BPS black holes, wall-crossing and mock modular forms of higher depth - I

Boris Pioline

LABORATOIRE DE PHYSIQUE THEORIQUE ET HAUTES ENERGIES

Workshop on Moonshine, Erwin Schrödinger Institute, Vienna, 13/09/2018

based on arXiv:1804.06928,1808.08479 with Sergei Alexandrov,

and earlier works 1605.05945,1702.05497 with S. Banerjee and J. Manschot

Precision counting of $\mathcal{N}=4$ BPS black holes I

- Our goal is precision counting of BPS black holes in $\mathcal{N}=2$ string vacua. For perspective, I will first recall aspects of the $\mathcal{N}=4$ story, which should be more familiar to Moonshine practitioners.
- In $\mathcal{N}=4$ string vacua, such as type II strings compactified on $K_3 \times T_2$, heterotic strings on T^6 or orbifolds thereof, the BPS indices $\Omega(\gamma, z)$ counting 1/4-BPS states with charge $\gamma=(Q,P)$ in a vacuum with moduli $z \in \mathcal{M}_4$ at spatial infinity are given by Fourier coefficients of a meromorphic Siegel modular form,

$$\Omega(\gamma, z) = \oint_{\mathcal{C}(\gamma,z)} \frac{e^{2\pi \mathrm{i} \mathrm{Tr}(\tau\cdot\gamma\otimes\gamma)}}{\Phi(\tau)} , \qquad \gamma\otimes\gamma = \begin{pmatrix} Q^2 & Q\cdot P \\ Q\cdot P & P^2 \end{pmatrix}$$

Dijkgraaf Verlinde Verlinde '96; David Jatkar Sen '05-06; ...

Precision counting of $\mathcal{N} = 4$ BPS black holes II

- When z crosses real codimension-1 walls

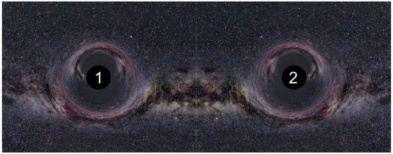
$$W(\gamma_L, \gamma_R) = \{z \in \mathcal{M}_4, M(\gamma_L + \gamma_R) = M(\gamma_L) + M(\gamma_R)\}$$

where γ_L, γ_R are 1/2-BPS charge vectors, the contour $\mathcal{C}(\gamma, z)$ crosses a pole of $1/\Phi(\tau)$, so that the index Ω jumps according to the primitive wall-crossing formula

$$\Delta\Omega(\gamma_L + \gamma_R) = \langle \gamma_L, \gamma_R \rangle \, \Omega(\gamma_L) \, \Omega(\gamma_R)$$

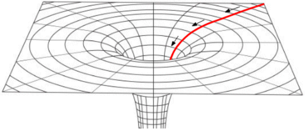
Denef Moore '07; Cheng, Verlinde '07; Sen '07-08

corresponding to contributions of bound states of two 1/2-BPS black holes.



Precision counting of $\mathcal{N} = 4$ BPS black holes III

- One may extract the contributions of single-centered black holes by evaluating $\Omega(\gamma, z)$ at the attractor point z_γ, where two-centered bound states are not allowed to form.



- The attractor indices $\Omega_*(\gamma) = \Omega(\gamma, z_\gamma)$ turn out to be Fourier coefficients of a vector-valued mock modular form. *[Dabholkar Murthy Zagier '12]*
- An interesting question is to derive $\Omega_*(\gamma)$ from holography in $AdS_2 \times S^2$, and understand the origin of the non-holomorphic correction term in the modular completion. *[Murthy BP'18]*

Precision counting of $\mathcal{N}=2$ BPS black holes I

In $\mathcal{N}=2$ string vacua, such as type II strings compactified on a CY threefold $\mathfrak{Y}$, the situation is far more complicated, due to the fact that

- The moduli space of scalars is no longer a symmetric space, instead

$$\mathcal{M}_4 = \mathcal{M}_V \times \mathcal{M}_H$$

  where $\mathcal{M}_V$ receives worldsheet instanton corrections (in IIA), and $\mathcal{M}_H$ receives both worldsheet instanton (in IIB), Euclidean D-brane instantons and NS5-brane instantons (in both)

