

Counting Calabi-Yau black holes with mock modular forms

Boris Pioline

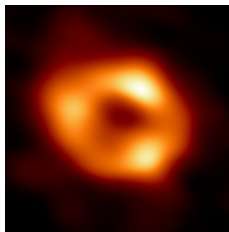


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Introduction

- A central goal for any theory of quantum gravity is to provide a **microscopic explanation** of the **thermodynamical entropy of black holes** in General Relativity [*Bekenstein'72, Hawking'74*]

$$S_{BH} = \frac{1}{4G_N} A$$

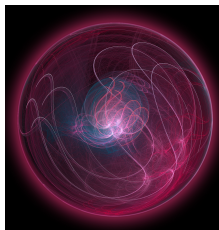


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Black hole microstates as wrapped D-branes

- Back in 1996, Strominger and Vafa showed that String Theory provides a quantitative description in the case of **BPS black holes in vacua with extended SUSY**: at weak coupling, BPS states are **bound states of D-branes wrapped on minimal cycles** of the internal Calabi-Yau manifold.

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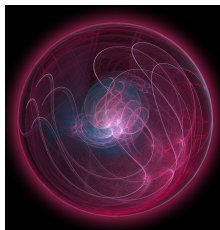


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- Besides confirming the consistency of string theory as a theory of quantum gravity, this has opened up many fruitful connections with mathematics.

- In the context of type IIA strings compactified on a Calabi-Yau three-fold X , BPS states are described mathematically by **stable objects in the derived category of coherent sheaves** $\mathcal{C} = D^b\text{Coh}X$. The Chern character $\gamma = (\text{ch}_0, \text{ch}_1, \text{ch}_2, \text{ch}_3)$ is identified as the electromagnetic charge, or D6-D4-D2-D0-brane charge.

BPS indices and Donaldson-Thomas invariants

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- The problem becomes a question in **Donaldson-Thomas theory**: for fixed $\gamma \in K(X)$, compute the **generalized DT invariant** $\Omega_Z(\gamma)$ counting **(semi)stable objects** of class γ , and determine its growth as $|\gamma| \rightarrow \infty$.

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- Importantly, $\Omega_z(\gamma)$ depends on the moduli of X , or more generally on a choice of **Bridgeland stability condition** $z \in \text{Stab}\mathcal{C}$. The **chamber structure** is fairly simple for $X = T^6$ or $X = K3 \times T^2$, but very intricate for a general CY 3-fold.

Modularity of Donaldson-Thomas invariants

- Physical arguments predict that **suitable generating series of DT invariants** (those counting D4-D2-D0 bound states in a suitable chamber) should have specific **modular properties**. This gives very good control on their asymptotic growth, and allows to test agreement with the BH prediction $\Omega_Z(\gamma) \simeq e^{S_{BH}(\gamma)}$.

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- More precisely, these generating series are expected to be **mock modular**, similar to Ramanujan's mock theta series. The modular anomaly can be repaired by adding a universal non-holomorphic correction, determined recursively from generating series with lower D4-brane charge [*Alexandrov BP Manschot'16-20*].

In this talk, I will explain how to combine knowledge of standard **Gromov-Witten invariants** (counting curves in X) and **wall-crossing arguments** to **rigorously compute many DT invariants**, and check mock modularity to **high precision**

S. Alexandrov, S. Feyzbakhsh, A. Klemm, BP, T. Schimannek, arXiv:2301.08066

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- 1 Reminder of enumerative invariants on CY3: GW, GV, DT, PT...
- 2 Mock modularity of D4-D2-D0 generating series
- 3 From rank 1 to rank 0 DT invariants, and back
- 4 Testing modularity on X_5 and other hypergeometric models
- 5 Conclusion and open problems

- Let X be a smooth, projective CY threefold. The **Gromov-Witten invariants** $GW_{\beta}^{(g)}$ count genus g curves Σ with class $\beta \in H_2(X, \mathbb{Z})$. They depend only on the symplectic structure (or Kähler moduli) of X and in general take rational values.

