Exact BPS couplings and black hole counting

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based on works with: S. Alexandrov, S. Banerjee, G. Bossard, C. Cosnier-Horeau, E. d'Hoker, M. B. Green, A. Kleinschmidt, J. Manschot, D. Persson, R. Russo, ...

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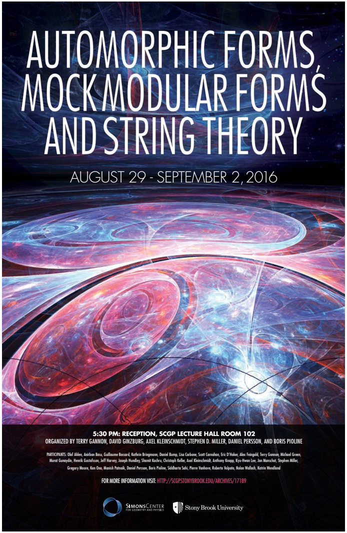

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- Importantly, the BPS index $\Omega(Q, z)$ is discontinuous across real codimension-one walls in moduli space, due to the (dis)appearance of multi-centered black hole bound states.
- For $\mathcal{N} = 4$, subtracting contributions from two-centered bound states, the indices counting single-centered black holes are Fourier coefficients of mock Jacobi forms *[Dabholkar Murthy Zagier'12]*.

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- A second complication is that multi-centered black hole bound states with arbitrary number of constituents may now contribute to the index.

Denef '00; Denef Moore '07

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- A famous example is the t Hooft-Polyakov monopole in $D=4$, which descends to the instanton responsible for confinement in 3D QED *[Polyakov 1977]*

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- BPS couplings are interactions with $f + 2n < \mathcal{N}$, which only get corrections from instantons preserving some fraction of SUSY:

$\mathcal{N}$	k	$(\mathcal{N} - k)/\mathcal{N}$	BPS couplings
32	16	$1/2$	$\mathcal{R}^4$
32	24	$1/4$	$\nabla^4\mathcal{R}^4$
32	28	$1/8$	$\nabla^6\mathcal{R}^4$
16	8	$1/2$	$F^4, \mathcal{R}^2$
16	12	$1/4$	$\nabla^2 F^4, F^2\mathcal{R}^2$
8	4	$1/2$	$(\nabla\phi)^2$

BPS indices from large radius limit

- The coefficients of these couplings are functions $f^{(D)}(R, z, \phi)$ of the radius R, moduli z in dimension $D+1$, and holonomies ϕ of the n gauge fields along the circle:

$$\mathcal{M}_D \sim \mathbb{R}^+ \times \mathcal{M}_{D+1} \times \mathcal{T}_n$$

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- In presence of gravity, the dual of the Kaluza-Klein gauge field $g_{\mu,D+1}$ leads to an additional scalar σ, the NUT potential, which lives in a circle bundle over $\mathcal{T}_{2n}$,

$$\mathcal{M}_3 \sim \mathbb{R}^+ \times \mathcal{M}_4 \times \mathcal{T}_{2n} \times S^1$$

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- In the limit $R \to \infty$, $f^{(D)}(R, z, \phi)$ is expected to behave as

$$f^{(D)}(R, z, \phi) \sim R\, f^{(D+1)}(z) + \sum_{Q \in \Lambda \backslash \{0\}} \Omega_n(Q, z)\, e^{-R\mathcal{M}(Q,z) + \mathrm{i}\, Q\cdot\phi} + \dots$$

  where $\mathcal{M}(Q, z)$ is the BPS mass, $\Omega_n(Q, z) \sim \mathrm{Tr}(-1)^{2J_3}(2J_3)^n$ is the helicity supertrace counting BPS states with $k = 2n$ fermionic zero-modes.

