Counting Calabi-Yau black holes with mock modular forms

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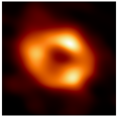
Introduction

- A central goal for any theory of quantum gravity is to provide a microscopic explanation of the thermodynamical entropy of black holes in General Relativity *[Bekenstein'72, Hawking'74]*

$$S_{BH} = \frac{A}{4G_N}$$



$$S_{BH} \stackrel{?}{=} \log \Omega$$

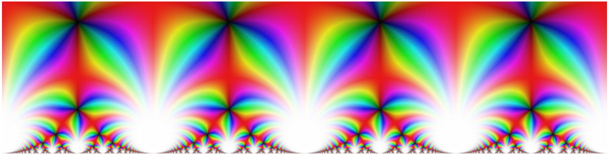
- As shown by *[Strominger Vafa'96,...]*, String Theory provides a quantitative description in the case of BPS black holes in vacua with extended SUSY: at weak string coupling, black hole micro-states arise as bound states of D-branes wrapped on cycles of the internal manifold.
- Besides confirming the consistency of string theory as a theory of quantum gravity, this has opened up many fruitful connections with mathematics.

BPS indices and Donaldson-Thomas invariants

- In the context of type IIA strings compactified on a Calabi-Yau three-fold X, BPS states are described mathematically by stable objects in the derived category of coherent sheaves $\mathcal{C} = D^b\mathrm{Coh}X$. The Chern character $\gamma = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2, \mathrm{ch}_3)$ is identified as the electromagnetic charge, or D6-D4-D2-D0-brane charge.
- The problem becomes a question in Donaldson-Thomas theory: for fixed $\gamma \in K(X)$, compute the generalized DT invariant $\Omega_z(\gamma)$ counting (semi)stable objects of class γ, and determine its growth as $|\gamma| \to \infty$.
- Importantly, $\Omega_z(\gamma)$ depends on the moduli of X, or more generally on a choice of Bridgeland stability condition $z \in \mathrm{Stab}\,\mathcal{C}$. The chamber structure is fairly simple for $X = T^6$ or $X = K3 \times T^2$, but very intricate for a general CY 3-fold.

Modularity of Donaldson-Thomas invariants

- Physical arguments predict that suitable generating series of DT invariants (those counting D4-D2-D0 bound states in a suitable chamber) should have specific modular properties. This gives very good control on their asymptotic growth, and allows to test agreement with the BH prediction $\Omega_z(\gamma) \simeq e^{S_{BH}(\gamma)}$.



- Recall that $f(\tau) = \sum_{n\geq 0} c(n) q^{n-\Delta}$ (with $q = e^{2\pi i\tau}$, $\mathrm{Im}\tau > 0$) is a *modular form* of weight w if $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subset SL(2,\mathbb{Z})$,

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^w f(\tau) \quad \Rightarrow \quad c(n) \sim \exp\left(4\pi\sqrt{\Delta n}\right)$$

Mock Modularity of Donaldson-Thomas invariants

- More precisely, these generating series are expected to be (higher depth) mock modular, similar to Ramanujan's mock theta series. The modular anomaly can be repaired by adding a universal non-holomorphic correction *[Alexandrov BP Manschot'16-20]*.
- A (depth one) mock modular form of weight w transforms inhomogeneously under $\Gamma \subset SL(2,\mathbb{Z})$,

$$f\left(\tfrac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^w \left[f(\tau) - \int_{-d/c}^{\mathrm{i}\infty} \overline{g(-\bar\rho)}(\tau+\rho)^{-w}\mathrm{d}\rho\right]$$

where the shadow $g(\tau)$ is an ordinary modular form of weight $2-w$. Equivalently, the non-holomorphic completion

$$\widehat{f}(\tau,\bar\tau) := f(\tau) + \int_{-\bar\tau}^{\mathrm{i}\infty} \overline{g(-\bar\rho)}(\tau+\rho)^{-w}\mathrm{d}\rho\ , \quad \tau_2^w \partial_{\bar\tau}\widehat{f}(\tau,\bar\tau) \propto \overline{g(\tau)}$$

transforms like a modular form *[Zagier'1973, Zwegers'02]*

Outline

In this talk, I will explain how to combine knowledge of standard Gromov-Witten invariants (counting curves in X) and wall-crossing arguments to rigorously compute many DT invariants, and check mock modularity to high precision

S. Alexandrov, S. Feyzbakhsh, A. Klemm, BP, T. Schimannek, arXiv:2301.08066

S. Alexandrov, S. Feyzbakhsh, A. Klemm, BP, arXiv:2312.12629

1. Reminder of enumerative invariants on CY3: GW, GV, DT, PT. . .
2. Mock modularity of D4-D2-D0 generating series
3. From rank 1 to rank 0 DT invariants, and back
4. Testing modularity on X_5 and other hypergeometric models
5. Conclusion and open problems