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- Physically, they determine certain **higher-derivative couplings** in the low energy effective action, which depend only on the (complexified) Kähler moduli t and receive **worldsheet instanton corrections**: $F_g(t) = \sum_\beta \text{GW}_\beta^{(g)} e^{2\pi i t \cdot \beta}$ [Antoniadis Gava Narain Taylor'93]

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- The first two F_0 and F_1 can be computed using **mirror symmetry**. **Holomorphic anomaly equations** along with suitable boundary conditions allow to determine $F_{g \geq 2}$ up to a certain genus g_{int} ($= 53$ for the quintic threefold X_5) [Bershadsky Cecotti Ooguri Vafa'93; Huang Klemm Quackenbush'06]

Gopakumar-Vafa invariants

- Gromov-Witten invariants turn out to be determined by a set of integer invariants $GV_{\beta}^{(g)}$ via [Gopakumar Vafa'98, Ionel Parker'13]

$$\sum_{g=0}^{\infty} \sum_{\beta} GW_{\beta}^{(g)} \lambda^{2g-2} e^{2\pi i t \cdot \beta} = \sum_{g=0}^{\infty} \sum_{k=1}^{\infty} \sum_{\beta} \frac{GV_{\beta/k}^{(g)}}{k} \left(2 \sin \frac{k\lambda}{2}\right)^{2g-2} e^{2\pi i k t \cdot \beta}$$

For $g = 0$, this reduces to [Candelas de la Ossa Greene Parkes'93]

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- Physically, $GV_{\beta}^{(0)}$ counts **D2-D0 brane bound states** with D2 charge β , and arbitrary D0 charge n , coming from M2-branes wrapped on $[\beta] \times S^1$.
- Importantly, $GV_{\beta}^{(g)}$ vanishes for large enough $g \geq g_{\max}(\beta)$ (Castelnuovo bound) [Doan Ionel Walpuski'16].

Generalized Donaldson-Thomas invariants

- More generally, bound states of D6-D4-D2-D0 branes are described by stable objects in the **bounded derived category of coherent sheaves** $\mathcal{C} = D^b\text{Coh}(X)$ [Kontsevich'95, Douglas'01]. Objects are bounded complexes $E = (\cdots \rightarrow \mathcal{E}_{-1} \rightarrow \mathcal{E}_0 \rightarrow \mathcal{E}_1 \rightarrow \cdots)$ of coherent sheaves $\mathcal{E}_k$, graded by the total Chern character
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- Stability depends on a choice of stability condition $\sigma = (Z, \mathcal{A})$, where the central charge $Z \in \text{Hom}(\Gamma, \mathbb{C})$ and the heart $\mathcal{A} \subset \mathcal{C}$ satisfy various axioms [Bridgeland 2007], in particular

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 - 1 $\forall E \in \mathcal{A}, \text{Im}Z(E) \geq 0$
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 - 2 $\forall E \in \mathcal{A}, \text{Im} Z(E) = 0 \Rightarrow \text{Re} Z(E) < 0$
- The **generalized Donaldson-Thomas invariant** $\Omega_\sigma(\gamma)$ is roughly the weighted Euler number of the moduli space $M_\sigma(\gamma)$ of **semi-stable objects** $E \in \mathcal{A}$ with $\text{ch } E = \gamma$, where semi-stability means that $\arg Z(E') \leq \arg Z(E)$ for any subobject $E' \subset E$.

- The space of stability conditions $\text{Stab } \mathcal{C}$ is a complex manifold of dimension $\dim K_{\text{num}}(X) = 2b_2(X) + 2$, *unless it is empty*
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- Stability conditions in the vicinity of the large volume point can be constructed subject to a conjectural **Bogomolov-Gieseker-type inequality** introduced in [Bayer Macri Toda'11] – more on this later.

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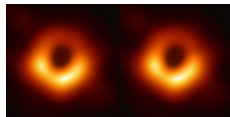
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- The BMT inequality (in its strong form) is very hard to prove for a general compact CY3, but has been proven for the quintic threefold X_5 [Li'18] and a couple of other examples [Koseki'20, Liu'21].

Generalized Donaldson-Thomas invariants

- $\Omega_\sigma(\gamma)$ may **jump** on co-dimension 1 walls in $\text{Stab } \mathcal{C}$ where some the central charge $Z(\gamma')$ of a subobject $E' \subset E$ becomes aligned with $Z(\gamma)$. The jump is governed by a universal **wall-crossing formula** [Joyce Song'08, Kontsevich Soibelman'08]. In simplest primitive case,

$$\Delta \Omega_\sigma(\gamma_1 + \gamma_2) = \langle \gamma_1, \gamma_2 \rangle \Omega_\sigma(\gamma_1) \Omega_\sigma(\gamma_2)$$

corresponding physically to the (dis)appearance of multi-centered black hole bound states [Denef Moore'07; Andriyash Denef Jafferis Moore'10; Manschot BP Sen'10]

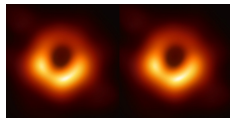


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- For $\gamma = (0, 0, \beta, n)$, $\Omega_\sigma(\gamma)$ coincides with $GV_\beta^{(0)}$ at large volume.