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  4. For gravitational theories in $D = 3$, contributions from Taub-NUT instantons of order $\mathcal{O}(e^{-R^2})$, needed to resolve the ambiguity of the divergent sum $\sum_Q e^{S_{BH}(Q) - R\mathcal{M}(Q)}$ [BP Vandoren (2009)]

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- The take-home message is that the BPS coupling $f^{(D)}(R, z, \varphi)$ provides a natural generating series of BPS indices in dimension $D+1$, similar in spirit to the naive black hole partition function $Z_n(R, z, \phi) = \sum_Q \Omega_n(Q, z) e^{-R\mathcal{M}(Q,z)+\mathrm{i}Q\cdot\phi}$ but better behaved.

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- In such cases, $f^{(D)}$ is an automorphic function under $G_D(\mathbb{Z})$. Its Fourier coefficients are automatically invariant under the subgroup $G_{D+1}(\mathbb{Z}) \subset G_D(\mathbb{Z})$, acting linearly on the charge Q, but further constrained by invariance under $G_D(\mathbb{Z})$.

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- For $\mathcal{N}=2$ vacua, viewing type IIA/$\mathcal{X}\times S^1$ as M/$\mathcal{X}\times T^2$, we expect the full spectrum of BPS states to be described by automorphic forms under $G_3(\mathbb{Z})=SL(2,\mathbb{Z})\ltimes \mathrm{Mon}(\mathcal{X})\ltimes H_{2n+1}$.

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- In third case, we find that generating series of DT invariants supported on divisors are mock modular forms of higher depth.

Outline

1. From BPS indices to BPS-saturated couplings
2. 1/8-BPS couplings in $\mathcal{N} = 8$ string vacua
3. 1/4-BPS couplings in $\mathcal{N} = 4$ string vacua
4. 1/2-BPS couplings in $\mathcal{N} = 2$ string vacua
5. Conclusion

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1/2-BPS and 1/4-BPS couplings in $\mathcal{N} = 8$ string vacua

- In type II string compactified on a torus T^d, the LEEA is expected to be invariant under $G_{D=10-d}(\mathbb{Z}) = E_{d+1}(\mathbb{Z})$, which extends both the T-duality group $SO(d, d, \mathbb{Z})$ and global diffeomorphisms $SL(d + 1, \mathbb{Z})$ of the M-theory torus. *[Hull Townsend 95,Witten 95]*

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- Supersymmetric Ward identities and known perturbative contributions uniquely determine the $\mathcal{R}^4$ and $\nabla^4\mathcal{R}^4$ couplings:

$$f^{(D)}_{\mathcal{R}^4} = 2\zeta(3)\,\mathcal{E}^{E_{d+1}(\mathbb{Z})}_{\frac{3}{2}\Lambda_1}\ , \quad f^{(D)}_{\nabla^4\mathcal{R}^4} = \zeta(5)\,\mathcal{E}^{E_{d+1}(\mathbb{Z})}_{\frac{5}{2}\Lambda_1}$$

where $\mathcal{E}^{G_D(\mathbb{Z})}_{s\lambda_k}$ is the Langlands-Eisenstein series

$$\mathcal{E}^{G_D(\mathbb{Z})}_{s\lambda_k} = \sum_{\gamma\in P_k(\mathbb{Z})\backslash G(\mathbb{Z})} y_k^{-2s}|_\gamma = \frac{1}{2\zeta(2s)} \sum_{\substack{\mathcal{Q}\in\Lambda_k \\ \mathcal{Q}\times\mathcal{Q}=0}} [\mathcal{M}(\mathcal{Q})]^{-s}$$

1/2-BPS and 1/4-BPS couplings in $\mathcal{N} = 8$ string vacua

- At weak coupling coupling, these reproduce the known tree-level, one-loop and two-loop contributions, plus infinite series of D-instanton corrections. *[Green Gutperle '97, ...]*

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- In the large radius limit, one recovers the expected $\mathcal{O}(e^{-R\mathcal{M}(Q)})$ contributions from 1/2-BPS and 1/4-BPS states in dimension $D+1$, respectively, weighted by the helicity supertraces $\Omega_8(Q)$ and $\Omega_{12}(Q)$, *[Green Miller Russo Vanhove '10, BP '10, Bossard BP '16]*

$$\Omega_8(Q) = \begin{cases} 1 & (Q \times Q = 0) \\ 0 & (Q \times Q \neq 0) \end{cases}$$

$$\Omega_{12}(Q) = \begin{cases} \sigma_3[\gcd(Q \times Q)] & (I_4'(Q) = 0, Q \times Q \neq 0) \\ 0 & (I_4'(Q) \neq 0) \end{cases}$$

where $Q \times Q$ is the Jordan quadratic product on Λ_{d+1} and $I_4'(Q)$ is the gradient of the quartic invariant $I_4(Q)$.