Gromov-Witten invariants

- Let X be a smooth, projective CY threefold. The Gromov-Witten invariants $\mathrm{GW}_\beta^{(g)}$ count genus g curves Σ with class $\beta \in H_2(X, \mathbb{Z})$. They depend only on the symplectic structure (or Kähler moduli) of X and in general take rational values.
- Physically, they determine certain higher-derivative couplings in the low energy effective action, which depend only on the (complexified) Kähler moduli t and receive worldsheet instanton corrections: $F_g(t) = \sum_\beta \mathrm{GW}_\beta^{(g)}\, e^{2\pi \mathrm{i} t \cdot \beta}$ *[Antoniadis Gava Narain Taylor'93]*
- The first two F_0 and F_1 can be computed using mirror symmetry. Holomorphic anomaly equations along with suitable boundary conditions allow to determine $F_{g\geq 2}$ up to a certain genus $g_{\rm int}$ ($= 53$ for the quintic threefold X_5) *[Bershadsky Cecotti Ooguri Vafa'93; Huang Klemm Quackenbush'06]*

Gopakumar-Vafa invariants

- Gromov-Witten invariants turn out to be determined by a set of integer invariants $\mathrm{GV}_\beta^{(g)}$ via [Gopakumar Vafa'98,Ionel Parker'13]

$$\sum_{g=0}^{\infty}\sum_{\beta}\mathrm{GW}_\beta^{(g)}\,\lambda^{2g-2}e^{2\pi\mathrm{i}t\cdot\beta}=\sum_{g=0}^{\infty}\sum_{k=1}^{\infty}\sum_{\beta}\frac{GV_\beta^{(g)}}{k}\left(2\sin\frac{k\lambda}{2}\right)^{2g-2}e^{2\pi\mathrm{i}kt\cdot\beta}$$

For $g=0$, this reduces to [Candelas de la Ossa Greene Parkes'93]

$$GW_\beta^{(0)}=\sum_{k|\beta}\frac{1}{k^3}GV_{\beta/k}^{(0)}$$

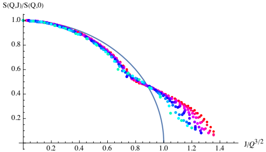
- Physically, $GV_\beta^{(0)}$ counts D2-D0 brane bound states with D2 charge β, and arbitrary D0 charge n ,while higher genus GV invariants keep track of their angular momentum.
- Importantly, $GV_\beta^{(g)}$ vanishes for large enough $g\geq g_{\max}(\beta)$ (Castelnuovo bound).

GV invariants and 5D rotating black holes

- Viewing type II string theory as M-theory on a circle, D2-branes lift to M2-branes wrapped on curve inside X, yielding BPS black holes in $\mathbb{R}^{1,4}$. These carry in general two angular momenta (j_L, j_R).
- Tracing over j_R, the number of BPS states with $m = j_L^z$ is

$$\Omega_{5D}(\beta, m) = \sum_{g=0}^{g_{\max}(\beta)} \binom{2g+2}{g+1+m} \mathrm{GV}_\beta^{(g)}$$

Katz Klemm Vafa'99



- There is some numerical evidence that $\Omega(\beta, m) \sim e^{2\pi\sqrt{\beta^3 - m^2}}$ for large β keeping m^2/β^3 fixed, in agreement with the BH entropy of 5D black holes *[Klemm Marino Tavanfar'07]*, with a transition to black rings at large angular momentum *[Halder Lin'23]*.

Generalized Donaldson-Thomas invariants

- More generally, bound states of D6-D4-D2-D0 branes are described by stable objects in the bounded derived category of coherent sheaves $\mathcal{C} = D^b\mathrm{Coh}(X)$ *[Kontsevich'95, Douglas'01]*. Objects are bounded complexes $E = (\cdots \to \mathcal{E}_{-1} \to \mathcal{E}_0 \to \mathcal{E}_1 \to \dots)$ of coherent sheaves $\mathcal{E}_k$, graded by the total Chern character $\gamma(E) = \sum_k (-1)^k \,\mathrm{ch}\,\mathcal{E}_k \in \Gamma$
- Stability depends on a choice of stability condition $\sigma = (Z, \mathcal{A})$, where the central charge $Z \in \mathrm{Hom}(\Gamma, \mathbb{C})$ and the heart $\mathcal{A} \subset \mathcal{C}$ satisfy various axioms *[Bridgeland 2007]*, in particular
  1. $\forall E \in \mathcal{A},\ \mathrm{Im} Z(E) \geq 0$
  2. $\forall E \in \mathcal{A},\ \mathrm{Im} Z(E) = 0 \Rightarrow \mathrm{Re} Z(E) < 0$
- The generalized Donaldson-Thomas invariant $\Omega_\sigma(\gamma)$ is roughly the weighted Euler number of the moduli space $M_\sigma(\gamma)$ of semi-stable objects $E \in \mathcal{A}$ with $\mathrm{ch}\, E = \gamma$, where semi-stability means that $\arg Z(E') \leq \arg Z(E)$ for any subobject $E' \subset E$.

Generalized Donaldson-Thomas invariants

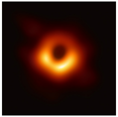
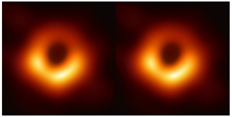
- The space of stability conditions $\mathrm{Stab}\,\mathcal{C}$ is a complex manifold of dimension $\dim K_{\rm num}(X) = 2b_2(X) + 2$, *unless it is empty* *[Bridgeland'07]*.
- Stability conditions in the vicinity of the large volume point can be constructed subject to a conjectural Bogomolov-Gieseker-type inequality introduced in *[Bayer Macri Toda'11]* – more on this later.
- The BMT inequality is very hard to prove for a general compact CY3, but has been proven for the quintic threefold X_5 *[Li'18]* and a couple of other examples *[Koseki'20, Liu'21]*.

