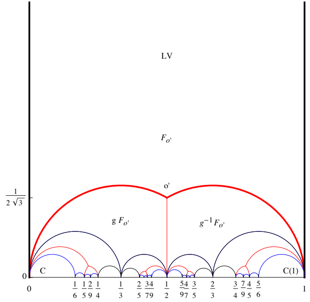
- The WCF ensures that the diagram is consistent, i.e. $\prod_{\gamma_i} \mathcal{U}(\gamma_i) = 1$ around any codimension 2 intersection. All DT invariants are determined from the initial rays, i.e. those containing the self-stability condition $(\zeta_\star(\gamma), \cdot) = \langle \gamma, \cdot \rangle$.
- Since $Z(\gamma)$ is holomorphic in the Kähler moduli, $\arg Z(\gamma)$ is preserved along the gradient flow of $|Z(\gamma)|$. Hence, the edges of attractor flow trees lie inside the rays $\mathcal{R}^+_\psi(\gamma_e)$, while vertices lie in $\mathcal{R}^+_\psi(\gamma_{L(v)}) \cap \mathcal{R}^-_\psi(\gamma_{L(v)})$.
- Thus, flow trees are subsets of scattering diagrams, determining which initial data scatter to produce an outgoing ray $\mathcal{R}^+_\psi(\gamma)$ passing through the desired point z, where $\Omega_z(k\gamma)$ can be read off.
- The Attractor Flow Tree Formula determines outgoing rays from incoming rays at each vertex. *[Argüz Bousseau '20]*.

A 2D slide of the orbifold scattering diagram



Kähler moduli space of $K_{\mathbb{P}^2}$

- The Kähler moduli space of $K_{\mathbb{P}^2}$ is the modular curve $X_1(3) = \mathbb{H}/\Gamma_1(3)$ parametrizing elliptic curves with level structure. It admits two cusps LV, C and one elliptic point o of order 3.
- The universal cover is parametrized by $\tau \in \mathbb{H}$:



$$
\begin{aligned}
Z_\tau(\gamma) &= -rT_D(\tau) + dT(\tau) - \mathrm{ch}_2 \\
T &= \int_\ell \lambda_{SW} \\
T_D &= \int_{\ell_D} \lambda_{SW}
\end{aligned}
$$

Central charge as Eichler integral

- Since $\partial_\tau \lambda_{SW}$ is holormorphic, its periods are proportional to $(1, \tau)$. Integrating on a path from o to τ, one finds the Eichler-type integral

$$\begin{pmatrix} T \\ T_D \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{3} \end{pmatrix} + \int_{\tau_o}^{\tau} \begin{pmatrix} 1 \\ u \end{pmatrix} C(u)\, \mathrm{d}u$$

  where $C(\tau) = \frac{\eta(\tau)^9}{\eta(3\tau)^3}$ is a weight 3 modular form for $\Gamma_1(3)$.

- This provides an computationally efficient analytic continuation of Z_τ throughout $\mathbb{H}$, and gives access to monodromies:

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} \quad \begin{pmatrix} 1 \\ T \\ T_D \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ m & d & c \\ m_D & b & a \end{pmatrix} \cdot \begin{pmatrix} 1 \\ T \\ T_D \end{pmatrix}$$

  where (m, m_D) are period integrals of C from τ_o to $\frac{a\tau_o - b}{c\tau_o - d}$.

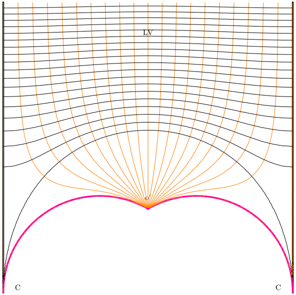
Central charge as Eichler integral

- At large volume, using $C = 1 - 9q + \ldots$ one finds

$$T = \tau + \mathcal{O}(q), \quad T_D = \frac{1}{2}\tau^2 + \frac{1}{8} + \mathcal{O}(q)$$

- For τ_2 large enough, one can use the $GL(2,\mathbb{R})^+$ action on stability conditions to absorb the $\mathcal{O}(q)$ corrections and work with

$$Z^{LV}_{(s,t)}(\gamma) = -\frac{r}{2}(s+\mathrm{i}t)^2 + d(s+\mathrm{i}t) - \mathrm{ch}_2 \ ,$$



