Modularity of Donaldson-Thomas invariants on Calabi-Yau threefolds

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"Arithmetic of CY manifolds"
Mainz Institute of Theoretical Physics, 20/03/2025

Main references

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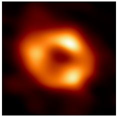
Introduction

- A driving force in high energy theory has been the quest for a microscopic explanation of the Bekenstein-Hawking entropy of black holes.

$$S_{BH} = \frac{A}{4G_N}$$



$$S_{BH} \stackrel{?}{=} \log \Omega$$

- As demonstrated by *[Strominger Vafa'95,...]*, String Theory provides a quantitative description in the context of BPS black holes in vacua with extended SUSY: at weak string coupling, black hole micro-states arise as bound states of D-branes wrapped on cycles of the internal manifold, and can (often) be counted accurately.
- Besides confirming the consistency of string theory as a theory of quantum gravity, this has opened up many fruitful connections with mathematics.

BPS indices and Donaldson-Thomas invariants

- In the context of type IIA strings compactified on a Calabi-Yau three-fold X, BPS states are described mathematically by stable objects in the derived category of coherent sheaves $\mathcal{C} = D^b\mathrm{Coh}X$. The Chern character $\gamma = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2, \mathrm{ch}_3)$ is identified as the electromagnetic charge, or D6-D4-D2-D0-brane charge.
- The problem becomes a question in Donaldson-Thomas theory: for fixed $\gamma \in K(X)$, compute the generalized DT invariant $\Omega_z(\gamma)$ counting (semi)stable objects of class γ for a Bridgeland stability condition $z \in \mathrm{Stab}\,\mathcal{C}$, and determine its growth as $|\gamma| \to \infty$.
- Physical arguments predict that suitable generating series of rank zero DT invariants (counting D4-D2-D0 bound states, $\mathrm{ch}_0 = 0$) should have specific mock modular properties. This gives very good control on their asymptotic growth, and allows to test whether it agrees with the BH prediction $\Omega_z(\gamma) \simeq e^{S_{BH}(\gamma)}$.

Introduction

- Today, I will explain how to combine knowledge of standard Gromov-Witten invariants (counting curves in X) and wall-crossing arguments to rigorously compute many rank 0 DT invariants, and check mock modularity to high precision.
- Conversely, postulating (mock) modularity one can compute an infinite number of rank 0 DT invariants, and obtain new constraints on Gromov-Witten invariants, allowing to compute them to higher genus than ever before.
- I will mostly restrict to one-parameter hypergeometric models such as the quintic threefold. Time permitting, I will discuss some multi-parameter models with K3 or elliptic fibrations at the end.

Gromov-Witten invariants

- Let X be a smooth, projective CY threefold. The Gromov-Witten invariants $\mathrm{GW}_\beta^{(g)}$ count genus g curves Σ with class $\beta \in H_2^{\rm eff}(X,\mathbb{Z})$. They depend only on the symplectic structure (or Kähler moduli) of X and take rational values.
- Physically, they determine certain protected couplings of the form $F_g(t) R^2 W^{2g-2}$ in the low energy effective action, which depend only on the complexified Kähler moduli t and receive worldsheet instanton corrections: $F_g(t) = \sum_\beta \mathrm{GW}_\beta^{(g)}\, e^{2\pi \mathrm{i} t\cdot\beta}$

Antoniadis Gava Narain Taylor'93

- The first two F_0 and F_1 can be computed using mirror symmetry. Holomorphic anomaly equations along with boundary conditions near the discriminant locus and MUM points allow to determine $F_{g\geq 2}$ up to a certain genus $g_{\rm int}$ ($=53$ for the quintic threefold X_5).

Bershadsky Cecotti Ooguri Vafa'93; Huang Klemm Quackenbush'06

Gopakumar-Vafa invariants

- While GW invariants take rational values, the Gopakumar-Vafa invariants $\mathrm{GV}_\beta^{(g)}$ defined by the 'multicover' formula

$$\sum_{g=0}^{\infty}\sum_{\beta}\mathrm{GW}_\beta^{(g)}\,\lambda^{2g-2}e^{2\pi\mathrm{i}t\cdot\beta}=\sum_{g=0}^{\infty}\sum_{k=1}^{\infty}\sum_{\beta}\frac{GV_\beta^{(g)}}{k}\left(2\sin\frac{k\lambda}{2}\right)^{2g-2}e^{2\pi\mathrm{i}kt\cdot\beta}$$

  take integer values. For $g=0$, $\mathrm{GW}_\beta^{(0)}=\sum_{k|\beta}\frac{1}{k^3}GV_{\beta/k}^{(0)}$. Moreover, $GV_\beta^{(g)}$ vanishes for large enough $g\geq g_{\max}(\beta)$ *[Ionel Parker'13]*

- Physically, $GV_\beta^{(0)}$ counts D2-D0 brane bound states with D2 charge β, and arbitrary D0 charge n, while $GV_\beta^{(g\geq 1)}$ keep track of their angular momentum (more on this below).
- The formula above arises from a one-loop Schwinger-type computation of the effective action in a constant graviphoton background $W\propto\lambda$ *[Gopakumar Vafa'98]*

GV invariants and 5D rotating black holes

- Viewing type II string theory as M-theory on a circle, D2-branes lift to M2-branes wrapped on a curve inside X, yielding BPS black holes in $\mathbb{R}^{1,4}$. These carry in general two angular momenta (j_L, j_R).
- Tracing over j_R, the number of BPS states with $m = j_L^z$ is *[Katz Klemm Vafa'99]*

$$\Omega_{5D}(\beta, m) = \sum_{g=0}^{g_{\max}(\beta)} \binom{2g+2}{g+1+m} \mathrm{GV}_\beta^{(g)}$$

There is numerical evidence that $\Omega(\beta, m) \sim e^{2\pi\sqrt{\beta^3 - m^2}}$ for large β keeping m^2/β^3 fixed, in agreement with the Bekenstein-Hawking entropy of 5D black holes ! *[Klemm Marino Tavanfar'07]*.