1/8-BPS couplings in $\mathcal{N} = 8$ string vacua

- The coupling $\nabla^6\mathcal{R}^4$ is not given by an Eisenstein series, since SUSY requires

$$\left(\Delta_{E_{d+1}} - \frac{6(D-6)(14-D)}{D-2}\right) f^{(D)}_{\nabla^6\mathcal{R}^4} = -[f^{(D)}_{\mathcal{R}^4}]^2$$

up to additional linear source terms in dimension $D = 4, 5, 6$ where the local and non-local parts of the 1PI effective action mix.

Green Vanhove '05, Green Russo Vanhove '10; BP '15; Bossard Verschinin '15

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- Upon decompactifying from $D=3$ to $D=4$, we expect contributions from 1/8-BPS black holes, weighted by $\Omega_{14}(Q) = c(I_4(Q)) \sim e^{\pi\sqrt{I_4(Q)}}$, where $c(n)$ are the coefficients of the weak holomorphic form

$$h(\rho) = \frac{\theta_4(2\rho)}{\eta^6(4\rho)} = \sum_{n\geq -1} c(n)\, q^n, \quad q = e^{2\pi i\rho}$$

Maldacena Moore Strominger '99; Shih Strominger Yin '05; BP '05; Sen '08

1/8-BPS couplings in $\mathcal{N} = 8$ string vacua

- In $D = 6$, a solution reproducing known perturbative contributions up to genus 3 is *[BP 2015]*

$$f^{(6)}_{\nabla^6\mathcal{R}^4} = \pi\, \mathrm{R.N.} \int_{\mathcal{F}_2} \mathrm{d}\mu_2\, \Gamma^{(2)}_{5,5}\, \varphi_{KZ} + \frac{8}{189} \mathcal{E}^{SO(5,5,\mathbb{Z})}_{4\Lambda_5}$$

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- Here φ_{KZ} is the Kawazumi-Zhang invariant, a real-analytic Siegel modular function which appears in the integrand of the genus-two $\nabla^6\mathcal{R}^4$ coupling *[d'Hoker Phong '01-05, d'Hoker Green '14]*

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1/8-BPS couplings in $\mathcal{N}=8$ string vacua

- In $D=6$, a solution reproducing known perturbative contributions up to genus 3 is *[BP 2015]*

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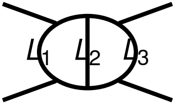
- In generic dimension, another proposal for the exact $\nabla^6\mathcal{R}^4$ coupling is obtained by covariantizing the two-loop supergravity amplitude *[Bossard Kleinschmidt '15; Bossard Kleinschmidt BP '20]*

Exact $\nabla^6\mathcal{R}^4$ coupling in $\mathcal{N}=8$ string vacua

- Performing the change of variables *[Green Kwon Vanhove '99]*

$$\begin{pmatrix} L_1+L_2 & L_2 \\ L_2 & L_2+L_3 \end{pmatrix} = \frac{1}{V\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}$$



and integrating over overall scale V, this produces

$$f^{(D)}_{\nabla^6\mathcal{R}^4} = \frac{8\pi^2}{3}\,\frac{\Gamma(d-2)}{\pi^{d-2}}\int_{\mathcal{F}}\frac{\mathrm{d}\tau_1\mathrm{d}\tau_2}{\tau_2^2}\,A(\tau)\sum'_{\substack{Q_1,Q_2\in\Lambda_{d+1}\\ Q_i\times Q_j=0}}\left[\frac{\tau_2}{G(Q_1+\tau Q_2,Q_1+\bar\tau Q_2)}\right]^{d-2}$$

where $A(\tau)$ is given in standard fundamental domain $\mathcal{F}$ by

$$A(\tau)=\frac{|\tau|^2-\tau_1+1}{\tau_2}+\frac{5\tau_1(\tau_1-1)(|\tau|^2-\tau_1)}{\tau_2^3}\,,\qquad \frac{A(\tau)}{V}=L_1+L_2+L_3-\frac{5L_1L_2L_3}{L_1L_2+L_2L_3+L_3L_1}$$

and extended to upper half-plane by modular-invariance.