Generalized Donaldson-Thomas invariants

- $\Omega_\sigma(\gamma)$ may jump on co-dimension 1 walls in $\mathrm{Stab}\,\mathcal{C}$ where some the central charge $Z(\gamma')$ of a subobject $E' \subset E$ becomes aligned with $Z(\gamma)$. The jump is governed by a universal wall-crossing formula *[Joyce Song'08, Kontsevich Soibelman'08]*. In simplest primitive case,

$$\Delta\Omega_\sigma(\gamma_1+\gamma_2) = \langle\gamma_1,\gamma_2\rangle\,\Omega_\sigma(\gamma_1)\,\Omega_\sigma(\gamma_2)$$

  corresponding physically to the (dis)appearance of multi-centered black hole bound states *[Denef Moore'07; Andriyash Denef Jafferis Moore'10; Manschot BP Sen'10]*

- For $\gamma = (0, 0, \beta, n)$, $\Omega_\sigma(\gamma)$ coincides with $GV^{(0)}_\beta$ at large volume.

GV invariants and D6-brane bound states

- For $\gamma = (-1, 0, \beta, -n)$ at large volume and B-field, stable objects have a much simpler mathematical description in terms of stable pairs $E : \mathcal{O}_X \xrightarrow{s} F$ *[Pandharipande Thomas'07]*:
  1. F is a pure 1-dimensional sheaf with $\mathrm{ch}_2 F = \beta$ and $\chi(F) = n$
  2. the section s has zero-dimensional kernel

  The PT invariant $\mathrm{PT}(\beta, n)$ is defined as the (weighted) Euler characteristic of the corresponding moduli space.
- Since a single D6-brane lifts to a Taub-NUT space in M-theory, which is locally flat, one expects that PT invariants are computable from GV invariants *[Dijkgraaf Vafa Verlinde'06]*.

GV invariants and D6-brane bound states

- More precisely, PT invariants are related to GV invariants by *[Maulik Nekrasov Okounkov Pandharipande'06]*

$$\sum_{\beta,n} \mathrm{PT}(\beta,n)\, e^{2\pi \mathrm{i} t\cdot\beta} q^n = \mathrm{Exp}\left(\sum_{\beta,g} \mathrm{GV}^{(g)}_{\beta} \, (\sqrt{q} - 1/\sqrt{q})^{2g-2} e^{2\pi \mathrm{i} t\cdot\beta} \right)$$

  where $\mathrm{Exp}(f(q)) = \exp(\sum_{n\geq 1} f(q^n))$ is the plethystic exponential.

- Under this relation, the Castelnuovo bound $GV_{\beta}^{(g\geq g_{\max}(\beta))} = 0$ is mapped to $PT(\beta, n \leq 1 - g_{\max}(\beta)) = 0$
- The main interest in this talk will be on rank 0 DT invariants $\Omega(0, p, \beta, n)$ counting D4-D2-D0 brane bound states supported on a divisor $\mathcal{D}$ with class $[\mathcal{D}] = p \in H_4(X, \mathbb{Z})$.

D4-D2-D0 indices as rank 0 DT invariants

- Viewing IIA=M/S^1, D4-D2-D0 branes on $\mathcal{D}$ arise from M5-branes wrapped on $\mathcal{D} \times S^1$. In the limit where S^1 is much larger than X, they are described by a two-dimensional superconformal field theory with $(0, 4)$ SUSY. *[Maldacena Strominger Witten'97]*
- D4-D2-D0 indices occur as Fourier coefficients in the elliptic genus $\mathrm{Tr}(-1)^F q^{L_0 - \frac{c_L}{24}} e^{2\pi \mathrm{i} q_a z^a}$. If the SCFT has a discrete spectrum, after theta series decomposition with respect to the elliptic variables z^a, one obtains a vector-valued modular form

$$h_{p,\mu}(\tau) := \sum_n \bar{\Omega}(0, p, \mu, n)\, q^{n - \frac{\chi(\mathcal{D})}{24} + \frac{1}{2}\mu^2 - \frac{1}{2}p\mu}$$

where μ takes values in the finite discriminant group Λ^*/Λ associated to $\Lambda = (H_4(X, \mathbb{Z}), \kappa_{ab} := \kappa_{abc} p^c)$.

Modularity of rank 0 DT invariants

- When $\mathcal{D}$ is very ample and irreducible, there are no walls extending to large volume, so the choice of chamber is irrelevant. The central charges are given by *[Maldacena Strominger Witten'97]*

$$\begin{cases} c_L = & p^3 + c_2(TX)\cdot p = \chi(\mathcal{D}) \\ c_R = & p^3 + \frac{1}{2}c_2(TX)\cdot p = 6\chi(\mathcal{O}_{\mathcal{D}}) \end{cases}$$

  Cardy's formula predicts a growth $\Omega(0,p,\beta,n\to\infty)\sim e^{2\pi\sqrt{p^3\,n}}$ in perfect agreement with Bekenstein-Hawking formula !
- Moreover, since the space of vector-valued weakly holomorphic modular form has finite dimension, the full series is completely determined by its polar coefficients, with $n+\frac{1}{2}\mu^2-\frac{1}{2}p\mu<\frac{\chi(\mathcal{D})}{24}$. *(Actually, the dimension can be smaller than the number of polar terms)*.