Generalized Donaldson-Thomas invariants

- More generally, bound states of D6-D4-D2-D0 branes are described by stable objects in the bounded derived category of coherent sheaves $D^b\mathrm{Coh}(X)$ *[Kontsevich'95, Douglas'01]*. Objects are complexes $E = (\dots \to \mathcal{E}_{-1} \to \mathcal{E}_0 \to \mathcal{E}_1 \to \dots)$ of sheaves $\mathcal{E}_k$, graded by the total Chern character $\gamma(E) = \sum_K (-1)^k \operatorname{ch} \mathcal{E}_k$.
- Stable objects are counted by the generalized Donaldson-Thomas invariant $\Omega_\sigma(\gamma)$, where $\gamma \in K_{\mathrm{num}}(\mathcal{C}) \sim \mathbb{Z}^{2b_2(X)+2}$ and $\sigma = (Z, \mathcal{A})$ is a stability condition in the sense of *[Bridgeland 2007]*. In particular,
  1. $\forall E \in \mathcal{A},\ \mathrm{Im} Z(E) \geq 0$
  2. $\forall E \in \mathcal{A}, \mathrm{Im} Z(E) = 0 \Rightarrow \mathrm{Re} Z(E) < 0$
- The space of stability conditions $\mathtt{Stab}\,\mathcal{C}$ is a complex manifold of dimension $\dim K_{\mathrm{num}}(X) = 2b_2(X) + 2$, *unless it is empty*.
- For X a projective CY3, stability conditions are only known to exist for the quintic threefold X_5 and a couple of other examples *[Li'18, Koseki'20, Liu'21]*

Generalized Donaldson-Thomas invariants

- $\Omega_\sigma(\gamma)$ is roughly the weighted Euler number of the moduli space of semi-stable objects $M_\sigma(\gamma)$, where semi-stability means that $\arg Z(E') \le \arg Z(E)$ for any subobject $E' \subset E$.
- $\Omega_\sigma(\gamma)$ may jump on co-dimension 1 walls in $\mathrm{Stab}\,\mathcal{C}$ where some the central charge $Z(\gamma')$ of a subobject $E' \subset E$ becomes aligned with $Z(\gamma)$. The jump is governed by a universal wall-crossing formula *[Joyce Song'08, Kontsevich Soibelman'08]*. In simplest primitive case,

$$\Delta\Omega_\sigma(\gamma_1+\gamma_2) = \langle\gamma_1,\gamma_2\rangle\,\Omega_\sigma(\gamma_1)\,\Omega_\sigma(\gamma_2)$$

- For $\gamma = (0, 0, \beta, n)$, $\Omega_\sigma(\gamma)$ coincides with $GV_\beta^{(0)}$ at large volume.

PT invariants and (anti)D6-D2-D0 brane bound states

- For $\gamma = (-1, 0, \beta, -n)$ at large volume and B-field, stable objects have a much simpler mathematical description in terms of stable pairs $E : \mathcal{O}_X \xrightarrow{s} F$ *[Pandharipande Thomas'07]*:
  1. F is a pure 1-dimensional sheaf with $\mathrm{ch}_1 F = \beta$ and $\chi(F) = n$
  2. the section s has zero-dimensional kernel

  The PT invariant $\mathrm{PT}(\beta, n)$ is defined as the (weighted) Euler characteristic of the corresponding moduli space.
- Since a single (anti)D6-brane lifts to a Taub-NUT space in M-theory, which is locally flat, one expects that PT invariants are determined by GV invariants *[Dijkgraaf Vafa Verlinde'06]*.

GV invariants and D6-brane bound states

- More precisely, PT invariants are related to GV invariants by *[Maulik Nekrasov Okounkov Pandharipande'06]*

$$\sum_{\beta,n} \mathrm{PT}(\beta,n)\, e^{2\pi \mathrm{i} t\cdot\beta} q^n = \mathrm{Exp}\left(\sum_{\beta,g} \mathrm{GV}_\beta^{(g)}\, (q^{1/2}-q^{-1/2})^{2g-2} e^{2\pi \mathrm{i} t\cdot\beta} \right)$$

  where $\mathrm{Exp}(f(q)) = \exp(\sum_{n\geq 1} f(q^n))$ is the plethystic exponential.
- Under this relation, the Castelnuovo bound $GV_\beta^{(g\geq g_{\max}(\beta))} = 0$ translates to $PT(\beta, n\leq 1-g_{\max}(\beta)) = 0$
- For n close to the Castelnuovo bound, one has $\mathrm{PT}(\beta,n) = \sum_{g=1}^{g_{\max}(\beta)} \binom{2g-2}{g-1-n} \mathrm{GV}_\beta^{(g)} + \mathcal{O}(GV^2)$, reminiscent of the KKV relation $\Omega_{5D}(\beta,m) = \sum_{g=0}^{g_{\max}(\beta)} \binom{2g+2}{g+1+m} \mathrm{GV}_\beta^{(g)}$.

D4-D2-D0 indices as rank 0 DT invariants

- The main interest in this talk will be on rank 0 DT invariants $\Omega(0,p,\beta,n)$ counting D4-D2-D0 brane bound states supported on a divisor $\mathcal{D}$ with class $[\mathcal{D}] = p \in H_4(X,\mathbb{Z})$.
- Viewing IIA=M/S^1, they arise from M5-branes wrapped on $\mathcal{D} \times S^1$. In the limit where S^1 is much larger than X, they are described by a two-dimensional superconformal field theory with $(0,4)$ SUSY.