GV invariants and D6-brane bound states

- For $\gamma = (-1, 0, \beta, -n)$ at large volume and B -field, stable objects have a much simpler mathematical description in terms of **stable pairs** $E : \mathcal{O}_X \xrightarrow{s} F$ [Pandharipande Thomas'07]:

The **PT invariant** $PT(\beta, n)$ is defined as the (weighted) Euler characteristic of the corresponding moduli space.

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- PT invariants are related to GV invariants by [Maulik Nekrasov Okounkov Pandharipande'06]

$$\sum_{\beta, n} \text{PT}(\beta, n) e^{2\pi i t \cdot \beta} q^n = \text{Exp} \left(\sum_{\beta, g} \text{GV}_{\beta}^{(g)} (\sqrt{q} - 1/\sqrt{q})^{2g-2} e^{2\pi i t \cdot \beta} \right)$$

where $\text{Exp}(f(q)) = \exp(\sum_{n \geq 1} \frac{1}{n} f(q^n))$ is the plethystic exponential.

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- Under this relation, the Castelnuovo bound $\text{GV}_{\beta}^{(g \geq g_{\max}(\beta))} = 0$ is mapped to $\text{PT}(\beta, n \leq 1 - g_{\max}(\beta)) = 0$

D4-D2-D0 indices as rank 0 DT invariants

- The main interest in this talk will be on **rank 0 DT invariants** $\Omega(0, p, \beta, n)$ counting D4-D2-D0 brane bound states supported on an effective divisor $\mathcal{D}$ with class $[\mathcal{D}] = p \in H_4(X, \mathbb{Z})$.

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- Viewing IIA= M/S^1 , D4-D2-D0 branes on $\mathcal{D}$ arise from **M5-branes** wrapped on $\mathcal{D} \times S^1$. In the limit where S^1 is much larger than X , they are described by a **two-dimensional superconformal field theory** with $(0, 4)$ SUSY. *[Maldacena Strominger Witten'97]*

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- D4-D2-D0 indices occur as Fourier coefficients in the **elliptic genus** $\text{Tr}(-1)^F q^{L_0 - \frac{c_L}{24}} e^{2\pi i q_a z^a}$. If the SCFT has a discrete spectrum, after theta series decomposition with respect to the elliptic variables z^a , one obtains a vector-valued modular form

$$h_{p,\mu}(\tau) := \sum_n \bar{\Omega}(0, p, \mu, n) q^{n - \frac{\chi(\mathcal{D})}{24} + \frac{1}{2}\mu^2 - \frac{1}{2}p\mu}, \quad \mu \in \Lambda^* / \Lambda$$

where Λ^* / Λ is the discriminant group associated to $\Lambda = (H_4(X, \mathbb{Z}), \kappa_{ab} := \kappa_{abc} p^c)$, with cardinality $|\det \kappa_{ab}|$.

Modularity of rank 0 DT invariants

- When $\mathcal{D}$ is **very ample** and **irreducible**, there are no walls extending to large volume, so the choice of chamber is irrelevant. The SCFT central charges are given by *[Maldacena Strominger Witten'97]*

$$\begin{cases} c_L = p^3 + c_2(TX) \cdot p = \chi(\mathcal{D}) , \\ c_R = p^3 + \frac{1}{2}c_2(TX) \cdot p = 6\chi(\mathcal{O}_{\mathcal{D}}) \end{cases}$$

Cardy's formula predicts a growth $\Omega(0, p, \beta, n \rightarrow \infty) \sim e^{2\pi\sqrt{p^3 n}}$ in perfect agreement with Bekenstein-Hawking formula !

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- Moreover, since the space of vector-valued weakly holomorphic modular form has finite dimension, the full series is completely determined by its **polar coefficients**, with $n + \frac{1}{2}\mu^2 - \frac{1}{2}p\mu < \frac{\chi(\mathcal{D})}{24}$.
(Actually, the dimension can be smaller than the number of polar terms).