- Fortunately, the BPS index and mass depend only on $\mathcal{M}_V$; in particular

$$M(\gamma, z) = |Z(\gamma, z)|$$

  where $Z(\gamma, z^a)$ is linear in γ and holomorphic in $z \in \mathcal{M}_V$

Precision counting of $\mathcal{N}=2$ BPS black holes II

- BPS bound states can involve an arbitrary number of BPS constituents with charges $\{\gamma_i\}$ such that $\gamma=\sum_i \gamma_i$. In particular, across a wall where $Z(\gamma_L) \parallel Z(\gamma_R)$, all indices $\Omega(\gamma,z)$ with $\gamma \in \mathrm{Span}(\gamma_L,\gamma_R)$ may jump.
- The jump $\Delta\Omega(N_L\gamma_L+N_R\gamma_R)$ was first computed by Joyce-Song and Kontsevich-Soibelman in the context of generalized Donaldson-Thomas invariants, which count stable coherent sheaves with $\gamma \sim \mathrm{ch}(\mathcal{E})$ and stability condition $Z(\gamma,z)$.
- The KS/JS wall-crossing formulae were (re)derived physically from the SUSY quantum mechanics of multi-centered black holes.

Denef Moore '07; de Boer et al '08; Andriyash et al '10, Manschot BP Sen '10

Precision counting of $\mathcal{N} = 2$ BPS black holes III

- The challenge is to compute $\Omega(\gamma, z)$ exactly, in some chamber and for an infinite class of charge vectors γ with $S_{\rm BH}(\gamma) > 0$.
- This may become feasible if the indices are Fourier coefficients of some quasi-modular generating function, with prescribed modular anomaly or modular completion.
- A natural sector is to consider D4-D2-D0 branes wrapped on a divisor $\mathcal{D} \subset \mathfrak{Y}$. In M-theory on $\mathfrak{Y} \times S_1$, this configuration lifts to an M5-brane wrapping $\mathcal{D} \times S_1$, described at low energy by a (0,4) 'black string SCFT' with computable central charges.

Maldacena Strominger Witten '97

Precision counting of $\mathcal{N}=2$ BPS black holes IV

- One expects that the generating function of the D4-D2-D0 indices

$$h_{p^a}(\tau,z) \sim \sum_{q_a,q_0} \Omega(0,p^a,q_a,q_0;z)\,\mathbf{e}(\tau q_0 + y^a q_a)$$

  is given by the elliptic genus of this SCFT, therefore (after performing the theta series decomposition to extract the sum over D2-brane fluxes q_a) by a vector-valued modular form of weight $w=-\frac{1}{2}b_2(\mathfrak{Y})-1$ and multiplier system.

  Gaiotto Strominger Yin '06, de Boer et al '06, Denef Moore '07

- This strategy was applied successfully to compute BPS indices for a single D4-brane on the quintic, using modularity plus explicit computations at small D0-brane charge.

  Gaiotto et al '05-06, Collinucci Wyder '08

Precision counting of $\mathcal{N} = 2$ BPS black holes V

- However, this expectation may break down for non-primitive D4-brane charge, or more generally when the D4-brane wraps a reducible divisor, due to wall-crossing.
- We shall be interested in the generating function of D4-D2-D0 BPS indices at the large volume attractor point

$$z^a_\infty(\gamma) = \lim_{\lambda\to+\infty} (-q^a + \mathrm{i}\lambda p^a)\,, \qquad \begin{cases} q^a = \kappa^{ab} q_b \\ \kappa_{ab} = \kappa_{abc} p^c \end{cases}$$

where D4-brane bound states are ruled out. We abuse notation and denote $\Omega_*(\gamma) = \Omega(\gamma, z^a_\infty(\gamma))$, which we call MSW invariants.