  [Maldacena Strominger Witten'97]
- D4-D2-D0 indices (in a suitable chamber) arise as Fourier coefficients of the elliptic genus ($q := e^{2\pi i\tau}$)

$$\mathrm{Tr}(-1)^{2J_3} q^{L_0-\frac{c_L}{24}} \bar{q}^{\bar{L}_0-\frac{c_R}{24}} e^{2\pi i q_a z^a} = \sum_{\mu\in\Lambda/\Lambda^*} h_{p,\mu}(\tau)\Theta_\mu(\tau,\bar{\tau},z)$$

$$h_{p^a,\mu_a}(\tau) := \sum_n \Omega(0,p^a,\mu_a,n)\, q^{n-\frac{\chi(\mathcal{D})}{24}+\frac{1}{2}\mu_a\kappa^{ab}\mu_b-\frac{1}{2}p^a\mu_a}$$

where $\Lambda = H_4(X,\mathbb{Z})$ equipped with $\kappa_{ab} := \kappa_{abc}p^c$.

Modularity of rank 0 DT invariants

- When $\mathcal{D}$ is very ample and irreducible, there are no walls extending to large volume, so the choice of chamber is irrelevant. The central charges of the SCFT are given by *[Maldacena Strominger Witten'97]*

$$\begin{cases} c_L = & p^3 + c_2(TX)\cdot p = \chi(\mathcal{D}) \, , \\ c_R = & p^3 + \frac{1}{2} c_2(TX)\cdot p = 6\chi(\mathcal{O}_{\mathcal{D}}) \end{cases}$$

  Cardy's formula predicts a growth $\Omega(0,p,\beta,n\to\infty) \sim e^{2\pi\sqrt{p^3\, n}}$ in perfect agreement with Bekenstein-Hawking formula
- The generating series $h_{p,\mu}(\tau)$ should be a vector-valued, weakly holomorphic modular form of weight $w = -\frac{1}{2}b_2(X) - 1$ in the Weil representation of the lattice Λ. It is then completely determined by its polar coefficients, with $n + \frac{1}{2}\mu_a\kappa^{ab}\mu_b - \frac{1}{2}p^a\mu_a < \frac{\chi(\mathcal{D})}{24}$.

Mock modularity of rank 0 DT invariants

- When $\mathcal{D}$ is reducible, the generating series $h_{p,\mu}(\tau)$ of DT invariants $\Omega_*(0,p,\beta,n)$ in a suitable ("large volume attractor") chamber is expected to be a vector-valued mock modular form of higher depth *[Alexandrov BP Manschot'16-20]*)
- Namely, there exists explicit non-holomorphic theta series $\Theta_n(\{p_i\},\tau,\bar\tau)$ such that (omitting the μ's for simplicity)

$$\widehat{h}_p(\tau,\bar\tau) = h_p(\tau) + \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \Theta_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^{n} h_{p_i}(\tau)$$

  transforms as a vv modular form of weight $-\frac{1}{2}b_2(X)-1$.
- The derivation relies on the study of instanton corrections to the low energy effective action after compactifying on a circle, and implementing $SL(2,\mathbb{Z})$ symmetry manifest from $IIA/S^1 = M/T^2$.

Testing mock modularity for one-parameter models

- Our aim will be to test this prediction for CY threefolds with Picard rank 1, by computing the first few coefficients in the q-expansion and determine the putative vector-valued modular form.
- This was first attempted by *[Gaiotto Strominger Yin '06-07]* for the quintic threefold X_5 and a few other hypergeometric models. They were able to guess the first few terms for unit D4-brane charge, and found a unique modular completion.
- We shall compute many terms rigorously, using recent results by *[Soheyla Fezbakhsh and Richard Thomas'20-22]* relating rank r DT invariants (for any r including $r = 0$, relevant for D4-D2-D0 bound states) to PT invariants, hence to GV invariants.

Alexandrov, Feyzbakhsh, Klemm, BP, Schimannek'23

From rank 1 to rank 0 DT invariants

- The key idea is to use wall-crossing in a family of weak stability conditions (aka tilt-stability) parametrized by $b + \mathrm{i}t \in \mathbb{H}$, with central charge

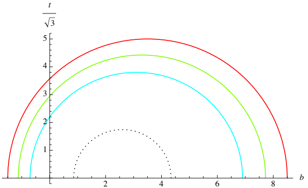
$$Z_{b,t}(E) = \frac{\mathrm{i}}{6} t^3 \,\mathrm{ch}_0^b(E) - \frac{1}{2} t^2 \,\mathrm{ch}_1^b(E) - \mathrm{i}t \,\mathrm{ch}_2^b(E) + 0 \,\mathrm{ch}_3^b(E)$$

  with $\mathrm{ch}_k^b(E) := \int_X H^{3-k} e^{-bH} \mathrm{ch}(E)$. The heart $\mathcal{A}_b$ is generated by length-two complexes $\mathcal{E}_{-1} \to \mathcal{F}_0$ with $(\mathcal{E}, \mathcal{F})$ slope semi-stable sheaves with $\mathrm{ch}_1^b(\mathcal{E}) > 0, \mathrm{ch}_1^b(\mathcal{F}) \leq 0$.

- Note that $Z_{b,t}(E)$ is obtained from $Z^{\mathrm{LV}}(E) = -\int_X e^{(b+\mathrm{i}t)H} \mathrm{ch}(E)$ by setting by hand the coefficient of ch_3^b to 0. In fact, tilt-stability provides the first step in constructing genuine stability conditions near the large volume point *[Bayer Macri Toda'11]*
- The KS/JS wall-crossing formulae hold for such weak stability conditions.

Rank 0 DT invariants from GV invariants

- Tilt stability agrees with slope stability at large volume, but the chamber structure is much simpler: walls are nested half-circles in the Poincaré upper half-plane spanned by $z = b + \mathrm{i}\frac{t}{\sqrt{3}}$.