Exact $\nabla^6\mathcal{R}^4$ coupling and BPS indices

- The agreement with weak coupling expansion (and alternative proposal in $D=6, d=4$) follows by observing *[BKP'20]*

$$\varphi_{KZ}(\Omega) = \sum_{\gamma\in(GL(2,\mathbb{Z})\ltimes\mathbb{Z}^3)\backslash Sp(4,\mathbb{Z})} \left(A(\tau)/V\right)|_\gamma$$

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$$\varphi_{KZ}(\Omega) = \int_{\mathcal{H}/\Gamma_0(4)} \frac{\mathrm{d}\rho\mathrm{d}\bar\rho}{\rho_2^2} \Gamma^{(1)}_{3,2}(\rho;\Omega) D_\rho h(\rho)\,, \quad h(\rho) = \frac{\theta_4(2\rho)}{\eta^6(4\rho)}$$

where $\Gamma_{3,2}$ is a genus-1 Siegel-Narain theta series *[BP 2015]*

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Outline

Duality $\mathcal{N} = 4$ string vacua

- A similar philosophy works for $\mathcal{N} = 4$ string vacua, such as heterotic string compactified on T^6, or type II on $K3 \times T^2$. The moduli space in $D = 4$ factorizes into

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$$\mathcal{M}_3 = \frac{O(24,8)}{O(24) \times O(8)} \supset \begin{cases} \mathbb{R}^+_R \times \mathcal{M}_4 \times \mathbb{R}^{56+1} \\ \mathbb{R}^+_{1/g_3^2} \times \frac{O(23,7)}{O(23)\times O(7)} \times \mathbb{R}^{23+7} \end{cases}$$

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- Accordingly, the U-duality group enhances from $G_4(\mathbb{Z}) = SL(2,\mathbb{Z}) \times O(22,6,\mathbb{Z})$ to $G_3(\mathbb{Z}) = O(24,8,\mathbb{Z})$ *[Sen 1994]*

1/2-BPS and 1/4-BPS couplings in $\mathcal{N} = 4$ string vacua

- The 4-derivative and 6-derivative couplings in $D = 3$

$$F_{abcd}(\Phi)\, \nabla\Phi^a\nabla\Phi^b\nabla\Phi^c\nabla\Phi^d + G_{ab,cd}(\Phi)\nabla(\nabla\Phi^a\nabla\Phi^b)\, \nabla(\nabla\Phi^c\nabla\Phi^d)$$

are expected to get contributions from 1/2-BPS and 1/4-BPS instantons, respectively. *[Bossard Cosnier-Horeau BP '16]*

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$$\mathcal{D}^2_{ef}\,F_{abcd} = 0\ , \quad \mathcal{D}^2_{ef}\,G_{ab,cd} = F_{abk(e}\,F_{f)cd}{}^{k}$$

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  where $\mathcal{D}^2_{ef}$ is a second order differential operator on $\mathcal{M}_3$.
- These constraints imply that F_{abcd} is perturbatively exact at one-loop, while $G_{ab,cd}$ is perturbatively exact at two-loop in heterotic perturbation theory. For brevity we focus on $G_{ab,cd}$.