Mock modularity of rank 0 DT invariants

- When $\mathcal{D}$ is reducible, the generating series $h_{p^a,\mu_a}(\tau)$ in a suitable ("large volume attractor") chamber is expected to be a mock modular form of higher depth *[Alexandrov BP Manschot'16-20]*)
- Namely, there exists explicit, universal non-holomorphic theta series $\Theta_n(\{p_i\},\tau,\bar\tau)$ such that (ignoring the μ's for simplicity)

$$\widehat{h}_p(\tau,\bar\tau) = h_p(\tau) + \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \Theta_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^{n} h_{p_i}(\tau)$$

transforms as a modular form. The completed series satisfy the holomorphic anomaly equation,

$$\partial_{\bar\tau}\widehat{h}_p(\tau,\bar\tau) = \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \widehat{\Theta}_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^{n} \widehat{h}_{p_i}(\tau,\bar\tau)$$

Mock modularity of rank 0 DT invariants

- For binary splittings, this reduces to mock modular forms encountered in the study of BPS dyons in Type II on $K3 \times T^2$, or in heterotic string on T^6 *[Dabholkar Murthy Zagier'12]*.
- The modular completion is constructed using similar ideas as in Zwegers's work on Ramanujan's mock theta series, namely replacing "step functions" with "generalized error functions" *[Alexandrov Banerjee BP Manschot'16]*.
- Our derivation relied on the study of instanton corrections to the QK metric on the moduli space after compactifying on a circle, and implementing $SL(2,\mathbb{Z})$ symmetry manifest from $IIA/S^1 = M/T^2$. A nice spin off of earlier research on hypermultiplet moduli spaces !

Alexandrov Banerjee Persson BP Manschot Saueressig Vandoren, 2008-19

Crash course on Indefinite theta series

- Θ_n and $\widehat{\Theta}_n$ belongs to the class of indefinite theta series

$$\vartheta_{\Phi,q}(\tau,\bar\tau) = \tau_2^{-\lambda} \sum_{k\in\Lambda+q} \Phi\left(\sqrt{2\tau_2}k\right)\, e^{-\mathrm{i}\pi\tau Q(k)}$$

  where (Λ, Q) is an even lattice of signature $(r, d-r)$, $q \in \Lambda^*/\Lambda$, $\lambda \in \mathbb{R}$. The series converges if $f(x) \equiv \Phi(x) e^{\frac{\pi}{2}Q(x)} \in L_1(\Lambda\otimes\mathbb{R})$.
- Theorem (Vignéras, 1978): $\{\vartheta_{\Phi,q}, q \in \Lambda^*/\Lambda\}$ transforms as a vector-valued modular form of weight $(\lambda + \frac{d}{2}, 0)$ provided
  - $R(x)f, R(\partial_x)f \in L_2(\Lambda\otimes\mathbb{R})$ for any polynomial $R(x)$ of degree ≤ 2
  - $\left[\partial_x^2 + 2\pi(x\partial_x - \lambda)\right]\Phi = 0$ [*]
- The operator $\partial_{\bar\tau}$ acts by sending $\Phi \to (x\partial_x - \lambda)\Phi$. Thus ϑ is holomorphic if Φ is homogeneous. But unless $r = 0$, $f(x)$ will fail to be square-integrable !

Indefinite theta series

- Example 1 (Siegel): $\Phi = e^{\pi Q(x_+)}$, where x_+ is the projection of x on a fixed plane of dimension r, satisfies [*] with $\lambda = -n$. ϑ_Φ is then the usual (non-holomorphic) Siegel-Narain theta series.
- Example 2 (Zwegers): In signature $(1, d-1)$, choose C, C' two vectors such that $Q(C), Q(C'), (C, C') > 0$, then

$$\widehat{\Phi}(x) = \mathrm{Erf}\left(\frac{(C,x)\sqrt{\pi}}{\sqrt{Q(C)}}\right) - \mathrm{Erf}\left(\frac{(C',x)\sqrt{\pi}}{\sqrt{Q(C')}}\right)$$

satisfies [*] with $\lambda = 0$. As $|x| \to \infty$, or if $Q(C) = Q(C') = 0$,

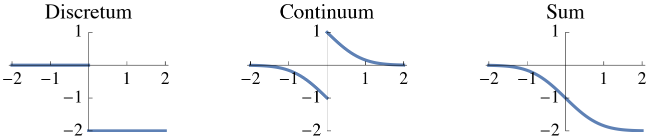
$$\widehat{\Phi}(x) \to \Phi(x) := \mathrm{sgn}(C, x) - \mathrm{sgn}(C', x)$$

- The theta series $\Theta_2(\{p_1, p_2\})$, $\widehat{\Theta}_2(\{p_1, p_2\})$ fall in this class. The generalization to $n > 2$ involves generalized error functions.

Alexandrov Banerjee Manschot BP 2016; Nazaroglu 2016

Non-holomorphic completion from Witten index

- Physically, the non-holomorphic corrections arise from the spectral asymmetry in the continuum of scattering states in the supersymmetric quantum mechanics of n BPS black holes.