Mock modularity of rank 0 DT invariants

- When $\mathcal{D}$ is reducible, the generating series $h_{p^a, \mu_a}(\tau)$ in a suitable ("large volume attractor") chamber is expected to be a **mock modular form of higher depth** [Alexandrov BP Manschot'16-20])

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- Namely, there exists explicit, universal **non-holomorphic theta series** $\Theta_n(\{p_i\}, \tau, \bar{\tau})$ such that (omitting the μ 's for brevity)

$$\hat{h}_p(\tau, \bar{\tau}) = h_p(\tau) + \sum_{p=\sum_{i=1}^{n \geq 2} p_i} \Theta_n(\{p_i\}, \tau, \bar{\tau}) \prod_{i=1}^n h_{p_i}(\tau)$$

transforms as a modular form. The completed series satisfy the **holomorphic anomaly equation**,

$$\partial_{\bar{\tau}} \hat{h}_p(\tau, \bar{\tau}) = \sum_{p=\sum_{i=1}^{n \geq 2} p_i} \hat{\Theta}_n(\{p_i\}, \tau, \bar{\tau}) \prod_{i=1}^n \hat{h}_{p_i}(\tau, \bar{\tau})$$

- For binary splittings, this reduces to the depth one mock modular forms encountered in the study of BPS dyons in Type II on $K3 \times T^2$, or in heterotic string on T^6 [Dabholkar Murthy Zagier'12].

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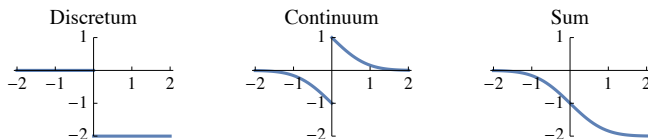
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- Our derivation relied on the study of **instanton corrections to the QK metric** on the moduli space after compactifying on a circle, and implementing $SL(2, \mathbb{Z})$ symmetry manifest from $IIA/S^1 = M/T^2$. A nice spin off of earlier research on hypermultiplet moduli spaces !

Alexandrov Banerjee Persson BP Manschot Saueressig Vandoren, 2008-19

Non-holomorphic completion from Witten index

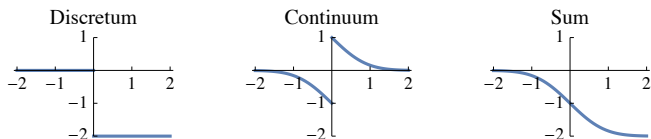
- Physically, the non-holomorphic corrections arise from the spectral asymmetry in the continuum of scattering states in the supersymmetric quantum mechanics of n BPS black holes.



BP 2015; Murthy BP 2018

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- Using localization, one can actually compute the Witten index for any n , and reproduce the full modular completion ! (at least for collinear D4-brane charges) *[BP Raj, to appear soon]*

Testing mock modularity for one-parameter models

- In the remainder of this talk, we shall test these modularity predictions for CY threefolds with Picard rank 1 (i.e. $b_2(X) = 1$), by computing the first few coefficients in the q -expansion and determine the putative vector-valued (mock) modular form.

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- This was first attempted by *[Gaiotto Strominger Yin '06-07]* for the quintic threefold X_5 and a few other hypergeometric models. They were able to guess the first few terms for unit D4-brane charge, and found a unique modular completion.
- We shall compute many terms rigorously, using recent results by *[Soheyla Fezbakhsh and Richard Thomas'20-22]* relating **rank r DT invariants** (including $r = 0$, counting D4-D2-D0 bound states) to **PT invariants**, hence to GV invariants.

Alexandrov, Fezbakhsh, Klemm, BP, Schimannek'23

From rank 1 to rank 0 DT invariants

- The key idea is to study wall-crossing in the space of Bridgeland stability conditions, away from the physical slice. For any $b + it \in \mathbb{H}$, consider the central charge

$$Z_{b,t}(E) = \frac{i}{6} t^3 \operatorname{ch}_0^b(E) - \frac{1}{2} t^2 \operatorname{ch}_1^b(E) - it \operatorname{ch}_2^b(E) + 0 \operatorname{ch}_3^b(E)$$

with $\operatorname{ch}_k^b(E) := \int_X H^{3-k} e^{-bH} \operatorname{ch}(E)$. With a suitable choice of heart (defined by tilting with respect to the slope $\frac{\operatorname{ch}_1^b(E)}{\operatorname{rk}(E)}$), this defines a **weak** stability condition called **tilt-stability**.

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- Note that $Z_{b,t}(E)$ is obtained from $Z^{\text{LV}}(E) = - \int_X e^{(b+it)H} \operatorname{ch}(E)$ by setting by hand the coefficient of ch_3^b to 0. In fact, tilt-stability is the first step in constructing genuine stability conditions near the large volume point [Bayer Macri Toda'11]

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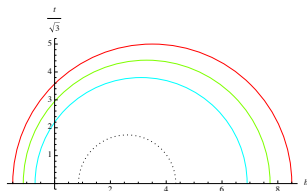
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- The KS/JS wall-crossing formulae still hold for such weak stability conditions.