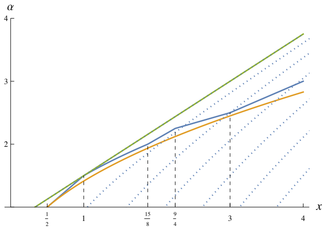
- Importantly, tilt-semistable objects E satisfy a conjectural inequality on Chern classes $C_i := \int_X \mathrm{ch}_i(E).H^{3-i}$ *[Bayer Macri Toda'11; Bayer Macri Stellari'16]*

$$(C_1^2 - 2C_0C_2)(\tfrac{1}{2}b^2 + \tfrac{1}{6}t^2) + (3C_0C_3 - C_1C_2)b + (2C_2^2 - 3C_1C_3) \geq 0$$

Rank 0 DT invariants from PT invariants

- By studying wall-crossing between the empty chamber provided by BMT bound and large volume, *[Feyzbakhsh Thomas'20-22]* show that D4-D2-D0 indices can be computed from PT invariants, and vice-versa.
- Let (X, H) be a smooth polarised CY threefold with $\mathrm{Pic}(X) = \mathbb{Z}.H$ satisfying the BMT conjecture. Aim: compute $PT(\beta, m) = \lim_{t\to\infty} \Omega_{b,t}(-1, 0, \beta, -m)$ by wall-crossing.
- Fix $m \in \mathbb{Z}, \beta \in H_2(X, \mathbb{Z})$ and define $x = \frac{\beta.H}{H^3}, \alpha = -\frac{3m}{2\beta.H}$ and

$$f(x) := \begin{cases} x + \frac{1}{2} & \text{if } 0 < x < 1 \\ \sqrt{2x + \frac{1}{4}} & \text{if } 1 < x < \frac{15}{8} \\ \frac{2}{3}x + \frac{3}{4} & \text{if } \frac{15}{8} \leq x < \frac{9}{4} \\ \frac{1}{3}x + \frac{3}{2} & \text{if } \frac{9}{4} \leq x < 3 \\ \frac{1}{2}x + 1 & \text{if } 3 \leq x \end{cases}$$



A new explicit formula (S. Feyzbakhsh'23)

Theorem (wall-crossing for $\gamma = (-1, 0, \beta, -m)$):

- If $\alpha > f(x)$ then the stable pair invariant $\mathrm{PT}(\beta, m)$ equals

$$\sum_{(\beta', m')} (-1)^{\chi_{\beta',m'}} \chi_{\beta',m'} \mathrm{PT}(\beta', m')\, \Omega\left(0, 1, \beta - \beta' + \frac{H^2}{2}, m' - m - \beta'.H + \frac{H^3}{6}\right)$$

  where $\chi_{\beta',m'} = \beta.H + \beta'.H + m - m' - \frac{H^3}{6} - \frac{1}{12} c_2(X).H$.

- The sum runs over $(\beta', m') \in H_2(X, \mathbb{Z}) \oplus H_0(X, \mathbb{Z})$ such that

$$0 \leq \beta'.H \leq \frac{H^3}{2} + \frac{3mH^3}{2\beta.H} + \beta.H$$

$$-\frac{(\beta'.H)^2}{2H^3} - \frac{\beta'.H}{2} \leq m' \leq \frac{(\beta.H - \beta'.H)^2}{2H^3} + \frac{\beta.H + \beta'.H}{2} + m$$

  In particular, $\beta'.H < \beta.H$.

Corollary (Castelnuovo bound): $\mathrm{PT}(\beta, m) = 0$ unless $m \geq -\frac{(\beta.H)^2}{2H^3} - \frac{\beta.H}{2}$

Modularity for one-modulus compact CY

- Using the previous theorem and known GV invariants, we could compute a large number of coefficients in the generating series of Abelian (=unit D4-brane charge) rank 0 DT invariants in one-parameter hypergeometric threefolds, including the quintic X_5.
- In all cases (except $X_{3,2,2}, X_{2,2,2,2}$ where current knowledge of GV invariants is insufficient), we found a linear combination of the following vv modular forms matching all computed coeffs:

$$\frac{E_4^a E_6^b}{\eta^{4\kappa+c_2}} D^\ell(\vartheta_\mu^{(\kappa)}) \quad \text{with} \quad \vartheta_\mu^{(\kappa)} = \sum_{k \in \mathbb{Z}+\frac{\mu}{\kappa}+\frac{1}{2}} q^{\frac{1}{2}\kappa k^2}, \quad \kappa := H^3$$

where $D = 2\pi\mathrm{i}\partial_\tau - \frac{w}{12}E_2$, and $4a + 6b + 2\ell - 2\kappa - \frac{1}{2}c_2 = -2$.

Modularity for one-modulus compact CY

X	χ_X	κ	$c_2(TX)$	$\chi(\mathcal{O}_\mathcal{D})$	n_1	C_1
$X_5(1^5)$	-200	5	50	5	7	0
$X_6(1^4,2)$	-204	3	42	4	4	0
$X_8(1^4,4)$	-296	2	44	4	4	0
$X_{10}(1^3,2,5)$	-288	1	34	3	2	0
$X_{4,3}(1^5,2)$	-156	6	48	5	9	0
$X_{4,4}(1^4,2^2)$	-144	4	40	4	6	1
$X_{6,2}(1^5,3)$	-256	4	52	5	7	0
$X_{6,4}(1^3,2^2,3)$	-156	2	32	3	3	0
$X_{6,6}(1^2,2^2,3^2)$	-120	1	22	2	1	0
$X_{3,3}(1^6)$	-144	9	54	6	14	1
$X_{4,2}(1^6)$	-176	8	56	6	15	1
$X_{3,2,2}(1^7)$	-144	12	60	7	21	1
$X_{2,2,2,2}(1^8)$	-128	16	64	8	33	3