de Boer et al 08, Andriyash 08, Manschot 09

Modularity from S-duality I

- To determine the precise modular properties of generalized DT invariants, one can focus on a particular BPS-saturated coupling in the low-energy action of type IIA/$\mathfrak{Y} \times S_1(R)$, which receives contributions from Euclidean BPS black holes wrapped on S_1.
  [Gunaydin Neitzke BP Waldron '05]
- Namely, in $D = 3$ the moduli space factorizes as $\mathcal{M}_3 = \widetilde{\mathcal{M}}_V \times \mathcal{M}_H$, where both factors are quaternion-Kähler manifolds. As $R \to \infty$,

$$\widetilde{\mathcal{M}}_V \sim \text{c-map}(\mathcal{M}_V) + \sum_\gamma \Omega(\gamma, z^a)\, e^{-RM(\gamma)} + \dots$$

Cecotti Ferrara Girardello '89, Ferrara Sabharwal '90; Alexandrov BP Vandoren '08

- Since IIA/$\mathfrak{Y} \times S_1 =$ M/$\mathfrak{Y} \times T^2$, $\widetilde{\mathcal{M}}_V$ must admit an isometric action of $SL(2,\mathbb{Z})$, which stays unbroken in the large volume limit.

Modularity from S-duality II

- Main point: *this requirement implies that the generating function of DT invariants satisfies the MSW modularity constraints*, at least when the divisor $\mathcal{D}$ wrapped by the D4-brane is irreducible.
- When $\mathcal{D}$ is a sum of $n \geq 2$ irreducible divisors, the generating function acquires a specific modular anomaly: they are now mock modular forms of depth $n - 1$. *[Alexandrov Banerjee Manschot BP '16, Alexandrov BP '18]*
- *Remark: $\widetilde{\mathcal{M}}_V$ is also the hypermultiplet moduli space $\mathcal{M}_H$ in type IIB string theory compactified on $\mathfrak{Y}$, with $SL(2,\mathbb{Z})$ being the usual type IIB S-duality in $D = 10$. Counting D4-D2-D0 bound states is equivalent to computing D3-D1-D(-1) instanton corrections to $\mathcal{M}_H$.*

Alexandrov, Banerjee, Manschot, Persson, BP, Saueressig, Vandoren '08-18

Outline

Outline

Vector multiplet moduli space in $D = 3$ I

- The VM moduli space $\mathcal{M} = \widetilde{\mathcal{M}}_V$ in M-theory compactified on $\mathfrak{Y} \times T^2$ has dimension $4b_2 + 4$:
  - τ: complex structure of T^2
  - t^a: Kähler moduli of $\mathfrak{Y}$ on a basis γ^a, $a = 1 \dots b_2$ of $H_2(\mathfrak{Y}, \mathbb{Z})$
  - (b^a, c^a): period of the 3-form on $\gamma^a \times S_1$
  - $\tilde{c}_a$: period of 6-form on $\gamma_a \times T^2$, γ_a basis of $H_4(\mathfrak{Y}, \mathbb{Z})$
  - $(\tilde{c}_0, \psi)$: dual of the KK gravitons
- In IIA/$\mathfrak{Y} \times S_1(R)$, the moduli $(\zeta^\Lambda, \tilde{\zeta}_\Lambda) \sim (\tau_1, c^a, \tilde{c}_a, \tilde{c}_0)$ defined via the classical mirror map are fibered over the complexified Kähler moduli space parametrized by $z^a = b^a + \mathrm{i} t^a$, and transform as a vector under the monodromy group $\Gamma \subset Sp(2b_2 + 2, \mathbb{Z})$.

Böhm Günther Herrmann Louis '99

Vector multiplet moduli space in $D = 3$ II

- In the large volume limit $t^a \to \infty$, $\mathcal{M}$ reduces to the c-map of the special Kähler space with prepotential

$$F(X) = -\frac{1}{6}\kappa_{abc}\frac{X^a X^b X^c}{X^0}\,, \qquad \frac{X^a}{X^0} = z^a = b^a + \mathrm{i}t^a$$

  It admits an isometric action of $SL(2,\mathbb{R})$:

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}\,, \qquad t^a \mapsto |c\tau + d|\, t^a, \qquad \begin{pmatrix} c^a \\ b^a \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} c^a \\ b^a \end{pmatrix},$$

$$\tilde{c}_a \mapsto \tilde{c}_a\,, \qquad \begin{pmatrix} \tilde{c}_0 \\ \psi \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \tilde{c}_0 \\ \psi \end{pmatrix}$$

- $SL(2,\mathbb{R})$ is broken by worldsheet and D-instantons to $SL(2,\mathbb{Z})$

  Robles-Llana Rocek Saueressig Theis Vandoren '05

- In absence of KK monopoles (or NS5-D5 instantons in IIB picture), the continuous isometries along $(\tilde{c}_0, \psi)$ are unbroken.