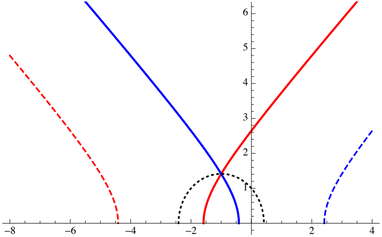
$$s = \frac{\mathrm{Im}\, T_D}{\mathrm{Im}\, T}$$

$$\frac{1}{2}(s^2+t^2) = -\frac{\mathrm{Im}(T\bar{T}_D)}{\mathrm{Im}\, T}$$

$$\mathcal{A} = \{E \overset{d}{\to} F, \mu(E) \le s, \mu(F) \ge s\}$$

Large volume scattering diagram

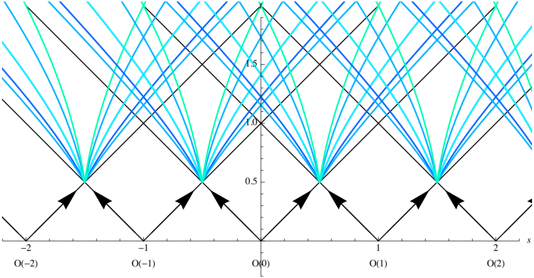
- For the stability conditions $(Z^{LV_{(s,t)}}, \mathcal{A}(s))$, *[Bousseau'19]* constructed the scattering diagram $\mathcal{D}_\psi$ in (s,t) upper half-plane for $\psi = 0$. The diagram for $\psi \neq 0$ is obtained by transforming $(s,t) \mapsto (s - t\tan\psi, t/\cos\psi)$. We shall restrict to $\psi = 0$.
- The rays $\mathcal{R}^+(\gamma)$ are branches of hyperbola asymptoting to $t = \pm(s - \frac{d}{r})$ for $r \neq 0$, or vertical lines when $r = 0$. Walls of marginal stability $\mathcal{W}(\gamma, \gamma')$ are half-circles centered on real axis.



- Think of $\mathcal{R}^+(\gamma)$ as the worldline of a particle of charge r, mass $m^2 = \frac{1}{2}d^2 - r\,\mathrm{ch}_2$ moving in a constant electric field !

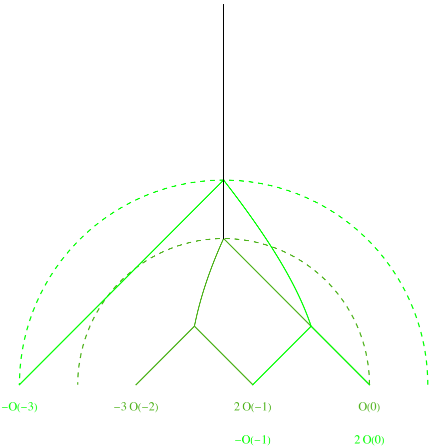
Large volume scattering diagram

- Initial rays correspond to $\mathcal{O}(m)$ and $\mathcal{O}(m)[1]$, *ie* (anti)D4-branes with m units of flux, emanating from $(s,t) = (m,0)$ on the boundary where the central charge vanishes.



- The first scatterings occur for $t \geq \frac{1}{2}$, after each constituent has moved by $|\Delta s| \geq \frac{1}{2}$. The monotonicity of the 'electric potential' $\varphi(\gamma) = d - sr$ along the flow, allows to bound the number and charge of constituents.

Flow trees for $\gamma = [0, 4, 1)$

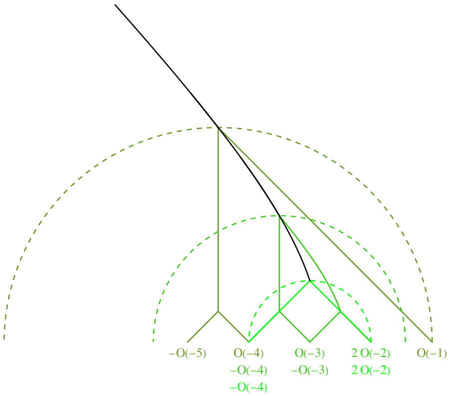


- $\{\{-3\mathcal{O}(-2), 2\mathcal{O}(-1)\}, \mathcal{O}\}$:
  $3\mathcal{O}(-2) \to 2\mathcal{O}(-1) \oplus \mathcal{O} \to E$
  $K_3(2,3)K_{12}(1,1) \to -156$

- $\{-\mathcal{O}(-3), \{-\mathcal{O}(-1), 2\mathcal{O}\}\}$:
  $\mathcal{O}(-3) \oplus \mathcal{O}(-1) \to 2\mathcal{O} \to E$
  $K_3(1,2)K_{12}(1,1) \to -36$