Modular predictions for the quintic threefold

- Using known $GV_\beta^{(g\leq 53)}$ we can compute more than 20 terms:

$$h_0 = q^{-\frac{55}{24}} \Big(\underline{5 - 800q + 58500q^2} + 5817125q^3 + 75474060100q^4 + 28096675153255q^5 + 3756542229485475q^6 + 277591744202815875q^7 + 13610985014709888750q^8 + \dots\Big),$$

$$h_{\pm 1} = q^{-\frac{55}{24}+\frac{3}{5}} \Big(\underline{0 + 8625q} - 1138500q^2 + 3777474000q^3 + 3102750380125q^4 + 577727215123000q^5 + \dots\Big)$$

$$h_{\pm 2} = q^{-\frac{55}{24}+\frac{2}{5}} \Big(\underline{0 + 0q} - 1218500q^2 + 441969250q^3 + 953712511250q^4 + 217571250023750q^5 + 22258695264509625q^6 + \dots\Big)$$

Modular predictions for the quintic threefold

- The space of vv modular forms has dimension 7. Remarkably, all terms above are reproduced by *[Gaiotto Strominger Yin'06]*

$$h_\mu = \frac{1}{\eta^{55+15}} \Big[-\frac{222887 E_4^8 + 1093010 E_4^5 E_6^2 + 177095 E_4^2 E_6^4}{35831808} + \frac{25\left(458287 E_4^6 E_6 + 967810 E_4^3 E_6^3 + 66895 E_6^5\right)}{53747712}\, D + \frac{25\left(155587 E_4^7 + 1054810 E_4^4 E_6^2 + 282595 E_4 E_6^4\right)}{8957952}\, D^2 \Big] \vartheta_\mu^{(5)}$$

- Physically, polar coefficients are expected to arise as bound states of D6-brane and anti D6-branes *[Denef Moore'07]*. Indeed, they are often consistent with the naive ansatz

$$\Omega(0,1,\beta,n) = (-1)^{\chi(\mathcal{O}_\mathcal{D})-\beta.H-n+1}\left(\chi(\mathcal{O}_\mathcal{D}) - \beta.H - n\right) DT(\beta,n)$$

corresponding to bound states $(D6 + \beta D2 + nD0, \overline{D6(-1)})$

Collinucci Wyder'08, Alexandrov Gaddam Manschot BP'22

Mock modularity for non-Abelian D4-D2-D0 indices

- For D4-D2-D0 indices with $N=2$ units of D4-brane charge, $\{h_{2,\mu}, \mu\in\mathbb{Z}/(2\kappa\mathbb{Z})\}$ should transform as a vv mock modular form with modular completion

$$\widehat{h}_{2,\mu}(\tau,\bar\tau) = h_{2,\mu}(\tau) + \sum_{\mu_1,\mu_2=0}^{\kappa-1} \delta^{(\kappa)}_{\mu_1+\mu_2-\mu}\, \Theta^{(\kappa)}_{\mu_2-\mu_1+\kappa}\, h_{1,\mu_1}\, h_{1,\mu_2}$$

where (denoting $\beta(x) = 2|x|^{-1/2} e^{-\pi x} - 2\pi \mathrm{Erfc}(\sqrt{\pi|x|})$)

$$\Theta^{(\kappa)}_{\mu}(\tau,\bar\tau) := \frac{(-1)^\mu}{8\pi} \sum_{k\in 2\kappa\mathbb{Z}+\mu} |k|\, \beta\left(\frac{\tau_2 k^2}{\kappa}\right) e^{-\frac{\pi i \tau}{2\kappa}\, k^2},$$

$$\partial_{\bar\tau}\Theta^{(\kappa)}_{\mu} = \frac{(-1)^\mu\sqrt{\kappa}}{16\pi i \tau_2^{3/2}} \sum_{k\in 2\kappa\mathbb{Z}+\mu} e^{\frac{-\pi i\bar\tau}{2\kappa}\, k^2}$$

Mock modularity for non-Abelian D4-D2-D0 indices

- Suppose there exists a holomorphic function $g_\mu^{(\kappa)}$ such that $\Theta_\mu^{(\kappa)} + g_\mu^{(\kappa)}$ transforms as a vv modular form. Then

$$\widetilde{h}_{2,\mu}(\tau,\bar\tau) = h_{2,\mu}(\tau) - \sum_{\mu_1,\mu_2=0}^{\kappa-1} \delta^{(\kappa)}_{\mu_1+\mu_2-\mu}\, g^{(\kappa)}_{\mu_2-\mu_1+\kappa}\, h_{1,\mu_1}\, h_{1,\mu_2}$$

will be an ordinary weak holomorphic vv modular form, hence uniquely determined by its polar part.

- For $\kappa = 1$, the series $\Theta_\mu^{(1)}$ is the one appearing in the modular completion of the generating series of Hurwitz class numbers *[Hirzebruch Zagier 1973]* (or rank 2 Vafa-Witten invariants on $\mathbb{P}^2$)

$$H_0(\tau) = -\tfrac{1}{12} + \tfrac{1}{2}q + q^2 + \tfrac{4}{3}q^3 + \tfrac{3}{2}q^4 + \dots$$
$$H_1(\tau) = q^{\frac{3}{4}}\left(\tfrac{1}{3} + q + q^2 + 2q^3 + q^4 + \dots\right)$$

Thus we can choose $g_\mu^{(1)} = H_\mu(\tau)$.