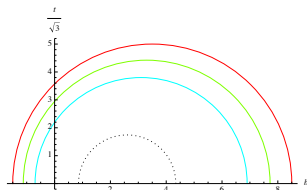
Rank 0 DT invariants from GV invariants

- Tilt stability agrees with slope stability at large volume, but the chamber structure is much simpler: walls are **nested half-circles** in the Poincaré upper half-plane spanned by $z = b + i\frac{t}{\sqrt{3}}$.



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- Importantly, for any tilt-semistable object E there is a **conjectural inequality** on Chern classes $C_i := \int_X \text{ch}_i(E) \cdot H^{3-i}$ [Bayer Macri Toda'11; Bayer Macri Stellari'16]

$$(C_1^2 - 2C_0C_2)(\frac{1}{2}b^2 + \frac{1}{6}t^2) + (3C_0C_3 - C_1C_2)b + (2C_2^2 - 3C_1C_3) \geq 0$$

- In particular, if the discriminant $\Delta(C)$ at $t = 0$ is positive, there exists an empty chamber ! $\Delta(\gamma)$ is quartic in the charges,

$$\Delta(C) = 8C_0C_2^3 + 6C_1^3C_3 + 9C_0^2C_3^2 - 3C_1^2C_2^2 - 18C_0C_1C_2C_3 \geq 0$$

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- Remarkably, $\Delta(C)$ is proportional to (minus) the quartic invariant $I_4(Q)$ which determines the entropy $S_{BH} \sim \pi \sqrt{I_4(Q)}$ of single-centered black holes ! In particular, *an empty chamber exists whenever single-centered black hole are ruled out !*

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- Consider an anti-D6-brane with charge $\gamma = (-1, 0, \beta, -n)$ such that $\Delta(C) > 0$. By studying wall-crossing between the empty chamber where $\Omega_{b,t}(\gamma) = 0$ and the large volume chamber where $\Omega_{b,t}(\gamma) = \text{PT}(\beta, m)$, one can extract the indices of the D4-D2-D0 branes emitted at each wall !

A new explicit formula (S. Feyzbakhsh'23)

Theorem Let (X, H) be a smooth polarised CY threefold with $\text{Pic}(X) = \mathbb{Z}.H$ satisfying the BMT conjecture. There is $f(x)$ such that

- If $\frac{m}{\beta.H} < f(\frac{\beta.H}{H})$ then the stable pair invariant $\text{PT}(\beta, m) =$

$$\sum_{(m', \beta')} (-1)^{\chi_{m', \beta'}} \chi_{m', \beta'} \text{PT}(\beta', m') \Omega\left(0, 1, \frac{H^2}{2} - \beta' + \beta, \frac{H^3}{6} + m' - m - \beta'.H\right)$$

where $\chi_{m', \beta'} = \beta.H + \beta'.H + m - m' - \frac{H^3}{6} - \frac{1}{12}c_2(X).H$.

- The sum runs over $(\beta', m') \in H_2(X, \mathbb{Z}) \oplus H_0(X, \mathbb{Z})$ such that

$$0 \leq \beta'.H \leq \frac{H^3}{2} + \frac{3mH^3}{2\beta.H} + \beta.H$$
$$-\frac{(\beta'.H)^2}{2H^3} - \frac{\beta'.H}{2} \leq m' \leq \frac{(\beta.H - \beta'.H)^2}{2H^3} + \frac{\beta.H + \beta'.H}{2} + m$$

In particular, $\beta'.H < \beta.H$.

Corollary (Castelnuovo bound): $\text{PT}(\beta, m) = 0$ unless $m \geq -\frac{(\beta.H)^2}{2H^3} - \frac{\beta.H}{2}$

- Using the theorem above and known GV invariants, we could compute a large number of coefficients in the generating series of Abelian (=unit D4-brane charge) rank 0 DT invariants in **one-parameter hypergeometric threefolds**, including the quintic X_5 .

Modularity for one-modulus compact CY

- Using the theorem above and known GV invariants, we could compute a large number of coefficients in the generating series of Abelian (=unit D4-brane charge) rank 0 DT invariants in **one-parameter hypergeometric threefolds**, including the quintic X_5 .
- In all cases (except $X_{3,2,2}$, $X_{2,2,2,2}$ where current knowledge of GV invariants is insufficient), we found a **unique vector-valued modular form** matching all computed coefficients.

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- For two examples $X = X_8$ and $X = X_{10}$, we could compute sufficiently many D4-D2-D0 indices with two units of D4-brane charge to identify a **unique depth-one mock modular form**.