BP 2015; Murthy BP 2018; BP Raj, in progress

Testing mock modularity for one-parameter models

- In the remainder of this talk, we shall test these modularity predictions for CY threefolds with Picard rank 1, by computing the first few coefficients in the q-expansion and determine the putative vector-valued (mock) modular form.
- This was first attempted by *[Gaiotto Strominger Yin '06-07]* for the quintic threefold X_5 and a few other hypergeometric models. They were able to guess the first few terms for unit D4-brane charge, and found a unique modular completion.
- We shall compute many terms rigorously, using recent results by *[Soheyla Fezbakhsh and Richard Thomas'20-22]* relating rank r DT invariants (including $r = 0$, counting D4-D2-D0 bound states) to PT invariants, hence to GV invariants.

Alexandrov, Feyzbakhsh, Klemm, BP, Schimannek'23

From rank 1 to rank 0 DT invariants

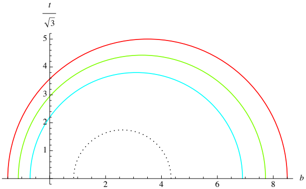
- The key idea is to study wall-crossing in the space of Bridgeland stability conditions, away from the physical slice. For any $b + \mathrm{i}t \in \mathbb{H}$, consider the central charge

$$Z_{b,t}(E) = \frac{\mathrm{i}}{6} t^3 \,\mathrm{ch}_0^b(E) - \frac{1}{2} t^2 \,\mathrm{ch}_1^b(E) - \mathrm{i}t \,\mathrm{ch}_2^b(E) + 0 \,\mathrm{ch}_3^b(E)$$

  with $\mathrm{ch}_k^b(E) := \int_X H^{3-k} e^{-bH} \mathrm{ch}(E)$. With a suitable choice of heart (defined by tilting with respect to the slope $\frac{\mathrm{ch}_1^b(E)}{\mathrm{rk}(E)}$), this defines a weak stability condition called tilt-stability.
- Note that $Z_{b,t}(E)$ is obtained from $Z^{\mathrm{LV}}(E) = -\int_X e^{(b+\mathrm{i}t)H} \mathrm{ch}(E)$ by setting by hand the coefficient of ch_3^b to 0. In fact, tilt-stability is the first step in constructing genuine stability conditions near the large volume point *[Bayer Macri Toda'11]*
- The KS/JS wall-crossing formulae still hold for such weak stability conditions.

Rank 0 DT invariants from GV invariants

- Tilt stability agrees with slope stability at large volume, but the chamber structure is much simpler: walls are nested half-circles in the Poincaré upper half-plane spanned by $z = b + \mathrm{i}\frac{t}{\sqrt{3}}$.



- Importantly, for any tilt-semistable object E there is a conjectural inequality on Chern classes $C_i := \int_X \mathrm{ch}_i(E).H^{3-i}$ *[Bayer Macri Toda'11; Bayer Macri Stellari'16]*

$$(C_1^2 - 2C_0C_2)(\tfrac{1}{2}b^2 + \tfrac{1}{6}t^2) + (3C_0C_3 - C_1C_2)b + (2C_2^2 - 3C_1C_3) \geq 0$$

Rank 0 DT invariants from GV invariants

- In particular, if the discriminant $\Delta(C)$ at $t=0$ is positive, there exists an empty chamber ! $\Delta(\gamma)$ is quartic in the charges,

$$\Delta(C) = 8C_0C_2^3 + 6C_1^3C_3 + 9C_0^2C_3^2 - 3C_1^2C_2^2 - 18C_0C_1C_2C_3 \geq 0$$

- Remarkably, $\Delta(C)$ is proportional to (minus) the quartic invariant $I_4(Q)$ which determines the entropy $S_{BH} \sim \pi\sqrt{I_4(Q)}$ of single-centered black holes ! In particular, *an empty chamber exists whenever single-centered black hole are ruled out !*
- Consider an anti-D6-brane with charge $\gamma = (-1, 0, \beta, -n)$ such that $\Delta(C) > 0$. By studying wall-crossing between the empty chamber where $\Omega_{b,t}(\gamma) = 0$ and the large volume chamber where $\Omega_{b,t}(\gamma) = \mathrm{PT}(\beta, m)$, one can extract the indices of the D4-D2-D0 branes emitted at each wall !

A new explicit formula (S. Feyzbakhsh'23)

<u>Theorem</u> Let (X, H) be a smooth polarised CY threefold with $\mathrm{Pic}(X) = \mathbb{Z}.H$ satisfying the BMT conjecture. There is $f(x)$ such that

- If $\frac{m}{\beta \cdot H} < f(\frac{\beta, H}{H})$ then the stable pair invariant $\mathrm{PT}(\beta, m) =$

$$\sum_{(m', \beta')} (-1)^{\chi_{m',\beta'}} \chi_{m',\beta'} \mathrm{PT}(\beta', m') \, \Omega \left(0, 1, \frac{H^2}{2} - \beta' + \beta \, , \, \frac{H^3}{6} + m' - m - \beta'.H\right)$$

  where $\chi_{m',\beta'} = \beta.H + \beta'.H + m - m' - \frac{H^3}{6} - \frac{1}{12} c_2(X).H$.
- The sum runs over $(\beta', m') \in H_2(X, \mathbb{Z}) \oplus H_0(X, \mathbb{Z})$ such that

$$0 \leq \beta'.H \leq \frac{H^3}{2} + \frac{3mH^3}{2\beta.H} + \beta.H$$

$$-\frac{(\beta'.H)^2}{2H^3} - \frac{\beta'.H}{2} \leq \; m' \; \leq \frac{(\beta.H - \beta'.H)^2}{2H^3} + \frac{\beta.H + \beta'.H}{2} + m$$

  In particular, $\beta'.H < \beta.H$.