Twistorial description of instanton corrections I

- Instanton corrections to the QK metric are most easily described in terms of a complex contact structure on the twistor space $\mathbb{P}^1_t \to \mathcal{Z} \to \mathcal{M}$. Locally, the contact 1-form can be written as

$$e^{\Phi}\left(\frac{\mathrm{d}t}{t} + \frac{p_+}{t} + p_3 + t p_-\right) = \mathrm{d}\alpha + \tilde{\xi}_\Lambda \mathrm{d}\xi^\Lambda$$

  where $p_\pm, p_3$ are the components of the $SU(2)$ part of the Levi-Civita connection on $\mathcal{M}$, $\alpha, \xi^\Lambda, \tilde{\xi}_\Lambda$ are local Darboux coordinates and $\Phi(t,x)$ is the contact potential.
- The contact structure is defined globally by specifying complex contact transformations on overlaps of Darboux coordinate patches.
- Key fact: any isometry of $\mathcal{M}$ lifts to a holomorphic contact transformation on $\mathcal{Z}$. *[Salamon, Le Brun]*

Twistorial description of instanton corrections II

- In the large volume limit, a single Darboux coordinate system suffices, away from $t = 0$ and $t = \infty$,

$$\begin{aligned} \xi^\Lambda &= \zeta^\Lambda + \frac{\tau_2}{2}\left(\bar{X}^\Lambda t - X^\Lambda\, t^{-1}\right) \qquad & \alpha = \psi + \dots \\ \tilde{\xi}_\Lambda &= \tilde{\zeta}_\Lambda + \frac{\tau_2}{2}\left(\bar{F}_\Lambda t - F_\Lambda\, t^{-1}\right) \qquad & F_\Lambda = \partial F/\partial X^\Lambda \end{aligned}$$

  and the contact potential is $e^\Phi = \frac{\tau_2^2}{12}(t^3)$ where $(t^3) \equiv \kappa_{abc} t^a t^b t^c$.

- Under $SL(2,\mathbb{R})$, with a suitable action on the $\mathbb{P}^1$ fiber, the Darboux coordinates transform by a complex contact transformation,

$$\xi^0 \mapsto \frac{a\xi_0 + b}{c\xi^0 + d}, \quad \xi^a \mapsto \frac{\xi^a}{c\xi^0 + d}, \dots, \quad e^\phi \mapsto \frac{e^\phi}{|c\tau + d|}.$$

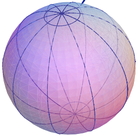
- It is advantageous to define $z = \frac{t+\mathrm{i}}{t-\mathrm{i}}$ so that the action of $SL(2,\mathbb{Z})$ on the $\mathbb{P}^1$ fiber simplifies to a phase rotation, $z \mapsto \frac{c\bar{\tau}+d}{|c\tau+d|}\, z$.

Twistorial description of instanton corrections III

- Instanton corrections induce discontinuities in Darboux coordinates along the BPS rays $\ell_\gamma = \{t \in \mathbb{P}^1, Z_\gamma/t \in \mathrm{i}\mathbb{R}^-\}$. The coordinates in each angular sector are solutions of the 'TBA eqs'

$$\mathcal{X}_\gamma(t) = \mathcal{X}^{\rm cl}_\gamma(t)\, \mathbf{e}\left(\frac{1}{8\pi^2} \sum_{\gamma'\in\Gamma} \bar\Omega(\gamma')\, \langle\gamma,\gamma'\rangle \int_{\ell_{\gamma'}} \frac{\mathrm{d}t'}{t'}\, \frac{t+t'}{t-t'}\, \mathcal{X}_{\gamma'}(t') \right)$$

where $\mathcal{X}_{(p^\Lambda,q_\Lambda)} = \mathbf{e}\left(p^\Lambda\tilde\xi_\Lambda - q_\Lambda\xi^\Lambda\right)$ are the 'holomorphic Fourier modes' and $\mathcal{X}^{\rm cl}_\gamma$ their classical (a.k.a semi-flat) limit.