BPS indices from Siegel modular forms

- Degeneracies of 1/4-BPS dyons are given by Fourier coefficients of a meromorphic Siegel modular form:

$$\Omega_6(Q,P;z) = (-1)^{Q\cdot P} \int_{\mathcal{C}} \mathrm{d}^3\Omega \frac{e^{\mathrm{i}\pi(\rho Q^2+\sigma P^2+2vQ\cdot P)}}{\Phi_{10}(\Omega)}$$

where $\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix} \in \mathcal{H}_2$, and Φ_{10} is the Igusa cusp form of weight 10 under $Sp(4,\mathbb{Z})$. *[Dijkgraaf Verlinde Verlinde '96; David Jatkar Sen '05-06]*

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- The integration contour is chosen as $\mathcal{C} = [0,1]^3 + \mathrm{i}\Omega_2^*$ with

$$\Omega_2^\star = \Lambda \left[\frac{1}{S_2}\begin{pmatrix} 1 & S_1 \\ S_1 & |S|^2 \end{pmatrix} + \frac{1}{|P_R \wedge Q_R|}\begin{pmatrix} |P_R|^2 & -P_R\cdot Q_R \\ -P_R\cdot Q_R & |Q_R|^2 \end{pmatrix} \right]$$

  with $\Lambda \gg 1$. This ensures that $\mathcal{C}$ crosses a zero of Φ_{10} whenever z crosses a wall of marginal stability. *[Cheng Verlinde '07]*

Wall-crossing from residues

- By virtue of

$$\frac{1}{\Phi_{10}(\Omega)} \overset{v\to 0}{\sim} \frac{1}{v^2} \times \frac{1}{\Delta(\rho)} \times \frac{1}{\Delta(\sigma)}$$

where $1/\Delta = \sum_{N\geq -1} c(N)\, q^N$ is the generating function of the BPS indices $\Omega_4(Q,P)$ counting 1/2-BPS states, the jump in $\Omega_6(Q,P;z)$ matches the contribution of bound states of two 1/2-BPS dyons:

$$\Delta\Omega_6(Q,P) = \pm(P_1 Q_2 - P_2 Q_1)\, \Omega_4(Q_1,P_1)\, \Omega_4(Q_2,P_2)$$

where $P_1 \parallel Q_1, P_2 \parallel Q_2, (Q,P) = (Q_1,P_1) + (Q_2,P_2)$.

Denef Moore '07

Perturbative $\nabla^2(\nabla\Phi)^4$ coupling in $D=3$

- The $\nabla^2(\nabla\Phi)^4$ coupling ireceives up to two-loop corrections,

$$g_3^6\, G_{ab,cd} = \frac{c_0}{g_3^2}\delta_{ab}\delta_{cd} + \delta_{ab} G^{(1)}_{cd} + g_3^2\, G^{(2)}_{ab,cd} + \mathcal{O}(e^{-1/g_3^2})$$

where the one-loop correction is given by *[Sakai Tanii '87]*

$$G^{(1)}_{ab} = \mathrm{RN}\int_{\mathcal{F}_1} \mathrm{d}\mu_1 \frac{\widehat{E}_2\,\Gamma_{23,7}[P_{ab}]}{\Delta}\,,$$

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- Here, P_{ab} and $P_{ab,cd}$ are quadratic and quartic polynomials in lattice vectors.

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Outline

Instanton corrections to QK metric

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- Since IIA/$\mathcal{X} \times S_1 =$ M/$\mathcal{X} \times T^2$, $\widetilde{\mathcal{M}}_3$ must admit an isometric action of $SL(2,\mathbb{Z})$. This puts powerful constraints on the indices $\Omega(\gamma, z)$.

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- For $\gamma = n[pt]$ supported on a point, S-duality requires $\Omega(\gamma, z) = -\chi_{\mathcal{X}}$ (independent of z)

Robles-Llana Rocek Saueressig Theis Vandoren '06

Constraining DT invariants from S-duality

- For $\gamma = \beta + n[pt]$ supported on a curve β, $\Omega(\gamma, z) = \sum_{d|\beta} \frac{1}{d^3} n^{(0)}_{\beta/d}$ hence coincides with genus-zero Gopakumar-Vafa invariant (independent of z)

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- For $\gamma = \mathcal{D} + \beta + n[pt]$ supported on an ample divisor $\mathcal{D}$, the generating series of attractor indices $h_{\mathcal{D},\beta}(\tau) = \sum_n \Omega_\star(\gamma) q^n$ should be a vector-valued weakly holomorphic modular form of prescribed (negative) weight and multiplier system.