Modularity for one-modulus compact CY

X	χ_X	κ	$c_2(TX)$	$\chi(\mathcal{O}_D)$	n_1	C_1	n_2	C_2
$X_5(1^5)$	-200	5	50	5	7	0	36	1
$X_6(1^4, 2)$	-204	3	42	4	4	0	19	1
$X_8(1^4, 4)$	-296	2	44	4	4	0	14	1
$X_{10}(1^3, 2, 5)$	-288	1	34	3	2	0	7	0
$X_{4,3}(1^5, 2)$	-156	6	48	5	9	0	42	0
$X_{4,4}(1^4, 2^2)$	-144	4	40	4	6	1	25	1
$X_{6,2}(1^5, 3)$	-256	4	52	5	7	0	30	1
$X_{6,4}(1^3, 2^2, 3)$	-156	2	32	3	3	0	11	1
$X_{6,6}(1^2, 2^2, 3^2)$	-120	1	22	2	1	0	5	0
$X_{3,3}(1^6)$	-144	9	54	6	14	1	78	3
$X_{4,2}(1^6)$	-176	8	56	6	15	1	69	3
$X_{3,2,2}(1^7)$	-144	12	60	7	21	1	117	0
$X_{2,2,2,2}(1^8)$	-128	16	64	8	33	3	185	4

Modular predictions for the quintic threefold

- Using known $GV_{\beta}^{(g \leq 53)}$ we can compute more than 20 terms:

$$h_0 = q^{-\frac{55}{24}} \left(\underline{5 - 800q + 58500q^2 + 5817125q^3 + 75474060100q^4} \right. \\ \left. + 28096675153255q^5 + 3756542229485475q^6 \right. \\ \left. + 277591744202815875q^7 + 13610985014709888750q^8 + \dots \right),$$

$$h_{\pm 1} = q^{-\frac{55}{24} + \frac{3}{5}} \left(\underline{0 + 8625q - 1138500q^2 + 3777474000q^3} \right. \\ \left. + 3102750380125q^4 + 577727215123000q^5 + \dots \right)$$

$$h_{\pm 2} = q^{-\frac{55}{24} + \frac{2}{5}} \left(\underline{0 + 0q - 1218500q^2 + 441969250q^3 + 953712511250q^4} \right. \\ \left. + 217571250023750q^5 + 22258695264509625q^6 + \dots \right)$$

Modular predictions for the quintic threefold

- The space of vv modular forms has dimension 7. Remarkably, all terms above are reproduced by *[Gaiotto Strominger Yin'06]*

$$h_{\mu} = \frac{1}{\eta^{55+15}} \left[-\frac{222887E_4^8+1093010E_4^5E_6^2+177095E_4^2E_6^4}{35831808} + \frac{25(458287E_4^6E_6+967810E_4^3E_6^3+66895E_6^5)}{53747712} D + \frac{25(155587E_4^7+1054810E_4^4E_6^2+282595E_4E_6^4)}{8957952} D^2 \right] v_{\mu}^{(5)}$$

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- Physically, polar coefficients are expected arise as **bound states of D6-brane and anti D6-branes** *[Denef Moore'07]*. Indeed, they are **often** consistent with the naive ansatz *[Alexandrov Gaddam Manschot BP'22]*

$$\Omega(0, 1, \beta, n) = \pm(\chi(\mathcal{O}_{\mathcal{D}}) - \beta.H - n) DT(\beta, n) PT(0, 0)$$

but deviations do occur ! *[Collinucci Wyder'08, van Herck Wyder'09]*

Mock modularity for non-Abelian D4-D2-D0 indices

- For D4-D2-D0 indices with $N = 2$ units of D4-brane charge, $\{h_{2,\mu}, \mu \in \mathbb{Z}/(2\kappa\mathbb{Z})\}$ should transform as a **vector-valued mock modular form** with modular completion

$$\widehat{h}_{2,\mu}(\tau, \bar{\tau}) = h_{2,\mu}(\tau) + \sum_{\mu_1, \mu_2=0}^{\kappa-1} \delta_{\mu_1+\mu_2-\mu}^{(\kappa)} \Theta_{\mu_2-\mu_1+\kappa}^{(\kappa)} h_{1,\mu_1} h_{1,\mu_2}$$

where $\Theta_{\mu}^{(\kappa)}$ is the Eichler integral of a unary theta series,

$$\partial_{\bar{\tau}} \Theta_{\mu}^{(\kappa)} = \frac{(-1)^{\mu} \sqrt{\kappa}}{16\pi i \tau_2^{3/2}} \sum_{k \in 2\kappa\mathbb{Z} + \mu} e^{\frac{-\pi i \bar{\tau}}{2\kappa} k^2}$$

and $\delta_{\mu}^{(\kappa)} = 1$ if $\mu \equiv 0 \pmod{\kappa}$, 0 otherwise.