- Here $\bar\Omega(\gamma) = \sum_{d|\gamma} \frac{1}{d^2}\, \Omega(\gamma/d)$ are the rational DT invariants. $\bar\Omega(\gamma)$ may jump across walls of marginal stability, but the wall-crossing formula ensures that the QK metric on $\mathcal{M}$ is smooth.

Gaiotto Moore Neitze '08; Alexandrov '09

Twistorial description of instanton corrections IV

- As $\tau_2 \to \infty$, the integrals over ℓ_γ are dominated by a saddle point at $t_\gamma = \mathrm{i} \arg Z_\gamma$, leading to corrections of order $e^{-\pi\tau_2|Z_\gamma|}$. Thus, one may solve the system iteratively, producing a multi-instanton series in the form of a sum over rooted trees.

Gaiotto Moore Neitze '08; Stoppa 11

- Having found $\mathcal{X}_\gamma$, hence $(\xi^\Lambda, \tilde\xi_\Lambda)$, the coordinate α and contact potential follow by one further integration, e.g.

$$e^\Phi = \frac{\tau_2^2}{8} \mathrm{Im}\left(X^\Lambda \bar F_\Lambda\right) + \frac{\mathrm{i}\tau_2}{16} \sum_\gamma \int_{\ell_\gamma} \frac{\mathrm{d}t}{t} \left(t^{-1} Z_\gamma - t \bar Z_\gamma\right) H_\gamma,$$

where $H_\gamma = \frac{\bar\Omega(\gamma)}{(2\pi)^2} \mathcal{X}_\gamma$.

Outline

Multi-instantons in the large volume limit I

- In the large volume limit $t^a \to \infty$, the saddle point $t_\gamma \to \pm\mathrm{i}$ so that $z_\gamma = -\mathrm{i}\,\frac{(q_a+(pb)_a)\,t^a}{(pt^2)}$. The QK metric on $\mathcal{M}$ admits a simplified twistorial description by zooming near $z \to 0$ keeping zt^a fixed.
- In addition, one needs to take account corrections to the mirror map, determined such that the standard $SL(2,\mathbb{Z})$ action on $t^a, (c^a, b^a), \dots$ lifts to a holomorphic action on $(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)$.

Multi-instantons in the large volume limit II

- Keeping only contributions from D4-branes (or D3-branes in IIB language), we find that the TBA equations reduce to

$$\boxed{H_\gamma(z) = H^{\rm cl}_\gamma(z)\, \exp\left[\sum_{\gamma'\in\Gamma_+} \int_{\ell_{\gamma'}} \mathrm{d}z'\, K_{\gamma\gamma'}(z,z')\, H_{\gamma'}(z')\right]}$$

where $\ell_\gamma = \mathbb{R} + \mathrm{i} z_\gamma$, $H_\gamma = \frac{\bar\Omega(\gamma)}{(2\pi)^2}\, \mathcal{X}_\gamma$,

$$K_{\gamma_1\gamma_2}(z_1,z_2) = 2\pi\left((tp_1p_2) + \frac{\mathrm{i}\langle\gamma_1,\gamma_2\rangle}{z_1 - z_2}\right)$$

$H^{\rm cl}_\gamma$ is obtained by replacing $\mathcal{X}_\gamma$ by its classical limit

$$\mathcal{X}^{\rm cl}_\gamma = e^{-\pi\tau_2 (pt)^2 + 2\pi\mathrm{i} p^a \tilde c_a - 2\pi\tau_2 (pt^2)(z^2 - 2z_\gamma) + \dots}$$