Mock modularity for non-Abelian D4-D2-D0 indices

X	χ_X	κ	c_2	$\chi(\mathcal{O}_{2\mathcal{D}})$	n_2	C_2
$X_5(1^5)$	-200	5	50	15	36	1
$X_6(1^4,2)$	-204	3	42	11	19	1
$X_8(1^4,4)$	-296	2	44	10	14	1
$X_{10}(1^3,2,5)$	-288	1	34	7	7	0
$X_{4,3}(1^5,2)$	-156	6	48	16	42	0
$X_{4,4}(1^4,2^2)$	-144	4	40	12	25	1
$X_{6,2}(1^5,3)$	-256	4	52	14	30	1
$X_{6,4}(1^3,2^2,3)$	-156	2	32	8	11	1
$X_{6,6}(1^2,2^2,3^2)$	-120	1	5	2	5	0
$X_{3,3}(1^6)$	-144	9	54	21	78	3
$X_{4,2}(1^6)$	-176	8	56	20	69	3
$X_{3,2,2}(1^7)$	-144	12	60	26	117	0
$X_{2,2,2,2}(1^8)$	-128	16	64	32	185	4

Mock modularity for non-Abelian D4-D2-D0 indices

- For X_{10}, we computed the 7 polar terms + 2 non-polar terms and found a unique mock modular form reproducing this data:

$$h_{2,\mu} = \frac{5397523E_4^{12}+70149738E_4^9E_6^2-12112656E_4^6E_6^4-61127530E_4^3E_6^6-2307075E_6^8}{46438023168\eta^{100}}\vartheta_\mu^{(1,2)}$$
$$+\frac{-10826123E_4^{10}E_6-14574207E_4^7E_6^3+20196255E_4^4E_6^5+5204075E_4E_6^7}{1934917632\eta^{100}}D\vartheta_\mu^{(1,2)}$$
$$+(-1)^{\mu+1}H_{\mu+1}(\tau)\,h_1(\tau)^2$$

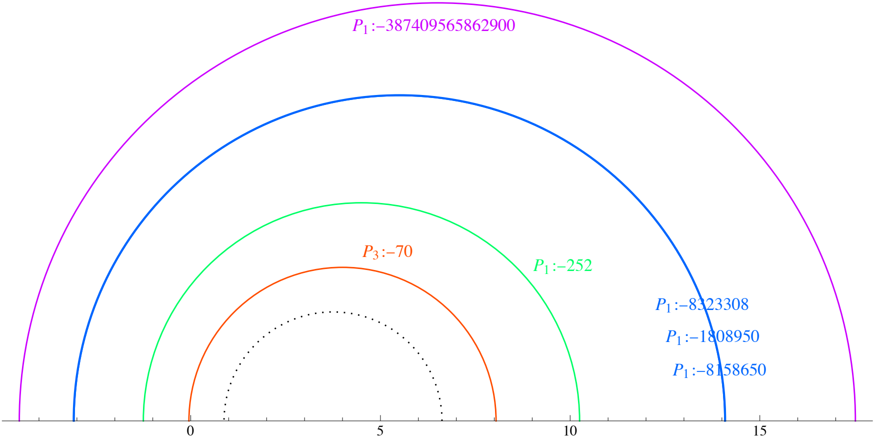
with $h_1 = \frac{203E_4^4+445E_4E_6^2}{216\,\eta^{35}} = q^{-\frac{35}{24}}(\underline{3-575q}+\dots)$, leading to integer DT invariants

$$h_{2,0}^{(\text{int})} = q^{-\frac{19}{6}}\left(\underline{7-1728q+203778q^2-13717632q^3}-23922034036q^4+\dots\right.$$
$$h_{2,1}^{(\text{int})} = q^{-\frac{35}{12}}\left(\underline{-6+1430q-1086092q^2}+208065204q^3+\dots\right)$$

- Similar results for X_8 *[S. Alexandrov, S. Feyzbakhsh, A. Klemm'23]*

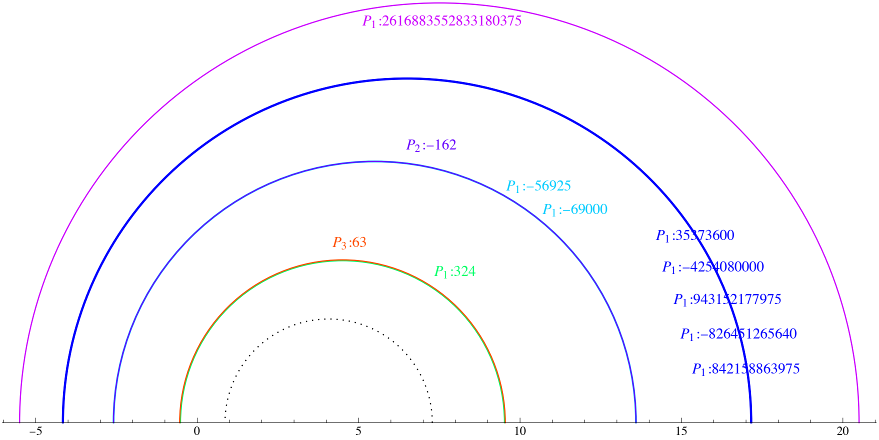
Computing the leading term in $h_{2,0}$ for X_{10}

Wall crossing for $\gamma = (-1, 6, 0, 15)$:



Computing the leading term in $h_{2,1}$ for X_{10}

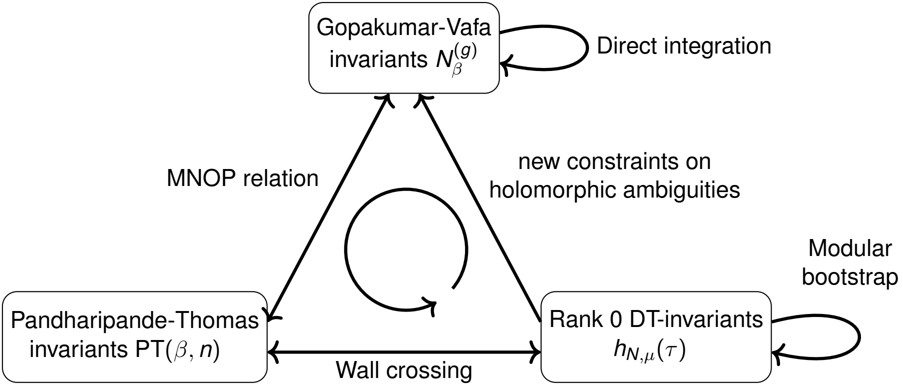
Wall crossing for $\gamma = (-1, 7, 0, 19)$:



Comments

- A particular solution of the modular anomaly equations is known for arbitrary $\kappa = H^3$ and D4-brane charge *[Alexandrov Bendriss'24]*, so given the polar coefficients, one can in principle determine the full mock modular generating series.
- Unfortunately, current knowledge of GV invariants remains insufficient to compute non-Abelian D4-D2-D0 invariants for other hypergeometric models, as well as Abelian D4-D2-D0 indices in other one-parameter examples in the AESZ list.
- Using the modular predictions for D4-D2-D0 invariants, we can run the algorithm in reverse to compute GV invariants to higher genus than hitherto possible (next slides).
- Deviations from the naive ansatz for polar terms remain to be understood *[van Herck Wyder'09']*
- Recently, *[McGovern'24]* has computed Abelian D4-D2-D0 invariants for several one-parameter models with non-trivial fondamental group (including the quotient $X_5/\mathbb{Z}_5$).

Quantum geometry from stability and modularity



Alexandrov Feyzbakhsh Klemm BP Schimannek'23

Quantum geometry from stability and modularity

X	χ_X	κ	type	$g_{\rm integ}$	$g^{(1)}_{\rm mod}$	$g^{(2)}_{\rm mod}$	$g_{\rm avail}$
$X_5(1^5)$	-200	5	F	53	69	80	64
$X_6(1^4,2)$	-204	3	F	48	66	84	48
$X_8(1^4,4)$	-296	2	F	60	84	112	64
$X_{10}(1^3,2,5)$	-288	1	F	50	70	95	68
$X_{4,3}(1^5,2)$	-156	6	F	20	24		24
$X_{6,4}(1^3,2^2,3)$	-156	2	F	14	17		17
$X_{6,6}(1^2,2^2,3^2)$	-120	1	K	18	22		22
$X_{4,4}(1^4,2^2)$	-144	4	K	26	34		34
$X_{3,3}(1^6)$	-144	9	K	29	33		33
$X_{4,2}(1^6)$	-176	8	C	50	66		50
$X_{6,2}(1^5,3)$	-256	4	C	63	78		49

http://www.th.physik.uni-bonn.de/Groups/Klemm/data.php

Modularity from geometry

- While modularity of D4-D2-D0 invariants is clear physically from the M5-brane picture, its mathematical origin is still mysterious. Presumably it should come from the action of some VOA on the cohomology of the moduli space of stable sheaves, in the spirit of *[Nakajima'94]*.
- When X admits a K3-fibration, using the relation to Noether-Lefschetz invariants one can show that modularity holds for vertical D4-brane charge. The modular anomaly disappears entirely due to $\kappa_{ab}p^b = 0$. *[Bouchard Creutzig Diaconescu Doran Quigley Sheshmani'16]*
- Similarly, when X admits a genus-one fibration, one can relate D4-D2-D0 invariants for a D4-brane wrapping the fiber to GW invariants via monodromy. Generating series of GW invariants are quasi-modular forms, consistent with $\kappa_{ab}p^a p^b = 0$. *[Klemm Manschot Wotschke'12; BP Schimannek, to appear.]*

Modularity for two-parameter K3-fibered models

- In recent work *[Doran BP Schimannek'24]*, we constructed a family of 2-parameter CY threefolds $X_m^{i,j]}$, fibered by Picard-rank 1 K3 surfaces Σ_m, and their mirror family $Y_m^{[i,j]}$, fibered by Picard-rank 19 K3 surfaces $\hat{\Sigma}_m$.
- In line with *[Doran-Harder-Thompson'17]*, both admit Tyurin degenerations corresponding to the K3-fibration on the mirror side. Moreover, they admit extremal transitions to 1-parameter models, including the 13 hypergeometric ones.
- Using mirror symmetry, we could compute vertical GV invariants and verify the modularity of NL invariants. Modularity for non-vertical D4-brane charge remains to be understood.

A family of Picard rank 2 K3-fibered threefolds $X_m^{[i,j]}$

$$h_{1,1} = 2$$

$$h_{1,2} = 22 + m(i^2 + j^2) - 2mij + h_{1,2}(F_m^{[i]}) + h_{1,2}(F_m^{[j]})$$

$$\kappa_{111} = 2m\left(\frac{1}{i} + \frac{1}{j}\right), \kappa_{112} = 2m,$$

$$\kappa_{122} = \kappa_{222} = 0$$

$$c_{2,1} = 2m(i+j) + 24\left(\frac{1}{i} + \frac{1}{j}\right)$$

$$c_{2,2} = 24$$

$$\mathrm{GV}_{0,1}^{(0)} = 2mij\,, \quad \mathrm{GV}_{0,k>0}^{(0)} = 0\,.$$

(m,i)	$h_{1,2}(F_m^{[i]})$	Construction of $F_m^{[i]}$
(1,1)	52	$\mathbb{P}_{1,1,1,1,3}[6]$
(1,2)	21	$\mathbb{P}_{1,1,1,2,3}[6]$
(2,1)	30	$\mathbb{P}^4[4]$
(2,2)	10	$\mathbb{P}_{1,1,1,1,2}[4]$
(2,4)	0	$\mathbb{P}^3$
(3,1)	20	$\mathbb{P}^5[2,3]$
(3,2)	5	$\mathbb{P}^4[3]$
(3,3)	0	$\mathbb{P}^4[2]$
(4,1)	14	$\mathbb{P}^6[2,2,2]$
(4,2)	2	$\mathbb{P}^5[2,2]$
(5,1)	10	$X^{2,5}_{\mathcal{O}(1)^{\oplus 2}\oplus\mathcal{O}(2)}$
(5,2)	0	$B_5 = X^{2,5}_{\mathcal{O}(1)^{\oplus 3}}$
(6,1)	7	$X^{2,5}_{\mathcal{S}(1)^\vee\oplus\mathcal{O}(1)}$
(7,1)	5	$X^{2,6}_{\mathcal{O}(1)^{\oplus 5}}$
(8,1)	3	$X^{3,6}_{\bigwedge^2\mathcal{S}^\vee\oplus\mathcal{O}(1)^{\oplus 3}}$
(9,1)	2	$X^{2,7}_{\mathcal{Q}^\vee(1)\oplus\mathcal{O}(1)^{\oplus 2}}$
(11,1)	0	$A_{22} = X^{3,7}_{(\bigwedge^2\mathcal{S}^\vee)^{\oplus 3}}$