<u>Corollary</u> (Castelnuovo bound): $\mathrm{PT}(\beta, m) = 0$ unless $m \geq -\frac{(\beta.H)^2}{2H^3} - \frac{\beta.H}{2}$

Modularity for one-modulus compact CY

- Using the theorem above and known GV invariants, we could compute a large number of coefficients in the generating series of Abelian (=unit D4-brane charge) rank 0 DT invariants in one-parameter hypergeometric threefolds, including the quintic X_5.
- In all cases (except $X_{3,2,2}, X_{2,2,2,2}$ where current knowledge of GV invariants is insufficient), we found a linear combination of the following vv modular forms matching all computed coeffs:

$$\frac{E_4^a E_6^b}{\eta^{4\kappa+c_2}} D^\ell(\vartheta_\mu^{(\kappa)}) \quad \text{with} \quad \vartheta_\mu^{(\kappa)} = \sum_{k \in \mathbb{Z}+\frac{\mu}{\kappa}+\frac{1}{2}} q^{\frac{1}{2}\kappa k^2}, \quad \kappa := H^3$$

where $D = 2\pi\mathrm{i}\partial_\tau - \frac{w}{12}E_2$, and $4a + 6b + 2\ell - 2\kappa - \frac{1}{2}c_2 = -2$.

Modularity for one-modulus compact CY

X	χ_X	κ	$c_2(TX)$	$\chi(\mathcal{O}_\mathcal{D})$	n_1	C_1
$X_5(1^5)$	-200	5	50	5	7	0
$X_6(1^4,2)$	-204	3	42	4	4	0
$X_8(1^4,4)$	-296	2	44	4	4	0
$X_{10}(1^3,2,5)$	-288	1	34	3	2	0
$X_{4,3}(1^5,2)$	-156	6	48	5	9	0
$X_{4,4}(1^4,2^2)$	-144	4	40	4	6	1
$X_{6,2}(1^5,3)$	-256	4	52	5	7	0
$X_{6,4}(1^3,2^2,3)$	-156	2	32	3	3	0
$X_{6,6}(1^2,2^2,3^2)$	-120	1	22	2	1	0
$X_{3,3}(1^6)$	-144	9	54	6	14	1
$X_{4,2}(1^6)$	-176	8	56	6	15	1
$X_{3,2,2}(1^7)$	-144	12	60	7	21	1
$X_{2,2,2,2}(1^8)$	-128	16	64	8	33	3

Modular predictions for the quintic threefold

- Using known $GV_\beta^{(g\leq 53)}$ we can compute more than 20 terms:

$$h_0 = q^{-\frac{55}{24}} \Big(\underline{5 - 800q + 58500q^2} + 5817125q^3 + 75474060100q^4$$
$$+ 28096675153255q^5 + 3756542229485475q^6$$
$$+ 277591744202815875q^7 + 13610985014709888750q^8 + \ldots\Big),$$
$$h_{\pm 1} = q^{-\frac{55}{24}+\frac{3}{5}} \Big(\underline{0 + 8625q} - 1138500q^2 + 3777474000q^3$$
$$+ 3102750380125q^4 + 577727215123000q^5 + \ldots\Big)$$
$$h_{\pm 2} = q^{-\frac{55}{24}+\frac{2}{5}} \Big(\underline{0 + 0q} - 1218500q^2 + 441969250q^3 + 953712511250q^4$$
$$+ 217571250023750q^5 + 22258695264509625q^6 + \ldots\Big)$$

Modular predictions for the quintic threefold

- The space of vv modular forms has dimension 7. Remarkably, all terms above are reproduced by *[Gaiotto Strominger Yin'06]*

$$h_\mu = \frac{1}{\eta^{55+15}} \left[-\frac{222887 E_4^8 + 1093010 E_4^5 E_6^2 + 177095 E_4^2 E_6^4}{35831808} + \frac{25\left(458287 E_4^6 E_6 + 967810 E_4^3 E_6^3 + 66895 E_6^5\right)}{53747712} D + \frac{25\left(155587 E_4^7 + 1054810 E_4^4 E_6^2 + 282595 E_4 E_6^4\right)}{8957952} D^2 \right] \vartheta_\mu^{(5)}$$

- Physically, polar coefficients are expected arise as bound states of D6-brane and anti D6-branes *[Denef Moore'07]*. Indeed, they are often consistent with the naive ansatz *[Alexandrov Gaddam Manschot BP'22]*

$$\Omega(0,1,\beta,n) = \pm(\chi(\mathcal{O}_{\mathcal{D}}) - \beta.H - n)\, DT(\beta,n) PT(0,0)$$

but deviations do occur ! *[Collinucci Wyder'08, van Herck Wyder'09]*

Mock modularity for non-Abelian D4-D2-D0 indices

- For D4-D2-D0 indices with $N = 2$ units of D4-brane charge, $\{h_{2,\mu}, \mu \in \mathbb{Z}/(2\kappa\mathbb{Z})\}$ should transform as a vector-valued mock modular form with modular completion