Multi-instantons in the large volume limit III

- The same expansion can be carried out for the contact potential:

$$e^{\Phi} = \frac{\tau_2^2}{12}(t^3) + \frac{\tau_2}{2}\mathrm{Re}\left(\mathcal{D}_{-\frac{3}{2}}\mathcal{G}\right) + \frac{1}{32\pi^2}\,\kappa_{abc}t^c\partial_{\tilde{c}_a}\mathcal{G}\,\partial_{\tilde{c}_b}\overline{\mathcal{G}}.$$

where $\mathcal{G}$ is the instanton generating function

$$\boxed{\mathcal{G} = \sum_{\gamma\in\Gamma_+}\int_{\ell_\gamma}\mathrm{d}z\,H_\gamma(z) - \frac{1}{2}\sum_{\gamma_1,\gamma_2\in\Gamma_+}\int_{\ell_{\gamma_1}}\mathrm{d}z_1\int_{\ell_{\gamma_2}}\mathrm{d}z_2\,K_{\gamma_1\gamma_2}(z_1,z_2)\,H_{\gamma_1}(z_1)H_{\gamma_2}(z_2)}$$

and $\mathcal{D}_{\mathfrak{h}}$ is the Maass raising operator

$$\mathcal{D}_{\mathfrak{h}} = \frac{1}{2\pi\mathrm{i}}\left(\partial_\tau + \frac{\mathfrak{h}}{2\mathrm{i}\tau_2} + \frac{\mathrm{i}t^a}{4\tau_2}\,\partial_{t^a}\right),$$

Multi-instantons in the large volume limit IV

- As $\tau_2 \to \infty$, the integral is dominated by a saddle point at $z = z_\gamma$, leading to exponentially suppressed corrections of order $e^{-\pi\tau_2(pt^2)}$.
- These equations can be solved iteratively,

$$H_{\gamma_1} = H^{\rm cl}_{\gamma_1} + \sum_{\gamma_2} K_{12} H^{\rm cl}_{\gamma_1} H^{\rm cl}_{\gamma_2} + \sum_{\gamma_2,\gamma_3} \left(\tfrac12 K_{12}K_{13} + K_{12}K_{23}\right) H^{\rm cl}_{\gamma_1} H^{\rm cl}_{\gamma_2} H^{\rm cl}_{\gamma_3} + \dots$$

$$\mathcal{G} = \sum_{\gamma} H^{\rm cl}_{\gamma} + \frac12 \sum_{\gamma_1,\gamma_2} K_{12} H^{\rm cl}_{\gamma_1} H^{\rm cl}_{\gamma_2} + \frac12 \sum_{\gamma_1,\gamma_2,\gamma_3} K_{12}K_{23} H^{\rm cl}_{\gamma_1} H^{\rm cl}_{\gamma_2} H^{\rm cl}_{\gamma_3} + \dots$$

where we denote $K_{ij} = K_{\gamma_i\gamma_j}(z_i, z_j)$ and omit the integrals.

Multi-instantons in the large volume limit V

- To all orders, the expansion is given by a sum over trees

$$\mathcal{G} = \sum_{n=1}^{\infty} \left(\prod_{i=1}^{n} \sum_{\gamma_i \in \Gamma_+} \int_{\ell_{\gamma_i}} \mathrm{d}z_i \, H^{\mathrm{cl}}_{\gamma_i}(z_i) \right) \sum_{\mathcal{T} \in \mathbb{T}_n} \frac{\prod_{e \in E_{\mathcal{T}}} K_{s(e)t(e)}}{|\mathrm{Aut}(\mathcal{T})|}$$

  where $\mathbb{T}_n$ is the set of (unrooted) trees with n vertices.
- One may show that jumps of $H^{\mathrm{cl}}(\gamma_i)$ across walls of marginal stability cancel against contributions of poles due to exchanging contours ℓ_{γ_i}, in such a way that $\mathcal{G}$ is smooth.

Modularity of the instanton generating function I

- Returning to the result for the contact potential,

$$e^{\Phi} = \frac{\tau_2^2}{12}(t^3) + \frac{\tau_2}{2}\mathrm{Re}\left(\mathcal{D}_{-\frac{3}{2}}\mathcal{G}\right) + \frac{1}{32\pi^2}\,\kappa_{abc}t^c\partial_{\tilde{c}_a}\mathcal{G}\,\partial_{\tilde{c}_b}\overline{\mathcal{G}},$$

  and requiring $e^{\phi} \mapsto \frac{e^{\phi}}{|c\tau+d|}$, we conclude that $\mathcal{G}$ should transform as a modular form of weight $(-\frac{3}{2},\frac{1}{2})$ (and specific multiplier system)
- In the one-instanton approximation, $\mathcal{G}$ coincides with the naive modified elliptic genus of the black string (0,4) SCFT, reproducing the modularity constraints of MSW.