Mock modularity for non-Abelian D4-D2-D0 indices

- Suppose there exists a holomorphic function $g_{\mu}^{(\kappa)}$ such that $\Theta_{\mu}^{(\kappa)} + g_{\mu}^{(\kappa)}$ transforms as a vv modular form. Then

$$\tilde{h}_{2,\mu}(\tau, \bar{\tau}) = h_{2,\mu}(\tau) - \sum_{\substack{\mu_1, \mu_2=0 \\ \mu_1+\mu_2=\mu}}^{\kappa-1} \delta_{\mu_1+\mu_2-\mu}^{(\kappa)} g_{\mu_2-\mu_1+\kappa}^{(\kappa)} h_{1,\mu_1} h_{1,\mu_2}$$

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- For $\kappa = 1$, the series $\Theta_\mu^{(1)}$ is the one appearing in the modular completion of the generating series of **Hurwitz class numbers** [Hirzebruch Zagier 1973] (or **rank 2 Vafa-Witten invariants on $\mathbb{P}^2$**)

$$H_0(\tau) = -\frac{1}{12} + \frac{1}{2}q + q^2 + \frac{4}{3}q^3 + \frac{3}{2}q^4 + \dots$$
$$H_1(\tau) = q^{\frac{3}{4}} \left(\frac{1}{3} + q + q^2 + 2q^3 + q^4 + \dots \right)$$

hence we can choose $g_\mu^{(1)} = H_\mu(\tau)$. See [Alexandrov Bendriss'25] for a general prescription valid for any $\kappa \geq 1, N \geq 2$.

Mock modularity for non-Abelian D4-D2-D0 indices

- For X_{10} , we computed the 7 polar terms + 4 non-polar terms and found a unique mock modular form reproducing this data:

$$h_{2,\mu} = \frac{5397523E_4^{12} + 70149738E_4^9E_6^2 - 12112656E_4^6E_6^4 - 61127530E_4^3E_6^6 - 2307075E_6^8}{46438023168\eta^{100}} \vartheta_\mu^{(1,2)} \\ + \frac{-10826123E_4^{10}E_6 - 14574207E_4^7E_6^3 + 20196255E_4^4E_6^5 + 5204075E_4E_6^7}{1934917632\eta^{100}} D\vartheta_\mu^{(1,2)} \\ + (-1)^{\mu+1} H_{\mu+1}(\tau) h_1(\tau)^2$$

with $h_1 = \frac{203E_4^4 + 445E_4E_6^2}{216\eta^{35}} = q^{-\frac{35}{24}} (3 - 575q + \dots)$, leading to integer DT invariants

$$h_{2,0}^{(\text{int})} = q^{-\frac{19}{6}} \left(\underline{7 - 1728q + 203778q^2 - 13717632q^3} - 23922034036q^4 + \dots \right) \\ h_{2,1}^{(\text{int})} = q^{-\frac{35}{12}} \left(\underline{-6 + 1430q - 1086092q^2 + 208065204q^3} + \dots \right)$$