$$\widehat{h}_{2,\mu}(\tau,\bar\tau) = h_{2,\mu}(\tau) + \sum_{\mu_1,\mu_2=0}^{\kappa-1} \delta^{(\kappa)}_{\mu_1+\mu_2-\mu}\, \Theta^{(\kappa)}_{\mu_2-\mu_1+\kappa}\, h_{1,\mu_1}\, h_{1,\mu_2}$$

where (denoting $\beta(x) = 2|x|^{-1/2} e^{-\pi x} - 2\pi \mathrm{Erfc}(\sqrt{\pi |x|})$)

$$\Theta^{(\kappa)}_{\mu}(\tau,\bar\tau) := \frac{(-1)^\mu}{8\pi} \sum_{k\in 2\kappa\mathbb{Z}+\mu} |k|\, \beta\left(\frac{\tau_2 k^2}{\kappa}\right) e^{-\frac{\pi \mathrm{i}\tau}{2\kappa}\, k^2},$$

$$\partial_{\bar\tau}\Theta^{(\kappa)}_{\mu} = \frac{(-1)^\mu \sqrt{\kappa}}{16\pi \mathrm{i} \tau_2^{3/2}} \sum_{k\in 2\kappa\mathbb{Z}+\mu} e^{\frac{-\pi \mathrm{i}\bar\tau}{2\kappa}\, k^2}$$

Mock modularity for non-Abelian D4-D2-D0 indices

- Suppose there exists a holomorphic function $g_\mu^{(\kappa)}$ such that $\Theta_\mu^{(\kappa)} + g_\mu^{(\kappa)}$ transforms as a vv modular form. Then

$$\widetilde{h}_{2,\mu}(\tau,\bar\tau) = h_{2,\mu}(\tau) - \sum_{\mu_1,\mu_2=0}^{\kappa-1} \delta^{(\kappa)}_{\mu_1+\mu_2-\mu}\, g^{(\kappa)}_{\mu_2-\mu_1+\kappa}\, h_{1,\mu_1}\, h_{1,\mu_2}$$

will be an ordinary weak holomorphic vv modular form, hence uniquely determined by its polar part.

- For $\kappa=1$, the series $\Theta_\mu^{(1)}$ is the one appearing in the modular completion of the generating series of Hurwitz class numbers *[Hirzebruch Zagier 1973]* (or rank 2 Vafa-Witten invariants on $\mathbb{P}^2$)

$$H_0(\tau) = -\tfrac{1}{12} + \tfrac{1}{2}q + q^2 + \tfrac{4}{3}q^3 + \tfrac{3}{2}q^4 + \dots$$
$$H_1(\tau) = q^{\frac{3}{4}}\left(\tfrac{1}{3} + q + q^2 + 2q^3 + q^4 + \dots\right)$$

Thus we can choose $g_\mu^{(1)} = H_\mu(\tau)$.

Mock modularity for non-Abelian D4-D2-D0 indices

X	χ_X	κ	c_2	$\chi(\mathcal{O}_{2\mathcal{D}})$	n_2	C_2
$X_5(1^5)$	-200	5	50	15	36	1
$X_6(1^4,2)$	-204	3	42	11	19	1
$X_8(1^4,4)$	-296	2	44	10	14	1
$X_{10}(1^3,2,5)$	-288	1	34	7	7	0
$X_{4,3}(1^5,2)$	-156	6	48	16	42	0
$X_{4,4}(1^4,2^2)$	-144	4	40	12	25	1
$X_{6,2}(1^5,3)$	-256	4	52	14	30	1
$X_{6,4}(1^3,2^2,3)$	-156	2	32	8	11	1
$X_{6,6}(1^2,2^2,3^2)$	-120	1	5	2	5	0
$X_{3,3}(1^6)$	-144	9	54	21	78	3
$X_{4,2}(1^6)$	-176	8	56	20	69	3
$X_{3,2,2}(1^7)$	-144	12	60	26	117	0
$X_{2,2,2,2}(1^8)$	-128	16	64	32	185	4

Mock modularity for non-Abelian D4-D2-D0 indices

- For X_{10}, we computed the 7 polar terms + 4 non-polar terms and found a unique mock modular form reproducing this data:

$$h_{2,\mu} = \frac{5397523E_4^{12}+70149738E_4^9E_6^2-12112656E_4^6E_6^4-61127530E_4^3E_6^6-2307075E_6^8}{46438023168\eta^{100}}\vartheta_\mu^{(1,2)} + \frac{-10826123E_4^{10}E_6-14574207E_4^7E_6^3+20196255E_4^4E_6^5+5204075E_4E_6^7}{1934917632\eta^{100}}D\vartheta_\mu^{(1,2)} + (-1)^{\mu+1}H_{\mu+1}(\tau)\,h_1(\tau)^2$$