Alexandrov Manschot BP '12

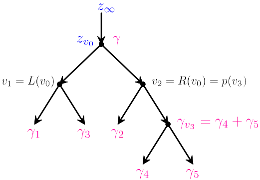
Outline

Tree flow formula I

- In order to spell out the constraints from modularity, we need to express the moduli-independent DT invariants $\Omega(\gamma, z^a)$ in terms of the attractor indices $\Omega_*(\gamma)$. For this, we may use the tree flow formula, inspired by the split attractor conjecture:

$$\bar{\Omega}(\gamma, z^a) = \sum_{\sum_{i=1}^n \gamma_i = \gamma} g_{\rm tr}(\{\gamma_i\}, z^a) \prod_{i=1}^n \bar{\Omega}_*(\gamma_i)$$

where the tree index is a sum over attractor flow trees,



$$g_{\rm tr}(\{\gamma_i\}, z^a) = \frac{1}{n!} \sum_{T \in \mathbb{T}_n^{\rm af}} \Delta(T)\, \kappa(T),$$

Denef Green Raugas '01; Denef Moore '07, Manschot 2010; Alexandrov BP '18

Tree flow formula II

- $\Delta(T) \in \{0, \pm 1\}$ ensures that only stable trees contribute,

$$\Delta(T) = \prod_{v \in V_T} \frac{1}{2} \left[\operatorname{sgn} \operatorname{Im}\left[Z_{\gamma_{L(v)}} \bar{Z}_{\gamma_{R(v)}}(z^a_{p(v)}) \right] + \operatorname{sgn}\langle \gamma_{L(v)}, \gamma_{R(v)} \rangle \right]$$

  where $z^a_{p(v)}$ are the moduli at the parent of the vertex v. The sign can be computed recursively in terms of the stability parameters $c_i = \operatorname{Im}\left[Z_{\gamma_i} \bar{Z}_{\gamma}(z^a) \right]$ using a discrete version of the attractor flow.

- $\kappa(T)$ is the bound state degeneracy:

$$\kappa(T) \equiv (-1)^{n-1} \prod_{v \in V_T} \kappa(\langle \gamma_{L(v)}, \gamma_{R(v)} \rangle), \qquad \kappa(x) = \frac{(-y)^x - (-y)^{-x}}{y - 1/y} \overset{y \to 1}{\longrightarrow} (-1)^x \, x$$

- The flow tree formula is manifestly consistent with the (refined) wall-crossing formula across walls of marginal stability. Apparent discontinuities across fake walls cancel after summing over trees.

Tree flow formula III

- Expressing $\Delta(T)$ in terms of asymptotic data, one finds products of sign functions whose arguments are polynomial in $\gamma_{ij} = \langle \gamma_i, \gamma_j \rangle$, and linear in the stability parameters c_i.
- After summing over trees and using sign identities such as

$$\mathrm{sgn}(x_1 + x_2) \times [\mathrm{sgn}(x_1) + \mathrm{sgn}(x_2)] = 1 + \mathrm{sgn}(x_1)\,\mathrm{sgn}(x_2)$$

  $g_{\rm tr}$ can be rewritten as a sum of products of sign functions whose arguments are linear both in γ_{ij} and c_i.
- To show this, we write the refined tree index as

$$g_{\rm tr}(\{\gamma_i, c_i\}, y) = \frac{(-1)^{n-1+\sum_{i<j}\gamma_{ij}}}{(y-y^{-1})^{n-1}}\,\mathrm{Sym}\left\{F_{\rm tr}(\{\gamma_i, c_i\})\, y^{\sum_{i<j}\gamma_{ij}}\right\},$$

  where the partial tree index $F_{\rm tr}$ is a sum over planar flow trees,

$$F_{\rm tr}(\{\gamma_i, c_i\}) = \sum_{T\in\mathbb{T}_n^{\text{af-pl}}} \Delta(T),$$