Alternative realizations

(m,i,j)	χ_X		CICY	Transition
$(1,1,1)$	-252	$\mathbb{P}^4_{1,1,2,2,6}[12]$		$X_{6,2}$
$(2,1,1)$	-168	$\mathbb{P}^4_{1,1,2,2,2}[8] = \left(\begin{array}{c\|cc} \mathbb{P}^4 & 4 & 1 \\ \mathbb{P}^1 & 0 & 2 \end{array}\right)$	7886, 7888	$X_{4,2}$
$(2,4,1)$	-168	$\left(\begin{array}{c\|cc} \mathbb{P}^4 & 4 & 1 \\ \mathbb{P}^1 & 1 & 1 \end{array}\right)$	7885	X_5
$(2,4,4)$	-168	$\left(\begin{array}{c\|c} \mathbb{P}^3 & 4 \\ \mathbb{P}^1 & 2 \end{array}\right)$	7887	X_8
$(3,1,1)$	-132	$\left(\begin{array}{c\|cccc} \mathbb{P}^6 & 3 & 2 & 1 & 1 \\ \mathbb{P}^1 & 0 & 0 & 1 & 1 \end{array}\right)$	7867, 7869	$X_{3,2,2}$
$(3,2,1)$	-120	$\left(\begin{array}{c\|ccc} \mathbb{P}^5 & 2 & 3 & 1 \\ \mathbb{P}^1 & 1 & 0 & 1 \end{array}\right)$	7840	$X_{3,3}$
$(3,2,2)$	-108	$\left(\begin{array}{c\|cc} \mathbb{P}^4 & 3 & 2 \\ \mathbb{P}^1 & 0 & 2 \end{array}\right)$	7806	$X_{4,3}$
$(3,3,1)$	-140	$\left(\begin{array}{c\|ccc} \mathbb{P}^5 & 2 & 3 & 1 \\ \mathbb{P}^1 & 0 & 1 & 1 \end{array}\right)$	7873	$X_{4,2}$
$(3,3,2)$	-128	$\left(\begin{array}{c\|cc} \mathbb{P}^4 & 3 & 2 \\ \mathbb{P}^1 & 1 & 1 \end{array}\right)$	7858	X_5
$(3,3,3)$	-148	$\left(\begin{array}{c\|cc} \mathbb{P}^4 & 3 & 2 \\ \mathbb{P}^1 & 2 & 0 \end{array}\right)$	7882	$X_{6,2}$
$(4,1,1)$	-112	$\left(\begin{array}{c\|cccc} \mathbb{P}^6 & 2 & 2 & 2 & 1 \\ \mathbb{P}^1 & 0 & 0 & 0 & 2 \end{array}\right)$	7819, 7823	$X_{2,2,2,2}$
$(4,2,1)$	-112	$\left(\begin{array}{c\|cccc} \mathbb{P}^6 & 2 & 2 & 2 & 1 \\ \mathbb{P}^1 & 0 & 0 & 1 & 1 \end{array}\right)$	7817	$X_{3,2,2}$
$(4,2,2)$	-112	$\left(\begin{array}{c\|ccc} \mathbb{P}^5 & 2 & 2 & 2 \\ \mathbb{P}^1 & 0 & 1 & 1 \end{array}\right)$	7816, 7822	$X_{4,2}$

Summary and open questions

- We provided overwhelming evidence that D4-D2-D0 indices possess mock modular properties. Can one prove this mathematically, for example by relating them to suitable Noether-Lefschetz type invariants, or by constructing some VOA acting on the cohomology of moduli space of stable objects ?
- Can one compute D4-D2-D0 invariants for multi-parameter CY threefolds using wall-crossing and make similar checks of modularity ? Can one follow these invariants through extremal transitions ?
- Is there any relation at all between the modularity of generating series of D4-D2-D0 indices, and the modularity of CY periods at attractor points ?

Thanks for your attention !

Back up: Physical interpretation of the BMT inequality

$$\boxed{(C_1^2 - 2C_0C_2)(\frac{1}{2}b^2 + \frac{1}{6}t^2) + (3C_0C_3 - C_1C_2)b + (2C_2^2 - 3C_1C_3) \geq 0}$$

- Requiring the existence of an empty chamber, the discriminant at $t = 0$ must be positive:

$$8C_0C_2^3 + 6C_1^3C_3 + 9C_0^2C_3^2 - 3C_1^2C_2^2 - 18C_0C_1C_2C_3 \geq 0$$

- In terms of the electric and magnetic charges

$$p^0 = C_0/\kappa, \quad p^1 = C_1/\kappa, \quad q_1 = -C_2 - \tfrac{c_2}{24\kappa}C_0, \quad q_0 = C_3 + \tfrac{c_2}{24\kappa}C_1$$

and ignoring the c_2-dependent terms, this becomes

$$\tfrac{8}{9\kappa}p^0q_1^3 - \tfrac{2}{3}\kappa q_0(p^1)^3 - (p^0q_0)^2 + \tfrac{1}{3}(p^1q_1)^2 - 2p^0p^1q_0q_1 \leq 0$$

hence an empty chamber arises whenever a single centered black hole solution is ruled out !