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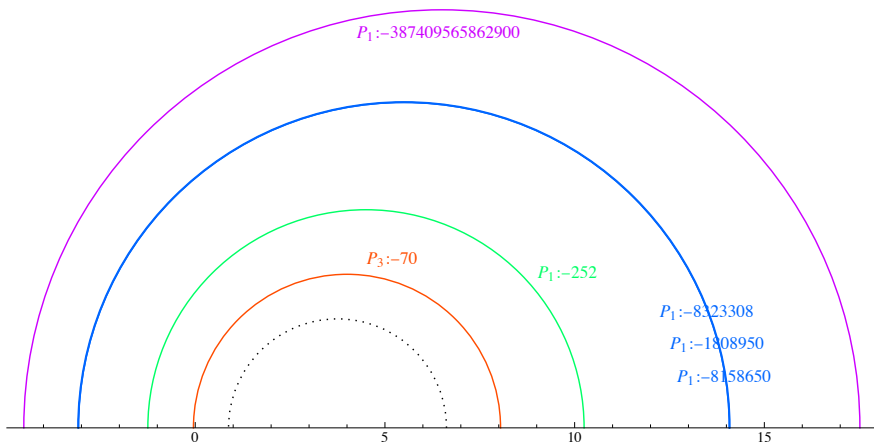
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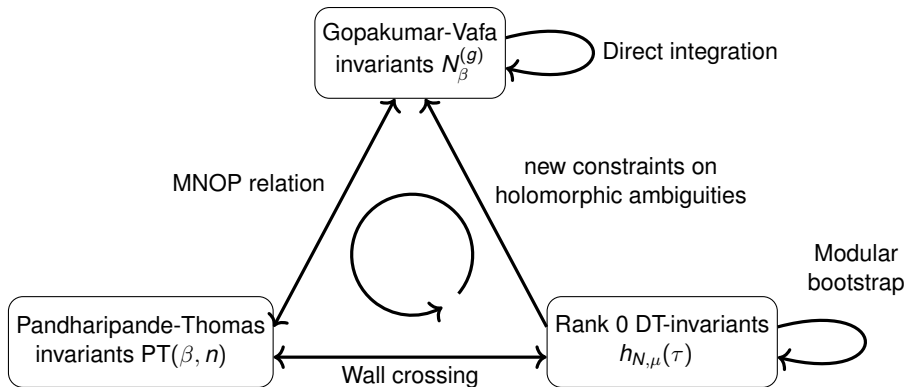
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- Similar results for X_8 [*S. Alexandrov, S. Feyzbakhsh, A. Klemm'23*]

Computing the leading term in $h_{2,0}$ for X_{10}



Quantum geometry from stability and modularity



Alexandrov Feyzbakhsh Klemm BP Schimannek'23

Quantum geometry from stability and modularity

X	χX	κ	type	g_{integ}	$g_{\text{mod}}^{(1)}$	$g_{\text{mod}}^{(2)}$	g_{avail}
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<http://www.th.physik.uni-bonn.de/Groups/Klemm/data.php>

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- When $X \xrightarrow{\pi} \mathbb{P}^1$ admits a **K3-fibration**, using the relation to **Noether-Lefschetz invariants** one can show that modularity holds for **vertical** D4-brane charge. The modular anomaly disappears due to $\kappa_{ab}p^b = 0$. [Bouchard Creutzig Diaconescu Doran Quigley Sheshmani'16]
- In [Doran BP Schimannek'24] we constructed a family of (in general non-toric) two-parameter K3-fibered threefolds $X_m^{[i,j]}$, whose mirror $Y_m^{[i,j]}$ is also K3-fibered, and which admit extremal transitions to 1-parameter models. We determined their NL invariants, as well as GV invariants at genus 0 and 1.

- Similarly, when $X \xrightarrow{\pi} B$ admits a **genus-one fibration** with N -section, one can relate rank 0 DT invariants for a D4-brane wrapping a pulled-back divisor $\pi^{-1}(D)$ to genus 0 GV invariants via a **relative conifold monodromy**. In this case, p^a is not ample but still nef, $\kappa_{ab}p^ap^b = 0$. See *[Klemm Manschot Wotschke'12]* for the elliptic ($N = 1$) case.

Mathematical origin of modularity

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- Generating series of GW invariants at fixed genus and base degree are quasi-modular forms for $\Gamma_1(N)$ *[Alim Scheidegger'12, Katz Klemm Huang'15, Cota Klemm Schimannek'19]*. After applying the conifold monodromy, one finds that the modular anomaly of genus 0 GW invariants matches that of rank 0 DT invariants, despite having different multi-cover effects ($\sum 1/d^3$ vs $\sum 1/d^2$).

- In fact, [Katz Klemm Huang'15] have conjectured that normalized generating series of PT invariants at fixed base degree are **meromorphic Jacobi forms**. The elliptic transformation was proven mathematically for reduced charges in [Oberdieck Shen'16]

$$\frac{Z_{\text{top}}(S, T, \lambda)}{Z_0(T, \lambda)} = 1 + \sum_{H \in H_2^{>0}(B, \mathbb{Z})} Z_H(T, \lambda) e^{2\pi i S \cdot H}$$

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- We show that the modular transformation follows from the **wave function behavior** of Z_{top} under a relative conifold monodromy [Aganagic Bouchard Klemm'06; BP Schimannek, to appear]

Summary and open questions

- We provided overwhelming evidence that D4-D2-D0 indices exhibit mock modular properties. Where does it come from mathematically ? Is there some VOA acting on the cohomology of moduli space of stable objects, à la *[Nakajima'94]* ?

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- A long-standing problem: incorporate NS5-instanton corrections to the QK metric on hypermultiplet moduli space, consistently with S-duality, beyond the linear analysis of *[Alexandrov Persson BP'10]*.

Thanks for your attention !