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- For γ supported on a reducible divisor $\mathcal{D} = \sum_{i=1}^n \mathcal{D}_i$, the same generating series $h_{\mathcal{D},\beta}(\tau)$ should be a vector-valued mock modular form of depth $n-1$ and same weight/multiplier system.

Alexandrov Banerjee Manschot BP '16-19

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Alexandrov Banerjee Manschot BP '16-19

- For γ supported on the full CY3, S-duality relates $\Omega(\gamma, z)$ to topological string partition function and ill-understood KKM effects.

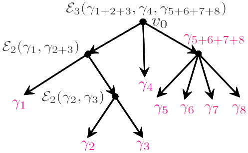
Mock modularity for DT invariants

- More explicitly (setting $\mathcal{D} = p$ and suppressing dependence on β) there exists explicit functions $R_n(\{\gamma_i\}, \tau_2)$, built out of generalized error functions $\mathcal{E}_v = \mathcal{E}_v^{(0)} + \mathcal{E}_v^{(+)}$) such that

$$\widehat{h}_p = h_p + \sum_{n=2}^{\infty} \sum_{\gamma=\sum_{i=1}^n \gamma_i} e^{\mathrm{i}\pi\tau Q_n(\{\gamma_i\})} R_n(\{\gamma_i\}, \tau_2) \prod_{i=1}^{n} h_{p_i}$$

transforms as a modular form of weight $-\frac{1}{2}b_2 - 1$. Here $Q_n(\{\gamma_i\}) = \kappa^{ab} q_a q_b - \sum_{i=1}^n \kappa_i^{ab} q_{i,a} q_{i,b}$ and $\kappa^{ab} = (\kappa_{abc} p^c)^{-1}$

$$R_n = \mathrm{Sym} \left\{ \sum_{T \in \mathbb{T}_n^S} (-1)^{n_T - 1} \mathcal{E}_{v_0}^{(+)} \prod_{v \in V_T \setminus \{v_0\}} \mathcal{E}_v^{(0)} \right\}$$



Alexandrov Banerjee Manschot BP '16-19

Mock modularity for DT invariants

- $\widehat{h}_p$ is modular but not holomorphic. Its anti-holomorphic derivative is entirely determined in terms of $\widehat{h}_{p_i}$,

$$\partial_{\bar\tau}\widehat{h}_p = \sum_{n=2}^{\infty} \sum_{\gamma=\sum_{i=1}^n \gamma_i} e^{\mathrm{i}\pi\tau Q_n(\{\gamma_i\})}\widehat{R}_n(\{\gamma_i\},\tau_2) \prod_{i=1}^{n} \widehat{h}_{p_i}$$

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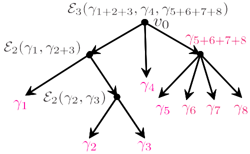

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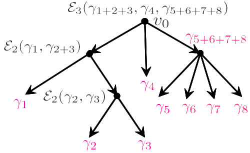

- In principle, one can use this information to determine h_p from the knowledge of the polar coefficients. In practice, this has only been done for one-parameter families of compact CY (such as the quintic) with primitive D4-brane charge [Gaiotto Strominger Yin '06]

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- The construction above predicts the non-holomorphic modular completion of the generating series of Vafa-Witten invariants for any Fano surface S and rank N ! *[Alexandrov BP Manschot '18-19]*
- For $N = 2$, $S = \mathbb{P}^2$, it reduces to the usual story about the generating series of Hurwitz class numbers.

Zagier '75; Klyashko '91; Yoshioka '94; Vafa Witten '94; Dabholkar Putrov Witten '20

Outline

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- The contributions from $D - \bar{D}$ instantons and Kaluza-Klein monopoles are not well understood yet. A first principle derivation from string field theory would be desirable.
- The hypermultiplet moduli space in type IIB/$\mathcal{X}$ is identical to the vector multiplet moduli space in type IIA on $\mathcal{X} \times S^1$, and determined by the same DT invariants, so this story may have implications for string phenomenology as well.