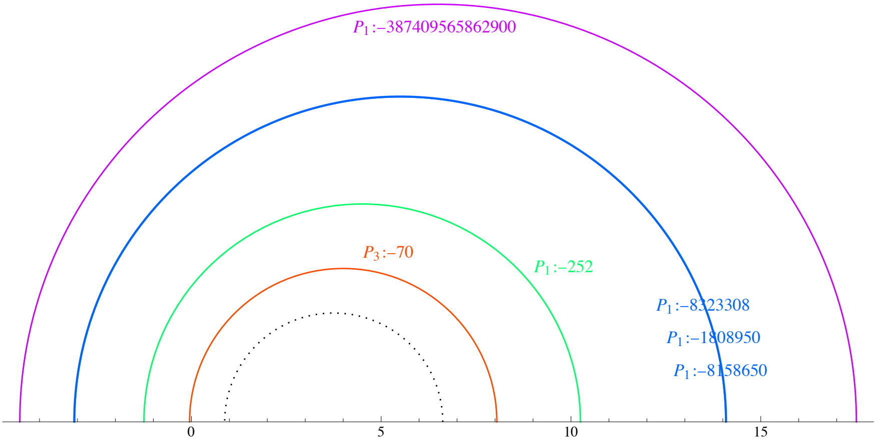
with $h_1 = \frac{203E_4^4+445E_4E_6^2}{216\,\eta^{35}} = q^{-\frac{35}{24}}(\underline{3-575q}+\dots)$, leading to integer DT invariants

$$h_{2,0}^{(\rm int)} = q^{-\frac{19}{6}}\left(\underline{7-1728q+203778q^2-13717632q^3}-23922034036q^4+\dots\right)$$

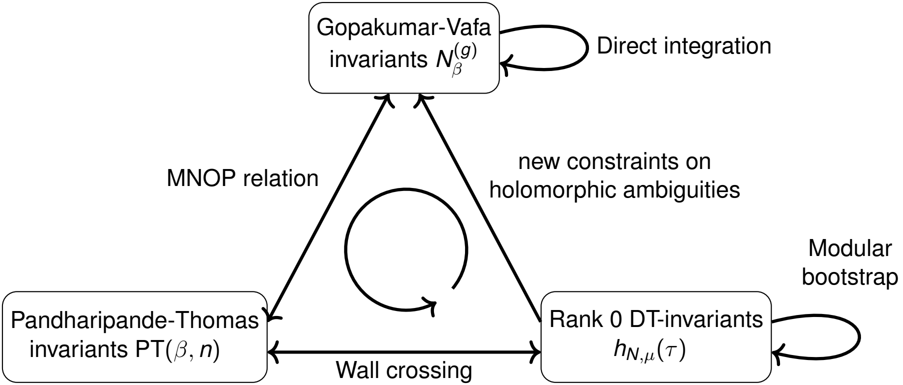
$$h_{2,1}^{(\rm int)} = q^{-\frac{35}{12}}\left(\underline{-6+1430q-1086092q^2}+208065204q^3+\dots\right)$$

- Similar results for X_8 *[S. Alexandrov, S. Feyzbakhsh, A. Klemm'23]*

Computing the leading term in $h_{2,0}$ for X_{10}



Quantum geometry from stability and modularity



Alexandrov Feyzbakhsh Klemm BP Schimannek'23

Quantum geometry from stability and modularity

X	χ_X	κ	type	g_{integ}	$g^{(1)}_{\text{mod}}$	$g^{(2)}_{\text{mod}}$	g_{avail}
$X_5(1^5)$	-200	5	F	53	69	80	64
$X_6(1^4,2)$	-204	3	F	48	66	84	48
$X_8(1^4,4)$	-296	2	F	60	84	112	66
$X_{10}(1^3,2,5)$	-288	1	F	50	70	95	72
$X_{4,3}(1^5,2)$	-156	6	F	20	24		24
$X_{6,4}(1^3,2^2,3)$	-156	2	F	14	17		17
$X_{6,6}(1^2,2^2,3^2)$	-120	1	K	18	22		26
$X_{4,4}(1^4,2^2)$	-144	4	K	26	34		34
$X_{3,3}(1^6)$	-144	9	K	29	33		33
$X_{4,2}(1^6)$	-176	8	C	50	66		64
$X_{6,2}(1^5,3)$	-256	4	C	63	78		49

http://www.th.physik.uni-bonn.de/Groups/Klemm/data.php

Summary and open questions

- We provided overwhelming evidence that D4-D2-D0 indices exhibit mock modular properties. Where does it come from mathematically ? Is there some VOA acting on the cohomology of moduli space of stable objects, à la *[Nakajima'94]* ?
- Can one test modularity in multi-parameter models, for example in genus-one fibrations or K3-fibrations ? Can one follow D4-D2-D0 invariants through extremal transitions ?
- Similar wall-crossing arguments also allow to compute higher rank DT invariants. Is there some higher rank version of *[MNOP'03]* ?
- A long-standing problem: incorporate NS5-instanton corrections to the QK metric on hypermultiplet moduli space, consistently with S-duality, beyond the linear analys of *[Alexandrov Persson BP'10]*.
- Thanks for your attention !

Back up slide: Modularity from geometry

- While modularity of D4-D2-D0 invariants is clear physically from the M5-brane picture, its mathematical origin is in general mysterious.
- When X admits a K3-fibration, using the relation to Noether-Lefschetz invariants one can show that modularity holds for vertical D4-brane charge. The modular anomaly disappears entirely due to $\kappa_{ab}p^b = 0$. *[Bouchard Creutzig Diaconescu Doran Quigley Sheshmani'16; Doran BP Schimannek'24]*
- Similarly, when X admits a genus-one fibration, one can relate D4-D2-D0 invariants for a D4-brane wrapping the fiber to GW invariants via Fourier-Mukai duality. Generating series of GW invariants are quasi-modular forms, consistent with $\kappa_{ab}p^a p^b = 0$. *[Klemm Manschot Wotschke'12; BP Schimannek, to appear.]*