Total: $\Omega_\infty(\gamma) = -192 = N_4^{(0)}$

Flow trees for $\gamma = [1, 0, -3)$

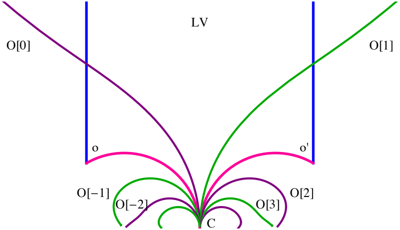


- $\{\{-\mathcal{O}(-5), \mathcal{O}(-4)\}, \mathcal{O}(-1)\}$
  $\mathcal{O}(-5) \to \mathcal{O}(-4) \oplus \mathcal{O}(-1) \to E$
  $K_3(1,1)^2 \to 9$
- $\{\{-\mathcal{O}(-4), \mathcal{O}(-3)\},$
  $\{-\mathcal{O}(-3), 2\mathcal{O}(-2)\}\}$
  $\mathcal{O}(-4) \oplus +\mathcal{O}(-3) \to$
  $\mathcal{O}(-3) \oplus +2\mathcal{O}(-2) \to E$
  $K_3(1,1)^2 K_3(1,2) \to 27$
- $\{-\mathcal{O}(-4), 2\mathcal{O}(-2)\}$
  $\mathcal{O}(-4) \to 2\mathcal{O}(-2) \to E$
  $K_6(1,2) \to 15$

Total: $\Omega_\infty(\gamma) = 51 = \chi(\mathbb{P}^{2[4]})$

Exact scattering diagram

- The full scattering diagram $\mathcal{D}^{\Pi}_{\psi}$ on the slice of Π-stability conditions should interpolate between $\mathcal{D}^{\rm LV}_{\psi}$ around $\tau = {\rm i}\infty$ and $\mathcal{D}^{o}_{\psi}$ around $\tau = \frac{e^{{\rm i}\pi/6}}{\sqrt{3}} + n$, and be invariant under the action of $\Gamma_1(3)$.
- Under $\tau \mapsto \frac{\tau}{3n\tau+1}$ with $n \in \mathbb{Z}$, $\mathcal{O} \mapsto \mathcal{O}[n]$. Hence we have an doubly infinite family of initial rays associated to $\mathcal{O}(m)[n]$.



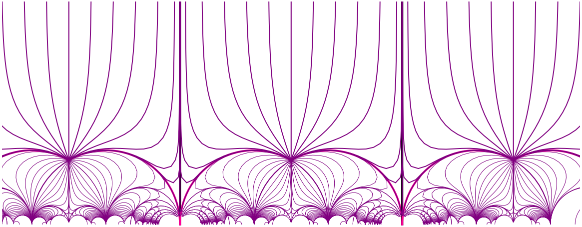
- For $|\tan\psi| < \frac{1}{2\mathcal{V}}$ where $\mathcal{V} = {\rm Im}\, T(0) = \frac{27}{4\pi^2}{\rm Im Li}_2(e^{2\pi{\rm i}/3}) \simeq 0.463$ only the rays associated to $\mathcal{O}(m)[0]$ and $\mathcal{O}(m)[1]$ escape to ${\rm i}\infty$, and merge onto rays in the large volume scattering diagram $\mathcal{D}^{\rm LV}_{\psi}$.

Exact scattering diagram

- In addition, there must be an infinite family of initial rays coming from $\tau = \frac{p}{q}$ with $q \neq 0 \mod 3$, corresponding to $\Gamma_1(3)$-images of $\mathcal{O}(0)$. This includes initial rays emitted at $\tau = n - \frac{1}{2}$, associated to $\Omega(n+1)$; for $\psi \sim \frac{\pi}{2}$, these merge onto initial rays of the orbifold scattering diagram.
- Since $\partial_\tau Z(\gamma) = (d - r\tau)C(\tau)$ and $C \neq 0$ for $\mathrm{Im}\tau > 0$, it appears that rays $\mathcal{R}_\psi(\gamma)$ can only end at $\tau = \frac{d}{r}$ such that $Z_\tau(\gamma)$ vanishes. This can be shown to hold for generic ψ, but when $\tan\psi \in \mathbb{Q}/\mathcal{V}$, a ray $\mathcal{R}(\gamma)$ emitted at $\tau = \frac{d}{r}$ might end up at $\tau' = \frac{d'}{r'}$.
- We conjecture that the only initial rays are the $\Gamma_1(3)$ images of the structure sheaf $\mathcal{O}$, each of them carrying $\Omega(k\gamma) = 1$ for $k = 1$, 0 otherwise.