Tree flow formula IV

- By definition, the partial tree index satisfies the recursion

$$F_{\rm tr}(\{\gamma_i, c_i\}) = \frac{1}{2}\sum_{\ell=1}^{n-1}(\mathrm{sgn}(S_\ell) - \mathrm{sgn}(\Gamma_{n\ell})) \times F_{\rm tr}(\{\gamma_i, c_i^{(\ell)}\}_{i=1}^{\ell})\, F_{\rm tr}(\{\gamma_i, c_i^{(\ell)}\}_{i=\ell+1}^{n}),$$

where $c_i^{(\ell)} = c_i - \frac{\beta_{ni}}{\Gamma_{n\ell}}\, S_\ell$,

$$S_\ell = \sum_{i=1}^{\ell} c_i, \quad \beta_{k\ell} = \sum_{i=1}^{k} \gamma_{i\ell}, \quad \Gamma_{k\ell} = \sum_{i=1}^{k}\sum_{j=1}^{\ell} \gamma_{ij}\ .$$

This sums over all ways of constructing a planar tree with n leaves by merging planar trees with ℓ and $n-\ell$ leaves.

Tree flow formula V

- Less obvious is the fact that it satisfies another recursion,

$$F_{\rm tr}(\{\gamma_i, c_i\}) = F_n^{(0)}(\{\gamma_i, c_i\}) - \sum_{\substack{n_1+\dots+n_m=n \\ n_k\geq 1,\ m<n}} F_{\rm tr}(\{\gamma'_k, c'_k\}_{k=1}^m) \prod_{k=1}^m F_{n_k}^{(\star)}(\gamma_{j_{k-1}+1}, \dots, \gamma_{j_k}),$$

where the sum runs over ordered partitions of n with m parts,

$$j_0 = 0, \qquad j_k = n_1 + \dots + n_k, \qquad \gamma'_k = \gamma_{j_{k-1}+1} + \dots + \gamma_{j_k}.$$

$$F_n^{(0)}(\{\gamma_i, c_i\}) = \frac{1}{2^{n-1}} \prod_{i=1}^{n-1} \mathrm{sgn}(S_i), \quad F_n^{(\star)}(\{\gamma_i\}) = \frac{1}{2^{n-1}} \prod_{i=1}^{n-1} \mathrm{sgn}(\Gamma_{ni}).$$

The virtue of this representation is that the sign functions have arguments which are now manifestly linear in γ_{ij} and c_i.

Upshot I

- S-duality dictates that $\mathcal{G}$ should be modular of weight $(-\frac{3}{2}, \frac{1}{2})$,

$$\mathcal{G} = \frac{1}{(2\pi)^2} \sum_{n=1}^{\infty} \left(\prod_{i=1}^{n} \sum_{\substack{\gamma_i \in \Gamma_+ \\ \mathcal{T} \in \mathbb{T}_n}} \frac{\bar{\Omega}(\gamma_i, z^a)}{|\mathrm{Aut}(\mathcal{T})|} \int_{\ell_{\gamma_i}} \mathrm{d}z_i \, \sigma_{\gamma_i} \mathcal{X}^{\mathrm{cl}}_{\gamma_i}(z_i) \prod_{e \in E_{\mathcal{T}}} K_{s(e)t(e)} \right)$$

$$\bar{\Omega}(\gamma, z^a) = \sum_{\gamma = \sum_{i=1}^{n} \gamma_i} g_{\mathrm{tr}}(\{\gamma_i, c_i\}) \prod_{j=1}^{n} \bar{\Omega}_*(\gamma_i)$$

  Since $\mathcal{X}^{\mathrm{cl}}(z_i)$ is Gaussian in z_i and K_{ij} are rational in $z_i - z_j$, the integrals over $z_i \in \mathbb{P}^1$ are generalized error functions !

  Alexandrov Banerjee Manschot BP 2016; Nazaroglu 2016

- In his talk, Sergei will explain how the modularity of $\mathcal{G}$ can be translated into modularity constraints for the generating functions of the MSW invariants $\bar{\Omega}_*(\gamma)$.

Upshot II

Thanks for your attention...

... and be ready for some serious weightlifting !