Exact scattering diagram - $\psi = \pm\frac{\pi}{2} \mod 2\pi$

- For $\psi = \pm\frac{\pi}{2}$, the diagram $\mathcal{D}^{\Pi}_{\psi}$ simplifies dramatically, since the loci $\mathrm{Im} Z_\tau(\gamma) = 0$ are lines of constant $s := \frac{\mathrm{Im}\, T_D}{\mathrm{Im}\, T} = \frac{d}{r}$.



- Hence, there is no wall-crossing between τ_o and $\tau = \mathrm{i}\infty$ when $-1 \le \frac{d}{r} \le 0$, explaining why the Gieseker index $\Omega_\infty(\gamma)$ agrees with the index in the anti-attractor chamber (where the orbifold quiver with potential reduces to the Beilinson quiver with relations)

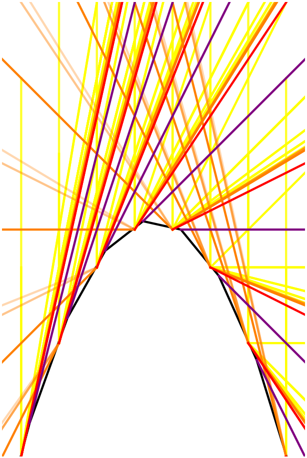
Scattering diagram in affine coordinates

- For $|\tan\psi| < \frac{1}{2\mathcal{V}}$ and fixed γ, the flow trees in $\mathcal{D}^{\Pi}_{\psi}$ are identical (topologically) to flow trees in $\mathcal{D}^{\rm LV}_{\psi}$. One way to show this is to map both of them to the plane

$$x = \frac{\mathrm{Re}\left(e^{-\mathrm{i}\psi}\, T\right)}{\cos\psi}\ , \quad y = -\frac{\mathrm{Re}\left(e^{-\mathrm{i}\psi}\, T_D\right)}{\cos\psi}\ ,$$

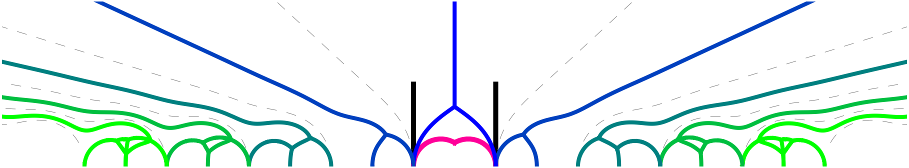
  such that $\mathcal{R}_{\psi}(\gamma)$ becomes a line segment $rx + dy - \mathrm{ch}_2 = 0$.
- The initial rays $\mathcal{R}_{\mathcal{O}(m)}$ are tangent to the parabola $y = -\frac{1}{2}x^2$ at $x = m$, but the origin of each ray is shifted to $x = m + \mathcal{V}\tan\psi$.
- In addition, there are initial rays associated to images of $\mathcal{O}(m)$ under $\Gamma_1(3)$, but those don't play a role if ψ is small enough.

Exact scattering diagram in (x, y) plane, $\psi = 1$

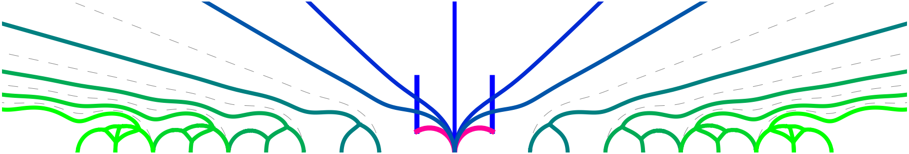


Exact scattering diagram, varying ψ

$\gamma = [0, 1, 1) = \text{ch}\,\mathcal{O}_C$:



$\gamma = [1, 0, 1) = \text{ch}\,\mathcal{O}$:



Conclusion - outlook

- The scattering diagram is the proper mathematical framework for the attractor tree conjecture in the case of local CY3. This is because the central charge is holomorphic, so the gradient flow preserves the phase of $Z(\gamma)$. Moreover, initial rays can only start from the boundary.
- This provides an effective way of computing DT invariants in any chamber, and a natural decomposition into elementary constituents. Mathematically, different trees should correspond to different strata in $\mathcal{M}_Z(\gamma)$, but the precise relation is not clear.
- It would be interesting to extend this description to other toric CY3, such as local del Pezzo surfaces.
- In the compact case, the phase of $Z(\gamma)$ is no longer constant along the flow and there can be attractor points with $\Omega_\star(\gamma) \neq 0$ at finite distance in Kähler moduli space...

Thanks for your attention !